

M. A.
Semester-I
ECONOMICS
COURSE CODE : ECO-103

Centre for Distance & Online Education

**UNIVERSITY OF JAMMU
JAMMU**



SELF LEARNING MATERIAL

M. A.ECONOMICS

Title : MATHEMATICAL METHODS IN ECONOMICS

Session 2025-26 Onwards

SEMESTER : I	No. of Credits : 6	UNIT : I - IV
COURSE CODE : ECO-103		LESSON NO. : 1-27

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M.A. ECONOMICS IST SEMESTER

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M.A Economics under Non-CBCS (CDOE)

Semester-1

Syllabus for the examination to be held in December 2025, 2026 & 2027

Course Code: ECO-103

Title – Mathematical Methods in Economics

Credits:6

Max. Marks: 100

Internal Assessment: 30

Time: 3 Hours

Semester Exam: 70

Course Outcomes

- To train the students in the use of mathematical tools to understand concepts in Economics presented in the form of mathematical models and express economic ideas in the same form.
- To enable the students to utilize the tools in subsequent courses especially those courses where the use of mathematics has become a norm.

UNIT-I: Equations and Differentiation

Numbers-natural, Integers, rational irrational complex, linear equations. Mathematical operations with Matrices, solution of simultaneous equations: Rank of the matrix, matrix inversion. Quadrants Equations, Eigen roots and Eigen Vectors. Concepts of limit and continuity, Economic examples and applications. Principles of differentiation, rules of differentiation, differentiation of implicit function, parametric function.

UNIT-II: Calculus

Partial and total differentiation, Expansion by Taylor Series Allied economic applications, Maxima and Minima- constrained and unconstrained, economic application. Principles of integration: Indefinite and definite. Application in economics theory, Economic application- Derivation of Consumer's Surplus, Producer's Surplus, Profit and utility maximization with one good, product and input, Derivation of demand curves for inputs, goods etc., Comparative statics and allied economic applications: combining calculus-Derivation of Slutsky Equation, IS-LM model, Cobb Douglas and CES production functions, elasticity of demand, supply substitution.

UNIT-III: Difference and Differentials

Differential Equations: definitions and concepts. Solution of first order and second order differential equations, Difference equations definitions and concepts Solution of first order and second order difference equations, Simultaneous Differential equations and phase diagrams. Application of difference and differential equations in Economics, Cobweb model, foreign trade multiplier model. Market model with stocks- National Income Model.

UNIT-IV : Linear Programming, Operations and Applications

Linear programming- Basic concepts, functions of a LP problem, Nature of feasible, basic and optimal solutions, Solution of a LP problem through graphical and simplex methods (Slack Surplus and artificial variables), Formulation of Dual and its interpretation, Input-Output Analysis Introduction, Input-Output transaction table, the technological Co- efficient matrix, solution of open model, The Hawkins-Simon conditions, solution for 2 and 3 industries, determination of equilibrium prices.

Note for Paper Setting

The term end examination shall be of 70 marks. There shall be two types of questions in each unit, two short answer type (each of 250 words) and two medium answer type (each of 600 words). The candidate will have to attempt one short answer type question and one medium answer type question from each unit. Each short answer type question shall carry 4.5 marks and each medium answer type question shall carry 13 marks.

Basic Reading List

1. Allen, R.G.D. (1976). Mathematical Analysis for Economists, Macmillan.
2. Chiang, A.C.(1974). Fundamental Methods of Mathematical Economics, McGraw Hill and Kogakusha, New Delhi.
3. Mehta & Madnani (1992). Mathematical for Economists, S. Chand, New Delhi.
4. Samuelson, P.A. (1967). Foundations of Economic Analysis, McGraw Hill, Tokyo.
5. Henderson & Quandt, Microeconomics: A Mathematical Approach, Tata McGraw Hill.
6. Baumol, W.J.(1970). Economic Dynamics, Macmillan, London.
7. Leonard and Von Long (1978). Introduction to Maths for students of Economics, Cambridge.

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Unit II: Calculus 8. Basic Concepts of Partial and Total Differentiation. 9. Expansion of Partial and Total Differentiation by Taylor Series and Applications. 10. Concept of Maxima and Minima, Constrained and Unconstrained (Economic	L 8-----Prof. Aroon Sharma L 9-----Bipan Singh L 10-12..... Prof. Aroon Sharma L 13-----Bipan Singh

Applications) 11. Principles of Integration, Indefinite and Definite (Economic Applications) 12. Comparative Statics and Its Economic Applications. 13. Cobb-Douglas and CES Production Function	
Unit III: Difference and Differential Equations 14. Differential Equations: Definitions and Concepts. 15. Solving First-Order Differential Equations. 16. Solving Second-Order Differential Equations. 17. Difference Equations: Definitions and Concepts. 18. Solving First-Order Difference Equations. 19. Solving Second-Order Difference Equations. 20. Simultaneous Differential Equations and Phase Diagrams. 21. Applications of Difference and Differential Equations in Economics.	Mohd. Yassir
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models. 27. Applications of Input-Output Analysis in Economics.	

M.A. Economics	Lesson No. 1
C.No. : Eco-103	Semester - 1st
	Unit I

INTRODUCTION TO NUMBERS AND BASIC CONCEPTS (NATURAL, INTEGERS, RATIONAL, IRRATIONAL AND COMPLEX NUMBERS)

STRUCTURE

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- 1.3 Natural Numbers
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- 1.5 Integers
- 1.6 Rational Numbers
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1.0 Objectives

The main objectives of this lesson are to enable you to:

- ☐ Define and differentiate among natural numbers, whole numbers, integers, rational numbers, irrational numbers, and complex numbers.
- ☐ Perform arithmetic operations (addition, subtraction, multiplication, division) on different types of numbers.
- ☐ Apply the concepts of modulus and conjugate in complex numbers.
- ☐ Convert rational numbers to decimal form and vice versa.
- ☐ Represent numbers on the number line and compare them.
- ☐ Identify standard forms and equivalent forms of rational numbers.
- ☐ Explore properties and rules related to the number system.

1.1 Learning outcomes

After going through this lesson, you will be able to:

- ☐ Classify a number as natural, whole, integer, rational, irrational, or complex.

- Apply the rules of arithmetic to rational and complex numbers.
- Represent rational and irrational numbers on the number line.
- Use number operations to solve real-life and mathematical problems.
- Use the concepts of modulus and conjugate in solving equations involving complex numbers.
- Distinguish between terminating and non-terminating decimals.
- Find rational numbers between two given numbers.

1.2 Introduction

The number system is a fundamental concept in mathematics, forming the basis of nearly all numerical computations and mathematical operations. From simple counting to complex problem-solving, an understanding of different kinds of numbers—including natural numbers, whole numbers, integers, rational numbers, irrational numbers, and complex numbers—is essential. This lesson introduces the classification, properties, and operations associated with various types of numbers, and explores their representations and applications.

1.3 Natural Numbers

Numbers of the form 1, 2, 3, 4 which are frequently used in counting are called natural numbers and the set of all natural numbers is denoted by the symbol N or Z . Natural numbers are also called cardinal numbers or positive integers.

$$\begin{aligned} N \text{ (or } Z) &= \{1, 2, \text{ and } 3 \text{ ---}\} = \{x: x \text{ is a natural number}\} \\ &= \{x: x \in Z\} \end{aligned}$$

There is no greatest natural number, for if 1 is added to any natural number, we get the next higher natural number, called its successor.

For, example,

$4 + 2 = 6$, again a natural number;

$6 + 21 = 27$, again a natural number;

$22 - 6 = 16$, again a natural number,

but $2 - 6$ is not defined in natural numbers.

Similarly, $4 \times 3 = 12$, again a natural number

$12 \times 3 = 36$, again a natural number;

$(12/2) = 6$ is a natural number but $(6/4)$ is not defined in natural numbers. Thus, we can say that

i) a) addition and multiplication of natural numbers again yield a natural number but

b) Subtraction and division of two natural numbers may or may not yield a natural number

i) The natural numbers can be represented on a number line as shown below.



ii) Two natural numbers can be added and multiplied in any order and the result obtained is always same. This does not hold for subtraction and division of natural numbers.

1.4 Whole Numbers

Whole numbers are the set of numbers that include all natural numbers (positive integers) and zero. The set of whole numbers is generally denoted by **W** and is written as:

$$\mathbf{W} = \{0, 1, 2, 3, 4, 5, \dots\}$$

There is no greatest whole number as in the case of natural numbers.

(i) The number 0 has the following properties:

- $a + 0 = a = 0 + a$
 - $a - 0 = a$ but $(0 - a)$ is not defined in whole numbers
 - $a \times 0 = 0 = 0 \times a$
 - Division by zero (0) is not defined.
- (ii) Four fundamental operations can be performed on whole numbers also as in the case of natural numbers (with restrictions for subtraction and division).
- (ii) Whole numbers can also be represented on the number line as follows:



1.5 Integers

Integers are the numbers consisting of all natural numbers, zero and negatives of natural numbers. The set of integers is denoted by 'I'. Thus,

$$I = \{\dots -3, -2, -1, 0, 1, 2, 3 \dots\}$$

$$I = \{x: x \in \mathbb{Z}\}$$

1, 2, 3... are called positive integers (+ve) and -3, -2, -1 are called negative integers (-ve). The set of all positive integers is denoted by I^+ , the set of all negative integers by I^- and the set of all integers excluding 0 {i.e. the set of all non-zero integers} is denoted by I_0 . Thus

$$I^+ = \{1, 2, 3, \dots\} = \{x: x \text{ is a +ve integer}\}$$

$$= \{x, x \in I^+\}$$

$$I^- = \{\dots -3, -2, -1 \dots\} = \{x: x \text{ is -ve integer}\}$$

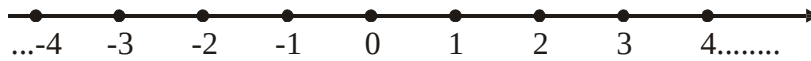
$$= \{x, x \in I^-\}$$

$$I_0 = \{\dots -3, -2, -1, 1, 2, 3, \dots\} = \{x: x \text{ is an integer excluding } 0\}$$

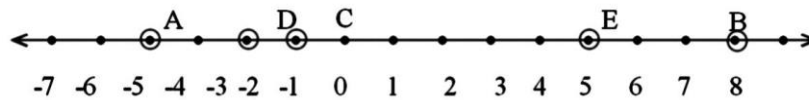
$$= \{x, x \in I_0\}$$

We extend the number line used for representing whole numbers to the left of

zero and mark points $-1, -2, -3, -4 \dots$ such that 1 and -1 , 2 and -2 , 3 and -3 are equidistant from zero and are in opposite directions of zero. Thus, we have the integer number line as follows:



We can now easily represent integers on the number line. For example, let us represent $-5, 7, -2, -3, 4$ on the number line. In the figure, the points A, B, C, D and E respectively represent $-5, 7, -2, -3$ and 4.



We note here that if an integer $a > b$, then 'a' will always be to the right of 'b', otherwise vise-versa.

For example, in the above figure $7 > 4$, therefore B lies to the right of E. Similarly, $-2 > -5$, therefore C (-2) lies to the right of A (-5).

Conversely, as $4 < 7$, therefore 4 lies to the left of 7 which is shown in the figure as E is to the left of B.

Therefore, for finding the greater (or smaller) of the two integers a and b, we follow the following rule:

- i) $a > b$, if a is to the right of b
- ii) $a < b$, if a is to the left of b

Example Identify natural numbers, whole numbers and integers from the following:-

15, 22, -6 , 7, -13 , 0, 12, -12 , 13, -31

Solution Natural numbers are: 7, 12, 13, 15 and 22

Whole numbers are: 0, 7, 12, 13, 15 and 22

Integers are: $-31, -13, -12, -6, 0, 7, 12, 13, 15$ and 22

1.6 Rational Numbers

All numbers of the form a/b , where a and b are integers and $b \neq 0$ are called rational numbers. These can be expressed either as finite decimals or recurring (infinite) decimals. For example $1/5 = 0.2$, $1/2 = 0.5$ and $1/3 = .333 \dots$. The set of all rational numbers is denoted by Q .

Thus,

$$Q = \{a/b: a \text{ and } b \text{ are integers and } b \neq 0\}$$

$$Q^+ = \{\text{All positive rational numbers}\}$$

$$= \{x: x \text{ is a +ve rational number}\}$$

$$Q^{-1} = \{\text{All negative rational numbers}\}$$

$$= \{x: x \text{ is a -ve rational number}\}$$

$$Q_0 = \{\text{All rational numbers excluding } 0\}$$

$$= \{\text{All non-zero rational numbers}\}$$

$$= \{x: x \text{ is a rational number} + x \neq 0\}$$

- i) A rational number a/b is said to be positive rational number if a and b are both positive or both negative integers

$$\text{Thus, } \frac{3}{4}, \frac{5}{6}, \frac{-3}{-2}, \frac{-8}{-6}, \frac{-12}{-57} \text{ are all positive rationals.}$$

- ii) If the integers a and b are of different signs, then a/b is said to be a negative rational number

$$\text{Thus, } \frac{-7}{2}, \frac{6}{-5}, \frac{-12}{3}, \frac{18}{3} \text{ are all negative rationals.}$$

1.6.1 Standard Form of a Rational Number

We know that numbers of the form $\frac{-a}{b}, \frac{a}{-b}, \frac{-a}{-b}$ & $\frac{a}{b}$ are all rational numbers, where a and b are positive integers.

$$\text{We can see that } \frac{-a}{b} = -\frac{a}{b}, \frac{-a}{-b} = \frac{-(-a)}{-(-b)} = \frac{a}{b}, \frac{a}{-b} = \frac{(-a)}{-(-b)} = \frac{-a}{b}$$

In each of the above cases, we have made the denominator q as positive. A rational number a/b , where a and b are integers and $b \neq 0$, in which b is positive (or made positive) and a and b are co-prime (i.e. when they do not have a common factor other than 1 and -1) is said to be in standard form.

Thus, the standard form of the rational number $\frac{2}{-3}$ is $\frac{-2}{3}$. Similarly $\frac{-5}{6}$ and $\frac{-3}{5}$ are rational numbers in standard form.

Note: “A rational number in standard form is also referred to as “a rational number in its lowest form”. In this lesson, we will be using these two terms interchangeably.

For example, rational number $\frac{18}{27}$ can be written as $\frac{2}{3}$ in the standard form (or the lowest form).

Similarly, $\frac{25}{-35}$ in standard form (or in lowest form) can be written as $\frac{-5}{7}$ (cancelling out 5 from both numerator and denominator).

Some Important Results

- (i) Every natural number is a rational number but the vice-versa is not always true.
- (ii) Every whole number and integer is a rational number but vice-versa is not always true.

Example Which of the following are rational numbers and which are not?

$$\frac{5}{3}, \frac{15}{7}, \frac{18}{5}, \frac{-7}{6}, -2, -17$$

Solution (i) -2 can be written as $\frac{-2}{2}$, which is of the form $\frac{p}{q}$, $q \neq 0$.
Therefore, -2 is a rational number.

(ii) $\frac{5}{3}$ is a rational number, as it is of the form $\frac{p}{q}$, $q \neq 0$.

(ii) -17 is also a rational number as it is of the form $\frac{-17}{1}$.

(iv) Similarly, $\frac{15}{7}$, $\frac{18}{5}$, $-\frac{7}{6}$ are all rational numbers according to the same argument.

1.6.2 Equivalent forms of Rational Number

A rational number can be written in an equivalent form by multiplying/dividing the numerator and denominator of the given rational number by the same number.

Example Write equivalent forms of $\frac{2}{3}$ rational number.

Solution The equivalent forms are as follows:

$$\frac{2}{3} = \frac{2 \times 2}{3 \times 2} = \frac{4}{6} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12} = \frac{2 \times 8}{3 \times 8} = \frac{16}{24}$$

are equivalent forms of the rational number $\frac{2}{3}$

$\frac{4}{6}$, $\frac{8}{12}$, $\frac{16}{24}$, etc. are equivalent forms of the rational number $\frac{2}{3}$

Similarly,

$$\frac{3}{8} = \frac{6}{16} = \frac{21}{56} = \frac{27}{72} \dots$$

and

$$\frac{4}{7} = \frac{8}{14} = \frac{12}{21} = \frac{28}{49} \dots$$

are equivalent forms of $\frac{3}{8}$ and $\frac{4}{7}$ respectively.

1.6.3 Comparison of Rational Numbers

In order to compare two rational numbers, we follow any of the following methods:

- (i) If two rational numbers, to be compared, have the same denominator, compare their numerators. The number having the greater numerator is the greater rational number. Thus for the two rational numbers $\frac{9}{17}$ and $\frac{5}{17}$ with the same positive denominator.
- $$17, \frac{9}{17} > \frac{5}{17} \text{ as } 9 > 5$$

$$\frac{9}{17} > \frac{5}{17}$$

- i) If two rational numbers are having different denominators, make their denominators equal by taking their equivalent form and then compare the numerators of the resulting rational numbers. The number having a greater numerator is greater rational number.

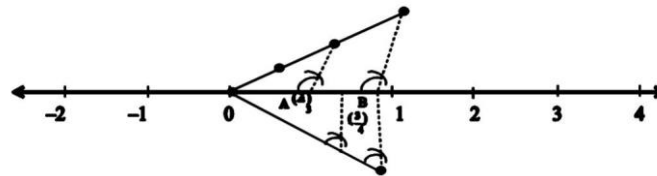
For example, to compare two rational numbers $\frac{3}{7}$ and $\frac{6}{11}$, we first make their denominators same in the following manner.

$$\frac{3 \times 11}{7 \times 11} = \frac{33}{77}, \frac{6 \times 7}{11 \times 7} = \frac{42}{77}$$

$$42 > 33, \frac{42}{77} > \frac{33}{77} \text{ or } \frac{6}{11} > \frac{3}{7}$$

- ii) By plotting two given rational numbers on the number line we see that the rational number to the right of the other rational number is greater.

For example, take $\frac{2}{3}$ and $\frac{3}{4}$, we plot these numbers on the number line as below:



$0 < \frac{2}{3} < 1$ and $0 < \frac{3}{4} < 1$. It means $\frac{2}{3}$ and $\frac{3}{4}$ both lie between 0 and 1.

1. By the method of dividing a line into equal number of parts, A

represents $\frac{2}{3}$ and B represents $\frac{3}{4}$ as B is to right of A, $\frac{3}{4} > \frac{2}{3}$ or

$$\frac{2}{3} > \frac{3}{4}.$$

Out of $\frac{2}{3}$ and $\frac{3}{4}$, $\frac{3}{4}$ is the greatest number.

1.6.5 Four Fundamental Operations on Rational Numbers

i) Addition of Rational Numbers

Consider the addition of rational numbers $\frac{p}{q}$ and $\frac{r}{s}$

$$\frac{p}{q} + \frac{r}{s} = \frac{ps}{qs} + \frac{rq}{sq} = \frac{ps + rq}{qs}$$

Example
$$\frac{3}{4} + \frac{2}{3} = \frac{3 \times 3 + 4 \times 2}{4 \times 3} = \frac{9+8}{12} = \frac{17}{12}$$

Solution
$$\frac{-4}{5} + \frac{7}{8} = \frac{-4 \times 8 + 5 \times 7}{5 \times 8} = \frac{35-32}{40} = \frac{3}{40}$$

From the above two cases, we generalise the following rule:

- (a) The addition of two rational numbers with common denominator is the rational number with common denominator and numerator as the sum of the numerators of the two rational numbers.
- (b) The sum of two rational numbers with different denominators is a rational number with the denominator equal to the product of the denominators of two rational numbers and the numerator equal to sum of the product of the numerator of first rational number with the denominator of second and the product of numerator of second rational number and the denominator of the first rational number.

ii) Subtraction of Rational Numbers

Consider the subtraction of rational numbers $\frac{p}{q}$ and $\frac{r}{s}$

$$\frac{p}{q} - \frac{r}{q} = \frac{p-r}{q} = \frac{ps-rq}{qs}$$

$$\frac{p}{q} - \frac{r}{s} = \frac{ps-rq}{qs}$$

Example

i)
$$\frac{7}{4} - \frac{1}{4} = \frac{7-1}{4} = \frac{6}{4} = \frac{3}{2}$$

Solution

$$i) \quad \frac{3}{5} + \frac{2}{12} = \frac{3 \times 12}{5 \times 12} + \frac{2 \times 5}{12 \times 5} = \frac{36}{60} + \frac{10}{60} = \frac{46}{60} = \frac{23}{30}$$

iii) Multiplication and Division of Rational Numbers

(i) Multiplication of two rational number $\left(\frac{p}{q}\right)$ and $\left(\frac{r}{s}\right)$, $q \neq 0, s$

$\neq 0$ is the rational number $\left(\frac{pr}{qs}\right)$ where $qs \neq 0$

$$= \frac{\text{product of numerators}}{\text{product of denominators}}$$

i) Division of two rational numbers $\left(\frac{p}{q}\right)$ and $\left(\frac{r}{s}\right)$, such that q

$\neq 0, s \neq 0$, is the rational number $\left(\frac{ps}{qr}\right)$, where $qr \neq 0$

$$\text{In other words } \left(\frac{p}{q}\right) \div \left(\frac{r}{s}\right) = \frac{p}{q} \times \frac{s}{r}$$

Example

Multiply the following rational numbers

Solution

$$i) \quad \frac{3}{7} \text{ and } \frac{2}{9} \quad ii) \quad \frac{5}{6} \text{ and } \frac{-2}{19} \quad iii) \quad \frac{7}{13} \text{ and } \frac{-2}{-5}$$

$$i) \quad \frac{3}{7} \times \frac{2}{9} = \frac{3 \times 2}{7 \times 9} = \frac{3 \times 2}{7 \times 3 \times 3} = \frac{2}{21}$$

$$\therefore \left(\frac{3}{7}\right) \times \left(\frac{2}{9}\right) = \frac{2}{21}$$

$$\text{i) } \frac{5}{6} \times \left(\frac{-2}{19} \right) = \frac{5 \times (-2)}{6 \times 19}$$

$$= \frac{2 \times 5}{2 \times 3 \times 19} = \frac{-5}{57}$$

$$\therefore \left(\frac{5}{6} \right) \times \left(\frac{-2}{19} \right) = \frac{-5}{57}$$

$$\text{ii) } \frac{7}{13} \times \left(\frac{-2}{-5} \right) = \frac{7 \times (-2)}{13 \times (-5)} = \frac{-14}{-65} = \frac{14}{65}$$

$$\therefore \left(\frac{7}{13} \right) \times \left(\frac{-2}{-5} \right) = \frac{14}{65}$$

Example Simplify the following:

Solution (a) $\left(\frac{2}{3} \right) \div \left(\frac{7}{12} \right)$

$$= \left(\frac{2}{3} \right) \times \left(\frac{12}{7} \right) = \frac{2 \times 12}{3 \times 7} = \frac{24}{21} = \frac{8}{7}$$

(b) $\left(\frac{9}{16} \right) \div \left(\frac{-105}{2} \right)$

$$= \left(\frac{9}{16} \right) \times \left(\frac{2}{-105} \right) = \frac{9 \times 2}{16 \times (-105)} = \frac{18}{-1680} = \frac{-3}{280}$$

(c) $\left(\frac{81}{77} \right) \div \left(\frac{27}{10} \right)$

$$= \left(\frac{81}{77} \right) \times \left(\frac{10}{27} \right) = \frac{81 \times 10}{77 \times 27} = \frac{810}{2079} = \frac{30}{231}$$

() ()

1.6.6 Decimal Representation of a Rational Number

The process of expressing a rational number into decimal form is to carry out the process of long division using decimal notation.

Example Represent each one of the following into a decimal number:

$$\begin{array}{lll} \text{i) } \frac{12}{5} & \text{ii) } \frac{-27}{25} & \text{iii) } \frac{13}{16} \end{array}$$

Solution

i) Using long division, we get:

$$\frac{12}{5} = 2.4$$

i) Using long division,

$$\frac{-27}{25} = -1.08$$

ii) Using long division,

$$\frac{13}{16} = 0.8125$$

From the above examples, it can be seen that the division process stops after a finite number of steps, when the remainder becomes zero and the resulting decimal number has a finite number of decimal places. Such decimals are known as terminating decimals.

Note: Note that in the above division, the denominators of the rational numbers had only 2 or 5 or both as the only prime factors.

Alternatively, we could have written

$$\frac{12}{5} = \frac{12 \times 2}{5 \times 2} = \frac{24}{10} = 2.4$$

And similarly for others.

In cases where the denominator has factors other than 2 or 5, the decimal representation starts repeating. Such decimals are called non-terminating repeating decimals. Thus, we see from examples below that the decimal representation of a rational number is

- (i) either a terminating decimal (and the remainder is zero after a finite number of steps)
- (ii) Or a non-terminating repeating decimal (where the division will never end) $\therefore$ Thus, a rational number is either a terminating decimal or a non-terminating repeating decimal.

Example Represent each one of the following into a decimal number:

$$\frac{7}{3} \quad \text{ii) } \frac{2}{7}$$

- i) Using long division, we get:

$$\frac{7}{3} = 2.333 = \overline{2.3}$$

- i) Using long division,

$$\frac{2}{7} = 0.285714285714 = \overline{0.285714}$$

Solution

Also, we can express decimal expansion of a Rational Number in p/q form:

Example Express the following in p/q form

- (i) 0.48
- (ii) 0.1357

Solution (i) $0.48 = \frac{48}{100} = \frac{12}{25}$

$$(ii) \quad 0.1375 = \frac{1375}{10000} = \frac{55}{400} = \frac{11}{80}$$

Example Express the following in p/q form

$$(i) 0.666666..... \quad (ii) 0.374374.....$$

Solution (i) Let $x = 0.666...$ (A)

$$10x = 0.666 \quad (B)$$

Subtracting (A) from (B)

$$10x - x = 6.666... - 0.666...$$

$$9x = 6$$

$$x = \frac{6}{9} = \frac{2}{3}$$

(ii) Let $x = 0.374374374...$ (A)

$$1000x = 374.374374 \quad (B)$$

Subtracting (A) from (B)

$$1000x - x = 374.374374... - 0.374374...$$

$$999x = 374$$

$$x = \frac{374}{999}$$

1.6.7 Rational Numbers between Two Rational Numbers

Is it possible to find a rational number between two given rational numbers? To explore this, consider the following examples.

Example 1.17: Find a rational number between $\frac{3}{4}$ and $\frac{6}{5}$.

Solution: Let us try to find the number

$$\frac{1}{2} \left(\frac{3}{4} + \frac{6}{5} \right)$$

$$= \frac{1}{2} \left(\frac{15+24}{20} \right) = \frac{1}{2} \left(\frac{39}{20} \right) = \frac{39}{40}$$

Now,

$$\frac{3}{4} = \frac{3 \times 10}{4 \times 10} = \frac{30}{40}$$

$$\frac{6}{5} = \frac{6 \times 8}{5 \times 8} = \frac{48}{40}$$

Obviously,

$$\frac{30}{40} < \frac{39}{40} < \frac{48}{40}$$

i.e., $\frac{39}{40}$ is a rational number between the rational numbers $\frac{3}{4}$ and $\frac{6}{5}$.

Note:

$$\frac{3}{4} = 0.75, \frac{39}{40} = 0.975, \frac{6}{5} = 1.2$$

$$\Rightarrow 0.75 < 0.975 < 1.2$$

1.7 Irrational Numbers

We have seen that the decimal expansion of a rational number is either terminating or is a non-terminating and repeating decimal. Are there decimals which are neither terminating nor non-terminating but repeating decimals? Consider the following decimal: 0.10 100 1000 10000 1..... (i) You can see that this decimal has a definite pattern and it can be written indefinitely, and there is no block of digits which is repeating. Thus, it is an example of a non-terminating and non-repeating decimal. A similar decimal is given as under: 0.1 2 3 4 5 6 7 8 9 10 11 12 13..... (ii) Can you write the next group

of digits in (i) and (ii)? The next six digits in (i) are 000001... and in (ii) they are 14 15 16 ... Such decimals as in (i) and (ii) represent irrational numbers. Thus, a decimal expansion which is neither terminating nor is repeating represents an irrational number

Example 1.19: Find an irrational number between 2 and 3.

Solution: Consider the number $\sqrt{2 \times 3}$.

We know that $\sqrt{6} \approx 2.45$.

$\therefore$ It lies between 2 and 3 and it is an irrational number.

Example 1.20: Find an irrational number lying between $\sqrt{3}$ and 2.

Solution: Consider the number

$$\frac{\sqrt{3} + 2}{2}$$

$$= 1 + \frac{\sqrt{3}}{2} \approx 1 + \frac{1.732}{2} = 1.866$$

$$\therefore \frac{\sqrt{3} + 2}{2} = 1.866 \text{ lies between } \sqrt{3} (\approx 1.732) \text{ and } 2$$

The required irrational number is $\frac{\sqrt{3} + 2}{2}$

Check Your Progress-I

Based on your understanding of the previous sections, solve these questions:

Q1. Identify rational numbers and integers from the following:

$$4, -3, \frac{5}{4}, 36, \frac{12}{7}, -\frac{8}{7}, \frac{15}{7}, -6$$

Q2. By making the following rational numbers with same denominator, simplify the following and specify whether the result in each case is a natural number, whole number, integer or a rational number:

(i) $3 + \frac{7}{3}$

(ii) $-3 + \frac{10}{4}$

(iii) $-8 - 13$

(iv) $12 - 12$

$\frac{9}{2} - \frac{1}{2}$

(v) $2 \frac{2}{2}$

(vi) $2 \times \frac{5}{7}$

(vii) $8 \div 3$

Q3. Which of the following rational numbers are integers?

$\frac{10}{5}$, $\frac{15}{15}$, $\frac{-5}{5}$, $\frac{13}{9}$, $\frac{27}{14}$, $\frac{7 \times 3}{-2}$, $\frac{-6}{2}$

Q4. Which of the following rational numbers are in lowest form?

$$\frac{8}{12}, \frac{5}{7}, \frac{-3}{12}, \frac{-6}{7}, \frac{\sqrt{5}}{\sqrt{27}}, \frac{15}{24}$$

Q5. Add the following rational numbers:

1. $\frac{3}{2} + \frac{5}{3}$

2. $\frac{17}{5} + \frac{7}{9}$

Q6. Simplify:

1. $\left(\frac{1}{3} + \frac{7}{5} - 2\frac{1}{6}\right)$

2. $\left(\frac{5}{2} + \frac{13}{4} - \frac{6}{3} - \frac{6}{4}\right)$

Q7. Write 3 rational numbers equivalent to given rational numbers:

$$\frac{2}{5}, \frac{-5}{6}, \frac{17}{3}$$

Q8. Convert $\frac{3}{8}$ into its decimal representation.

Q9. How do you identify if a rational number has a terminating decimal expansion?

Q10. What happens when two integers of the same sign are added?

Q 11. What are the properties of zero in whole number operations?

Q12. Why is there no greatest natural number?

Q13. What is the smallest natural number?

1.8 ComplexNumbers

All the numbers with which we have dealt so far were real numbers. However, some solutions in mathematics, such as solving quadratic equations require a new set of numbers. This new set of numbers is called the set of complex numbers.

If we solve the equation $x^2 = 4$ for x , we find the equation has two solutions.
 $x^2 = 4 \Rightarrow x = \sqrt{4} = 2$ or $x = -\sqrt{4} = -2$.

If we solve the equation $x^2 = -1$ in a similar way, we would expect it to have two solutions also.

$$x^2 = -1 \text{ should imply } x = \sqrt{-1} \text{ or } x = -\sqrt{-1}.$$

Each proposed solution of the equation $x^2 = -1$ involves the symbol $\sqrt{-1}$. For years it was believed that square roots of negative numbers denoted by $\sqrt{-5}$, $\sqrt{-2}$ and $\sqrt{-6}$ were nonsense. In the 17th century, these symbols were termed *imaginary numbers* by Rene Descartes (1596-1650). Now, the imaginary numbers are no longer thought to be impossible. In fact imaginary numbers have important uses in several branches mathematics and physics.

The number $\sqrt{-1}$ occurs so often in mathematics, that we give it a special symbol. We use better i to denote $\sqrt{-1}$. Since i stand for $\sqrt{-1}$, it immediately follows that $i^2 = -1$. The power of i with natural exponent produces an interesting pattern, as follows:

$$i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, i^6 = -1, i^7 = -i, i^8 = 1,$$

$$\text{also } i^{-1} = -i, i^{-2} = -1, i^{-3} = -i, i^{-4} = 1$$

A complex number is any number that can be put in the form **$a + bi$** ,

where a and b are real number and $i = \sqrt{-1}$.

The form $a + bi$ is called standard form for complex number. The number a is called the real part of the complex number. The number b is called imaginary part of the complex number.

We usually denote a complex number by z . We write $z = a + bi$. The real part of z is denoted by $\text{Re}(z)$ and the imaginary part of z is denoted by $\text{Im}(z)$.

1.8.1 Equality of two Complex Numbers

Two complex numbers are equal if and only if their real parts are equal and also their imaginary parts are equal.

Thus if, $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers,
then $z_1 = z_2$, that is, $a + bi = c + di$ if and only if $a = c$ and $b = d$.

Example (a) Find x and y if $3x + 4i = 12 - 8yi$
(b) Find a and b if $(4a - 3) + 7i = 5 + (2b - 1)i$

Solution Since the two complex numbers are equal, their real parts are equal and their imaginary parts are equal:

$$3x = 12 \text{ and } 4 = -8y$$

$$x = 4 \text{ and } y = -1/2$$

The real parts are $4a - 3$ and 5 . The imaginary parts are 7 and $2b - 1$.

$$4a - 3 = 5 \text{ and } 7 = 2b - 1$$

$$4a = 8 \text{ and } 2b = 8$$

$$a = 2 \text{ and } b = 4$$

i) Addition of two Complex numbers

Two complex numbers such as $z_1 = a + bi$ and $z_2 = c + di$ are added as if they are algebraic binomials:

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$$

Observe that $a + bi = (a + 0i) + (0 + bi)$. In other words, $a + bi$ is the sum of the real number a and the imaginary number bi .

Also observe that $z_1 + z_2$ is a complex number.

Example Add the following:

$$i) (3 + 4i) + (7 - 6i)$$

Solution $= (3 + 7) + (4 - 6)i = 10 - 2i$

$$ii) (8 - 3i) + (6 - 2i)$$

$$= (8 + 6) + (-3 - 2)i = 14 - 5i$$

ii) Subtraction of Complex Numbers

If $z_1 = a + bi$ and $z_2 = c + di$, we define $z_1 - z_2$ as $z_1 + (-z_2)$.

That is, $z_1 - z_2 = (a + bi) + ((-c) + (-d)i) = (a - c) + (b - d)i$

Example Subtract the following:

(i) $(-4 + 10i) + (-1 + 2i) = \dots$

(ii) $(-6 + 17i) + (4 - 11i) = \dots$

(iii) $(-4 + 2i) + (7 - 2i) = \dots$

(iv) $((3 - 5i) + (-3 + 5i)) = \dots$

Solution (i) $-5 + 12i$ (ii) $-2 + 6i$

(iii) 3 (iv) 0

iii) Multiplication of Complex Numbers

Two complex numbers such as $z_1 = a + bi$ and $z_2 = c + di$ are multiplied as if they were algebraic binomials, with $i^2 = -1$;

$$z_1 z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

By definition, product of two complex numbers is again a complex number. Also observe that $yi = (y + 0i)(0 + 1i)$ and is, thus the product of the real number y and the imaginary number i .

Example Multiply:

i) $(3 + 2i)(4 + 5i)$

Solution $= 12 + 15i + 8i + 10i^2$

$$= 12 + 15i + 8i - 10$$

$$= 2 + 23i [i^2 = -1]$$

i) $(2 + 5i)(7 + 3i)$

$$= 14 + 6i + 35i + 15i^2$$

$$= 14 + 6i + 35i - 15$$

$$= -1 + 41i \quad [i^2 = -1]$$

iv) Multiplicative Inverse of a Non-Zero Complex Number

If $a + ib \neq 0$ is any complex number, then there exists a complex number $x + iy$ such that

$$(a + ib)(x + iy) = 1 + 0i = \text{the multiplicative identity in } \mathbb{C}.$$

The number $x + iy$ is called the multiplicative inverse of $(a + ib)$ in $\mathbb{C}$.

$$\text{Now, } (a + ib)(x + iy) = 1 + 0i \Rightarrow (ax - by) + i(ay + bx) = 1 + 0i$$

$$\Rightarrow ax - by = 1 \text{ and } ay + bx = 0 \quad [\text{equality of two complex numbers}]$$

$$\Rightarrow ax - by - 1 = 0 \text{ and } ay + bx = 0$$

Solving these equations for x and y , we have

$$x = \frac{a}{a^2 + b^2} \quad (1)$$

$$y = \frac{b}{a^2 + b^2} \quad (2)$$

both of which exist in $\mathbb{R}$, because $(a + ib) \neq 0$ i.e., at least one of a, b is different from zero.

Thus, the multiplicative inverse is of $a + ib$ is

Thus, every non-zero complex number has a multiplicative inverse.

iv) Division In Complex Numbers

$$\text{If } Z_1 = x + iy \text{ and } Z_2 = a + ib \neq 0,$$

$$\text{Then, } \frac{Z_1}{Z_2} = \frac{x + iy}{a + ib} = (x + iy) \frac{1}{(a + ib)}$$

$$= (x + iy) \frac{a - ib}{a^2 + b^2}$$

$$= \frac{ax + by}{a^2 + b^2} + \frac{bx - ay}{a^2 + b^2}$$

Example If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then show that $a = 1$ and $b = 0$

Solution We have

$$\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1-i)(1+i)} = \frac{(1-i)^2}{(1)^2 - (i)^2}$$

$$= \frac{1 - 2i + i^2}{2}$$

$$= \frac{1 - 2i + 1}{2} = -i$$

$$\text{Thus, } \left(\frac{1-i}{1+i}\right)^{100} = (-i)^{100} = 1$$

$$a + ib = 1, a = 1 \text{ and } b = 0$$

1.9 Conjugate of a Complex Number

Definition : If $z = x + iy$, $x, y \in \mathbf{R}$ is a complex number, then the complex number $x - iy$ is called conjugate of z and is denoted by $\bar{z}$.

For instance,

$$\overline{2 + 3i} = 2 - 3i, \overline{3 + 4i} = 3 - 4i, \overline{i\sqrt{-1}} = -i\sqrt{-1}$$

$$\bar{3} = \overline{3 + 0i} = 3 - 0i = 3$$

1. $\bar{\bar{z}} = z$
2. $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
3. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
4. $\overline{\left(\frac{z_1}{z_2} \right)} = \frac{\bar{z}_1}{\bar{z}_2}, \text{ if } z_2 \neq 0$
5. If $z = a + ib$, then

$$z + \bar{z} = 2a = 2 \operatorname{Re}(z)$$

$$z - \bar{z} = 2ib = 2i \operatorname{Im}(z)$$
6. $z = \bar{z} \Leftrightarrow z$ is real
7. $z = -\bar{z} \Leftrightarrow z$ is imaginary

Some Properties of Complex Conjugat

1.10 Modulus of a Complex Number

Definition: If $z = x + iy$, $x, y \in \mathbb{R}$ is a complex number, then the real number $\sqrt{x^2 + y^2}$ is called the modulus of the complex number z , and is denoted by $|z|$.

For instance:

$$\text{If } z = 2 + 3i, |z| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$$

$$\text{If } z = 5 - 12i, |z| = \sqrt{5^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

Note that:

$$|z| = |\bar{z}| \text{ and } |z \cdot \bar{z}| = |z|^2 \text{ and if } c \text{ is a real number, then } |cZ| = |c||Z|$$

Some properties of modulus of complex numbers:

1. $|Z_1 \cdot Z_2| = |Z_1| \cdot |Z_2|$

D:\Himmat 2025\Less 2025\M.A. Semester-I Economics (ECO-103) 2025\M.A. Semester-I Economics (ECO-103) 2025 (Pages 1-135).p65 (**Set-HS**)

$$2. \quad |Z| = 0 \Leftrightarrow Z = 0$$

$$3. \quad \left| \frac{1}{Z} \right| = \frac{1}{|Z|} \text{ if } Z \neq 0$$

$$4. \quad |Z_1| - |Z_2| \leq |Z_1 - Z_2|$$

$$5. \quad \left| \frac{Z_1}{Z_2} \right| = \frac{|Z_1|}{|Z_2|} \text{ if } Z_2 \neq 0$$

$$6. \quad -|Z| \leq \operatorname{Re}(Z) \leq |Z|$$

$$7. \quad |Z_1 + Z_2|^2 = |Z_1|^2 + |Z_2|^2 + 2\operatorname{Re}(Z_1 Z_2)$$

Example : If $a + ib \neq 0$, show that

$$\left(\frac{a - ib}{a + ib} \right) \cdot \left(\frac{a + ib}{a - ib} \right) = 1$$

Solution: Let $Z = a + ib$, then $\bar{Z} = a - ib$

Since $|Z| = |\bar{Z}|$, we get

$$1 = \left| \frac{Z}{Z} \right| = \left| \frac{a - ib}{a + ib} \right|$$

$$\left| \frac{Z_1}{Z_2} \right| = \frac{|Z_1|}{|Z_2|}$$

Example: If $x + iy = \frac{a+ib}{c+id}$, then

$$x^2 + y^2 = \frac{a^2 + b^2}{c^2 + d^2}$$

Solution:

$$\Rightarrow (x + iy)^2 = \left| \frac{a + ib}{c + id} \right|^2$$

$$\Rightarrow |x + iy|^2 = \left| \frac{a + ib}{c + id} \right|^2 \Rightarrow |x + iy|^2 = \left(\frac{|a + ib|}{|c + id|} \right)^2$$

$$\Rightarrow (\sqrt{x^2 + y^2})^2 = \left(\frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}} \right)^2 \Rightarrow x^2 + y^2 = \frac{a^2 + b^2}{c^2 + d^2}$$

Check Your Progress-II

Based on your understanding of the previous sections, answer these questions in your own words:

Q1. Define a complex number and give the general form.

Q2. What is the imaginary unit i and what is i^2 ?

Q3. Write the values of i^1, i^2, i^3, i^4 ?

Q4. Add: $(3+4i) + (5+2i)$.

Q5. Subtract: $(6+7i) - (2+5i)$

Q6. Multiply: $(2+3i)(4+i)$

Q7. What is the conjugate of $5+2i$?

Q8. What is the modulus of $3+4i$?

Q9. Solve for x and y if $3x + 4i = 12 - 8i$

1.11 Let Us Sum Up

The lesson provides a comprehensive overview of the number system, beginning with natural and whole numbers and progressing to more complex types such as rational, irrational, and complex numbers. Students learn about the representation of numbers on the number line and perform operations like addition, subtraction, multiplication, and division. Special emphasis is placed on rational numbers' decimal forms and complex number operations such as conjugation and modulus. The lesson reinforces understanding through worked examples and "Check Your Progress" activities to assess and deepen learning.

1.12 Keywords

1. **Natural Numbers:** Counting numbers starting from 1 (i.e., 1, 2, 3, ...).
2. **Whole Numbers:** Natural numbers including 0 (i.e., 0, 1, 2, 3, ...).
3. **Integers:** All positive and negative whole numbers including 0 (e.g., -3, 0, 4).
4. **Rational Numbers:** Numbers that can be expressed as the ratio of two integers, where the denominator is not zero.

5. **Irrational Numbers:** Numbers that cannot be expressed as a ratio of two integers; their decimal expansions are non-terminating and non-repeating.
6. **Terminating Decimal:** A decimal number that ends after a finite number of digits.
7. **Non-Terminating Repeating Decimal:** A decimal that repeats the same digit or group of digits infinitely.
8. **Standard Form:** A rational number expressed with a positive denominator and in its lowest terms (i.e., numerator and denominator have no common factors except 1).
9. **Complex Numbers:** Numbers of the form $a + bi$, where a and b are real numbers and i is the imaginary unit ($i^2 = -1$).
10. **Imaginary Unit (i):** Defined as $i^2 = -1$, used to express square roots of negative numbers.
11. **Modulus of a Complex Number:** The absolute value or magnitude of a complex number, calculated as $\sqrt{a^2 + b^2}$ for $z = a + bi$.
12. **Conjugate of a Complex Number:** For a complex number $z = a + bi$, its conjugate is $a - bi$.
13. **Equivalent Rational Numbers:** Rational numbers that represent the same value when simplified or rewritten (e.g., $1/2$ and $2/4$).

1.13 Examination Oriented Questions

- Q1. What are cardinal numbers?
- Q2. Explain whether division of two natural numbers always results in a natural number.
- Q3. Can whole numbers be negative? Why or why not?
- Q4. What are the differences between positive and negative integers?
- Q5. What is an irrational number? Give two examples.

- Q6. Define the conjugate of a complex number $z = a+bi$.
- Q7. Find the conjugate of the complex number $z = 7-3i$
- Q8. If $z = 4 + 5i$, verify that z and $\bar{z}$.
- Q9. Show that the conjugate of a conjugate of a complex number is the number itself.
- Q10. Find the conjugate of $z = -6 + 11i$ and $w = 2-7i$ then compute $\overline{z+w}$
- Q11. If $z = a + bi$, show that $|z| = \sqrt{a^2 + b^2}$.
- Q12. For $z = \frac{1+i}{i-i}$, compute $|z|$.
- Q13. For $z = 2 + 3i$ and $w = 4 - i$, compute $|z| |w|$.
- Q14. Simplify the following:

$$1. \left(\frac{2}{3} \times \frac{7}{8} \times \frac{8}{25} \times \frac{37}{15} \right)$$

$$2. \left(\frac{3}{4} \div \frac{2}{4} \right) \times 21$$

- Q15. Subtract:

$$1. \frac{7}{15} \text{ from } \frac{13}{15}$$

$$2. \frac{7}{3} \text{ from } \frac{5}{3}$$

$$3. \frac{3}{7} \text{ from } \frac{9}{24}$$

- Q16. Multiply:

1. $\frac{2}{5}$ by $\frac{6}{6}$

2. $-\frac{3}{11}$ by $\frac{33}{35}$

3. $-\frac{11}{3}$ by $-\frac{27}{77}$

Q17. Divide:

1. $\frac{1}{2}$ by $\frac{1}{4}$

2. $\frac{7}{4}$ by $\frac{4}{5}$

3. $\frac{35}{33}$ by $-\frac{7}{22}$

Q18. Compare the following rational numbers by

(i) changing them to rational numbers in equivalent forms

(ii) using number line:

1. $\frac{2}{3}$ and $\frac{3}{4}$

2. $\frac{3}{5}$ and $\frac{7}{9}$

3. $\frac{2}{3}$ and $\frac{1}{2}$

4. $\frac{3}{7}$ and $\frac{5}{11}$

1.14 Hints to Check your Progress (CYP)

CYP- I

Q1. Refer Section 1.5 and Section 1.6

Q2. Refer Section 1.6.3

- Q3. Refer Section 1.6.1
- Q4. Refer Section 1.6.3
- Q5. Refer Section 1.6.5 (i)
- Q6. Refer Section 1.6.5 (iii)
- Q7. Refer Section 1.6.2
- Q8. Refer Section 1.6.6
- Q9. Refer Section 1.6.6
- Q10. Refer Section 1.5
- Q11. Refer Section 1.4
- Q12. Refer Section 1.3
- Q13. Refer Section 1.3

CYP – II

- Q1. Refer Section 1.8
- Q2. Refer Section 1.8
- Q3. Refer Section 1.8
- Q4. Refer Section 1.8.1 (i)
- Q5. Refer Section 1.8.1 (ii)
- Q6. Refer Section 1.8.1 (iii)
- Q7. Refer Section 1.9
- Q8. Refer Section 1.10
- Q9. Refer Section 1.6.6
- Q10. Refer Section 1.8.1

1.15 Suggested Readings

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M.A. Economics	Lesson No. 2
C.No. : Eco-103	Semester - 1st
	Unit I

LINEAR EQUATIONS AND SIMULTANEOUS EQUATIONS

STRUCTURE

- 2.0 Objectives
- 2.1 Learning outcomes
- 2.2 Introduction
- 2.3 Linear Equation: Definition
 - 2.4 Root of a Linear Equation
 - 2.5 Solution of a Linear Equation
- 2.6 Linear Equation in Two Variables
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- 2.7 Simultaneous Linear Equations in Three Variables in x, y and z
- 2.8 Economic Applications of Linear Equations
- 2.9 Let Us Sum Up
- 2.10 Keywords
- 2.11 Examination Oriented Questions
- 2.12 Hints to Check Your Progress

2.13 Suggested Readings

2.0 Objectives

The main objectives of this lesson are to enable you to:

- Understand the structure and types of linear equations.
- Solve linear equations in one, two, and three variables.
- Use algebraic techniques—elimination, substitution, cross multiplication, and matrix methods—for solving systems of equations.
- Apply linear models to economic concepts like demand and supply, taxation, and subsidies.
- Interpret and solve macroeconomic models using linear systems (e.g., income determination and IS-LM equilibrium).

2.1 Learning outcomes

After going through this lesson, you will be able to:

- Define and identify linear and simultaneous equations.
- Solve equations using multiple methods and apply them to economic problems.
- Determine equilibrium prices and quantities using demand and supply equations.
- Analyze the effects of government interventions like taxes and subsidies using algebraic models.
- Solve economic equilibrium conditions involving consumption, investment, income, and interest rates.
- Graphically and analytically interpret mathematical models in economics.

2.2 Introduction

Linear and simultaneous equations are foundational tools in economic modeling and analysis. These equations help economists capture and analyze relationships among variables such as price, quantity, income, investment, and consumption. From simple demand and supply models to complex macroeconomic frameworks like IS-LM models, linear and quadratic equations play a critical role in explaining how economic agents interact in markets and how equilibrium is achieved. This lesson explores methods of solving such equations and their broad economic applications—from market equilibrium to income determination and fiscal policy analysis.

2.3 Linear Equation: Definition

An equation of the form $ax + b = 0$ ($a \neq 0$), where a and b are constant and x , the variable or the unknown quantity is called a linear equation in x .

2.4 Root of a Linear Equation

The value of the variable which satisfies the equation i.e., which makes (LHS = RHS) is called a root of the equation. A linear equation has only one root.

2.5 Solution of a Linear Equation

The general linear equation in one unknown variable is

$$ax + b = 0 \quad (a \neq 0)$$

$$\Rightarrow ax = -b$$

$$\therefore x = \frac{-b}{a}$$

$$\text{Thus, } x = \frac{-b}{a} \text{ is the solution of } ax + b = 0.$$

Rule: The general method of solving a linear equation is to reduce it to the form $ax + b = 0$ and then solve it as above.

The following examples illustrate the solution of a linear equation.

Example Solve the equation:

$$3(x - 2) + 5 = x - 4(2 - x) - 3$$

Solution. The given equation is

$$3(x - 2) + 5 = x - 4(2 - x) - 3$$

$$\text{or } 3x - 6 + 5 = x - 8 + 4x - 3$$

$$\text{or } 3x - 1 = 5x - 11 \text{ or } 2x = 10$$

$$\therefore x = 5$$

Example Solve for x:

$$\frac{5x}{3} - \frac{x-2}{4} = 2 - \frac{1}{4} \left(x - \frac{2x-1}{3} \right)$$

Solution Opening the brackets, we have

$$\frac{5x}{3} - \frac{x-2}{4} = \frac{9}{4} - \frac{x}{2} + \frac{2x-1}{6}$$

$$\frac{20x - 3x + 6}{12} = \frac{27 - 6x + 4x - 2}{12}$$

$$\text{or } 20x - 3x + 6 = 27 - 6x + 4x - 2$$

$$\text{or } 20x - 3x + 6x - 4x = 27 - 2 - 6$$

$$\text{or } 19x = 19 \text{ or } x = \frac{19}{19} = 1$$

Example Solve: $\frac{1}{x+1} + \frac{1}{x+4} = \frac{1}{x+2} + \frac{1}{x+3}$

Solution
$$\frac{1}{x+1} + \frac{1}{x+4} = \frac{1}{x+2} + \frac{1}{x+3}$$

$$\text{or} \quad \frac{1}{x+1} - \frac{1}{x+3} = \frac{1}{x+2} - \frac{1}{x+4}$$

$$\text{or} \quad \frac{x+3-x-1}{(x+1)(x+3)} = \frac{x+4-x-2}{(x+2)(x+4)}$$

$$\text{or} \quad \frac{2}{x^2 + 4x + 3} = \frac{2}{x^2 + 6x + 8}$$

$$\text{or} \quad \frac{1}{x^2 + 4x + 3} = \frac{1}{x^2 + 6x + 8}$$

$$\text{or} \quad x^2 + 6x + 8 = x^2 + 4x + 3$$

$$\text{or} \quad 6x - 4x = 3 - 8$$

$$\text{or} \quad 2x = -5 \text{ or } x = -5/2$$

2.6 Linear Equation in Two Variables

An equation of the form $ax + by + c = 0$, where a , b and c are non-zero numbers, is called a linear equation in two variables. Thus, $5x - 3y + 7 = 0$ is the linear equation in two variables x and y .

2.6.1 Simultaneous Linear Equations in Two Variables

The equations of the type

$$ax + by + c = 0$$

$$px + qy + r = 0$$

which gives a particular pair of values of x and y satisfying them are called simultaneous linear equations in two variables.

Simultaneous linear equations should not be

- i) Inconsistent, as in $x + y = 2$, $x + y = 7$
- ii) Dependent, as in $x + y = 2$, $2x + 2y = 4$. Here the second equation

is derived from the first and does not provide any additional information about x and y .

262 Methods of Solving Simultaneous Linear Equations in Two Variables

There are various methods of solving the simultaneous linear equations in two variables:

- i) Elimination Method
- i) Substitution Method
- ii) Cross Multiplication Method
- iv) Graphic Method
- v) By Using Determinants and Matrices

Example Solve the following linear equation system:

$$4x - y = -2$$

$$3x + 4y = 27$$

Solution **Elimination Method**

The given equations are

$$4x - y = -2$$

$$3x - 4y = 27$$

Multiplying (i) by 4 and adding it to (ii), we get

$$3x - 4y = 27$$

$$16x - 4y = -8$$

By adding

$$19x = 19$$

$$\therefore x = 1$$

Putting the value of x in (i), we have

$$4(1) - y = -2$$

$$4 - y = -2$$

$$\therefore -y = -6 \text{ or } y = 6$$

Hence $x = 1$ and $y = 6$

Substitution Method

From (i) $y = 4x + 2$

Substituting this value of y in (ii) we get

$$3x + 4(4x + 2) = 27$$

$$\text{or } 3x + 16x + 8 = 27$$

$$\text{or } 3x + 16x = 19$$

$$\text{or } 19x = 19$$

$$\therefore x = 1$$

Now put this value of x in (iii)

$$y = 4(1) + 2 = 6$$

$$\therefore x = 1, y = 6$$

Cross Multiplication Method

The given equations may be written as

$$\begin{array}{r} -1 \\ 4 \end{array} \left| \begin{array}{c} \nearrow 2 \\ \searrow 27 \end{array} \right. \begin{array}{c} \nearrow 4 \\ \searrow 3 \end{array} \left. \begin{array}{c} \nearrow 1 \\ \searrow 4 \end{array} \right] = 0$$

$$4x - y + 2 = 0$$

$$3x + 4y - 27 = 0$$

Using cross multiplication, we have

$$\frac{(\quad)(\quad x - 27)}{-1 - 4} = \frac{(\quad)(\quad y - 4)}{2 - (-27)} = \frac{(\quad)(\quad 1)}{4 - (-1)}$$

$$\text{or } \frac{x}{27 - 8} = \frac{y}{6 + 108} = \frac{1}{16 + 3}$$

$$\text{or } \frac{x}{19} = \frac{y}{114} = \frac{1}{19}$$

$$\therefore x = \frac{19}{19} = 1, \text{ and } y = \frac{114}{19} = 6$$

$$\therefore x = 1 \text{ and } y = 6$$

Check your Progress-I

Based on your understanding of the previous section, solve these questions:

Q1. Solve the following equations:

$$\text{i) } \frac{x - y}{3} = \frac{y - 1}{4} \text{ and } \frac{4x - 5y}{7} = x - 7$$

$$\text{i) } \frac{2}{x} + \frac{3}{y} = 2 \text{ and } \frac{5}{x} + \frac{8}{y} = \frac{31}{6}$$

ii)

iv) $\frac{3}{x} + \frac{2}{y} + 1 = \frac{1}{x} + \frac{3}{y} = 10$

2.7 Simultaneous Linear Equations in Three Variables in x, y and z

When we are given three equations in three unknown variables x, y and z, then we delete one variable say x from three equations and get two equations in two variables y and z which we can solve by the previous method.

Example Solve the following equations:

$$x - y + z = 2 \qquad 2x - 3y + z = 1 \qquad 3x - y + 2z = 9$$

Solution The given equations are:

$$x - y + z = 2 \qquad \text{..... (i)}$$

$$2x - 3y + z = 1 \qquad \text{..... (ii)}$$

$$3x - y + 2z = 9 \qquad \text{..... (iii)}$$

Multiplying both sides of (i) by 2 and (ii) by 1, and subtract

$$2x - 2y + 2x = 4$$

$$2x - 3y + z = 1$$

$$\begin{array}{r} - \quad + \quad - \quad - \\ \hline \end{array} \quad \dots\dots\dots \text{(iv)}$$

$$y + z = 3$$

Multiplying both sides of (ii) by 3 and (iii) by 2, and subtract

$$6x - 3y + 3z = 3$$

$$6x - 2y + 4z = 18$$

$$\begin{array}{r} - \quad + \quad - \quad - \\ \hline \end{array} \quad \dots\dots\dots \text{(v)}$$

$$-7y - z = -15$$

Adding (iv) and (v), we get

$$y + z = 3$$

$$\text{Adding } -7y - z = -15$$

$$-6y = -12, \Rightarrow y = 2$$

Put $y = 2$ in (iv), we get $2 + z = 3$, $z = 3 - 2 = 1$

Put $y = 2$, $z = 1$ in (i), we get $x - 2 + 1 = 2$, $\therefore x = 3$

$$\therefore x = 3, y = 2, z = 1$$

Example Solve: $\frac{xy}{x+y} = 1$, $\frac{yz}{y+z} = 2$, $\frac{zx}{z+x} = 3$

Solution Taking reciprocals of both sides in each case, we have

$$\frac{x+y}{xy} = 1, \frac{y+z}{yz} = \frac{1}{2}, \frac{z+x}{zx} = \frac{1}{3}$$

$$\text{or } \frac{1}{y} + \frac{1}{x} = 1, \frac{1}{z} + \frac{1}{y} = \frac{1}{2}, \frac{1}{x} + \frac{1}{z} = \frac{1}{3}$$

Adding these equations, we get

$$2 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ \hline x & y & z & 11 \\ \hline \end{array} \right) = 2 \begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ \hline 2 & 2 & 2 & 22 \\ \hline \end{array}$$

$$\text{or } \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{11}{12}$$

From this equation, subtract each equations of (i) in turn. Thus,

$$\frac{1}{z} = \frac{11}{12} - 1 = -\frac{1}{12} \text{ or } z = -12$$

$$\frac{1}{x} = \frac{11}{12} - \frac{1}{2} = \frac{5}{12} \text{ or } x = \frac{12}{5}$$

$$\frac{1}{y} = \frac{11}{12} - \frac{1}{3} = \frac{7}{12} \text{ or } y = \frac{12}{7}$$

$$\therefore x = \frac{12}{5}, y = \frac{12}{7}, z = -12$$

2.8 Economic Applications of Linear Equations

i) Demand and Supply Analysis

Equilibrium in supply and demand analysis occurs when $Q_s = Q_d$. By equating the supply and demand functions, the equilibrium price and quantity can be determined.

Example Find the equilibrium price and quantity of the demand and supply laws are

$$3p + 2q = 16, 5p - 3q = 14$$

Solution The demand and supply laws are

$$3p + 2q = 16 \quad \text{..... (i)}$$

$$\text{and } 5p - 3q = 14 \quad \text{..... (ii)}$$

The equilibrium price and quantity are the positive values of p and q which satisfy both (i) and (ii)

Multiplying (i) by 3 and (ii) by 2 and adding

$$9p + 6q = 48$$

$$\frac{10p - 6q = 28}{19p = 76} \text{ (Adding)}$$

$$\therefore p = \frac{76}{19} = 4$$

Putting $p = 4$ in equation (i), we get

$$3 \times 4 + 2q = 16$$

$$\text{or } 2q = 16 - 12 = 4$$

$$\therefore q = 2$$

Hence, equilibrium price $p = 4$ and quantity $q = 2$.

Example If the demand and supply curves of two commodities A and B are given below find the equilibrium price and quantity.

$$D_1 = 10 - p_1 - 2p_2; S_1 = -3 + p_1 + p_2$$

$$D_2 = 6 - p_1 - p_2; S_2 = -2 + p_2$$

Solution For equilibrium, $D_1 = S_1$ and $D_2 = S_2$

$$\text{or } 10 - p_1 - 2p_2 = -3 + p_1 + p_2 \text{ and } 6 - p_1 - p_2 = -2 + p_2$$

$$\text{or } 2p_1 + 3p_2 = 13 \quad \dots\dots\dots (i)$$

$$\text{or } p_1 + 2p_2 = 8 \quad \dots\dots\dots (ii)$$

Multiply (ii) by 2 and subtract from (i), we get $p_2 = 3$

Substituting in (i), we get

$$2p_1 = 4, p_1 = 2$$

$$\text{At } p_1 = 2, p_2 = 3, D_1 = 10 - 2 - 6 = 2, D_2 = 6 - 2 - 3 = 1$$

$\therefore$ Equilibrium prices are $p_1 = 2, p_2 = 3$

and equilibrium quantities are $D_1 = 2, D_2 = 1$.

ii) Effect of Taxes and Subsidies on Equilibrium Price and Quantity

Specifically, if demand is $Q = a - bP$ and original supply is $Q = c + dP$, then imposing a tax t effectively changes supply to $Q = c + d(P - t)$ while a subsidy s changes it to $Q = c + d(P + s)$. Solving these paired equations yields closed-form expressions for the post-intervention equilibrium price and quantity, which transparently illustrate how the burden or benefit is split between consumers and producers and how output contracts under a tax or expands under a subsidy.

Example The demand and supply equations of a commodity are given by

$$q = \frac{5}{2} - \frac{1}{2}p \text{ and } q = 2p - 3$$

What will be the equilibrium price and quantity? What will be the new equilibrium price if a tax of Rs. 6/5 per unit is imposed on the commodity?

Solution The demand function is $q = \frac{5}{2} - \frac{1}{2}p$ (i)

and the supply function is $q = 2p - 3$ (ii)

a) The equilibrium price of a commodity is determined where the demand is equal to its supply.

$\therefore$ We solve these equations simultaneously.

$$\frac{5}{2} - \frac{1}{2}p = 2p - 3$$

$$\text{or } -\frac{1}{2}p - 2p = -3 - \frac{5}{2}, \text{ or } -\frac{5}{2}p = -\frac{11}{2}, p = \frac{11}{5}$$

Substituting $p = 11/5$ in (i), we get

$$q = \frac{5}{2} - \frac{1}{2} \left(\frac{11}{5} \right) = \frac{5}{2} - \frac{11}{10} = \frac{25 - 11}{10} = \frac{14}{10} = \frac{7}{5}$$

∴ The equilibrium price is Rs. 11/5 and the equilibrium quantity is 7/5.

- b) When a tax of Rs. 6/5 per unit is imposed on the commodity, then the new supply equation becomes.

$$q_1 = 2 \left(p_1 - \frac{6}{5} \right) - 3, \text{ or } q_1 = 2p_1 - (27/5) \quad \dots\dots\dots \text{(iii)}$$

$$\text{and new demand equation is } q_1 = \frac{5}{2} - \frac{1}{2} p_1 \quad \dots\dots\dots \text{(iv)}$$

Now again for equilibrium, putting demand equal to supply.

Solving (iii) and (iv) simultaneously, we get

$$2p_1 - \frac{27}{5} = \frac{5}{2} - \frac{1}{2} p_1$$

$$\text{or } 2p_1 + \frac{1}{2} p_1 = \frac{5}{2} + \frac{27}{5} \text{ or } \frac{5}{2} p_1 = \frac{79}{10} \text{ or } p_1 = \frac{79}{25}$$

When $p_1 = \frac{79}{25}$, then from (iv),

$$q_1 = \frac{5}{2} - \frac{1}{2} \left(\frac{79}{25} \right) = \frac{5}{2} - \frac{79}{50} = \frac{125 - 79}{50} = \frac{46}{50} = \frac{23}{25}$$

∴ Equilibrium price after tax imposition = Rs. 79/25.

Equilibrium quantity after tax imposition = 23/25.

iii) Income Determination Models

Income determination models generally express the equilibrium level of income
in a four-sector economy as

D:\Himmat 2025\Less 2025\M.A. Semester-I Economics (ECO-103) 2025\M.A. Semester-I Economics (ECO-103) 2025 (Pages 1-135).p65 (**Set-HS**)

$$Y = C + I + G + (X - Z)$$

where Y = income, C = consumption, I = investment, G = government expenditures, X = exports, and Z = imports. By substituting the information supplied in the problem, it is an easy matter to solve for the equilibrium level of income. Aggregating (summing) the variables on the right allows the equation to be graphed in two-dimensional space.

Example Assume a simple two-sector economy where $Y = C + I$, $C = C_0 + bY$, and $I = I_0$. Assume further that $C_0 = 85$, $b = 0.9$, and $I_0 = 55$. The equilibrium level of income can be calculated in terms of (1) the general parameters and (2) the specific values assigned to these parameters.

Solution The equilibrium equation is

$$Y = C + I$$

Solving for C and I ,

$$Y = C_0 + bY + I_0$$

Solving for Y ,

$$Y - bY = C_0 + I_0$$

$$(1 - b)Y = C_0 + I_0$$

$$Y = (C_0 + I_0) / (1 - b)$$

The solution in this form is called the reduced form. The reduced form (or solution equation) expresses the endogenous variable (here Y) as an explicit function of the exogenous variables (C_0 , I_0) and the parameters (b).

The specific equilibrium level of income can be calculated by substituting the numerical values for the parameters in either the original equation (a) or the reduced form (b):

$$A) \quad Y = C + bY + I = 85 + 0.9Y + 55$$

$$Y - 0.9Y = 140$$

$$0.1Y = 140$$

$$Y = 1400$$

$$\begin{aligned} \text{B)} \quad Y &= (C_0 + I_0) / (1 - b) \\ &= (85 + 55) / (1 - 0.9) \\ &= 140 / 0.1 = 1400 \end{aligned}$$

The term $1 / (1 - b)$ is called the autonomous expenditure multiplier in economics. It measures the multiple effect each dollar of autonomous spending has on the equilibrium level of income. Since $b = \text{MPC}$ in the income determination model, the multiplier $= 1 / (1 - \text{MPC})$.

iv) IS-LM Analysis

The IS schedule is a locus of points representing all the different combinations of interest rates and income levels consistent with equilibrium in the goods (commodity) market. The LM schedule is a locus of points representing all the different combinations of interest rates and income levels consistent with equilibrium in the money market. IS-LM analysis seeks to find the level of income and the rate of interest at which both the commodity market and the money market will be in equilibrium. This can be accomplished with the techniques used for solving simultaneous equations.

Example The commodity market for a simple two-sector economy is in equilibrium when $Y = C + I$. The money market is in equilibrium when the supply of money (M_s) equals the demand for money (M_d), which in turn is composed of the transaction-precautionary demand for money (M_t) and the speculative demand for money (M_z). Assume a two-sector economy where $C = 48 + 0.8Y$, $I = 98 - 75i$, $M_s = 250$, $M_t = 0.3Y$, and $M_z = 52 - 150i$.

Solution Commodity equilibrium (IS) exists when $Y = C + I$.

Substituting into the equation,

$$Y = 48 + 0.8Y + 98 - 75i$$

$$Y - 0.8Y = 146 - 75i$$

$$0.2Y + 75i - 146 = 0 \quad (1)$$

Monetary equilibrium (LM) exists when $M_s = M_t + M_z$.

Substituting into the equation,

$$250 = 0.3Y + 52 - 150i$$

$$0.3Y - 150i - 198 = 0 \quad (2)$$

A condition of simultaneous equilibrium in both markets can be found, then, by solving (1) and (2) simultaneously:

$$0.2Y + 75i - 146 = 0 \quad (1)$$

$$0.3Y - 150i - 198 = 0 \quad (2)$$

Multiply (1) by 2, add the result (3) to (2) to eliminate i , and solve for Y

$$0.4Y + 150i - 292 = 0 \quad (3)$$

$$0.3Y - 150i - 198 = 0$$

$$0.7Y - 490 = 0$$

$$Y = 700$$

Substitute $Y = 700$ in (1) or (2) to find i .

$$0.2Y + 75i - 146 = 0$$

$$0.2(700) + 75i - 146 = 0$$

$$140 + 75i - 146 = 0$$

$$75i = 6$$

$$75$$

$$i = \frac{6}{75} = 0.08$$

The commodity and money markets will be in simultaneous equilibrium when Y

= 700 and $i = 0.08$. At that point $C = 48 + 0.8(700) = 608$, $I = 98 - 75(0.08) = 92$, $M_t = 0.3(700) = 210$, and $M_z = 52 - 150(0.08) = 40$.

$$C + I = 608 + 92 = 700 \text{ and } M_t + M_z = 210 + 40 = 250 = M_s.$$

Check your Progress-II

Based on your understanding of the previous section, solve these questions:

- Q1. Given the following set of simultaneous equations for two related markets, beef (B) and pork (P), find the equilibrium conditions for each market, using the substitution method.

$$1) \quad Q_{dB} = 82 - 3P_B + P_P \quad Q_{sB} = -5 + 15P_B$$

$$2) \quad Q_{dP} = 92 + 2P_B - 4P_P \quad Q_{sP} = -6 + 32P_P$$

- Q2. Given: $Y = C + I + G$, $C = C_0 + bY$, $I = I_0$, and $G = G_0$, where $C_0 = 135$, $b = 0.8$, $I_0 = 75$, and $G_0 = 30$. (a) Find the equation for the equilibrium level of income in the reduced form. (b) Solve for the equilibrium level of income (1) directly and (2) with the reduced form.

- Q3. Solve for x:

$$i) \quad x + 2y + 3z = 22, \quad 2x - 3y + z = 5 \text{ and } 3x + 4y - 2z = 7$$

$$ii) \quad x + 3y + 4z = 1, \quad 2x + 7y + 3z = -5 \text{ and } 3x + 20y + 8z = -3$$

2.9 Let Us Sum Up

In this chapter, we explored the concept of linear equations, including their roots and methods of solution. Further, we examined linear equations in two and three variables, analyzing their structure and methods of solving such systems. The chapter also discussed the practical applications and relevance of linear equations in the field of economics, highlighting how these mathematical tools can be utilized for economic analysis and decision-making.

2.10 Keywords

1. **Linear Equation:** An equation of the form $ax + b = 0$, where $a \neq 0$. It represents a straight line in a graph and has only one solution.
2. **Root of an Equation:** The value of the unknown variable that satisfies the equation, making the Left-Hand Side (LHS) equal to the Right-Hand Side (RHS).
3. **Solution of a Linear Equation:** The process of finding the value of the variable x such that the equation $ax + b = 0$ holds true. The solution is $x = -b/a$.
4. **Simultaneous Linear Equations:** Two or more linear equations involving the same set of variables. For example:

$$\begin{cases} ax + by + c = 0 \\ px + qy + r = 0 \end{cases}$$
5. **Inconsistent System:** A system of equations that has no common solution (e.g., $x + y = 2$ and $x + y = 7$).

6. **Dependent System:** A system in which one equation is a multiple of the other, leading to infinitely many solutions (e.g., $x + y = 2$ and $2x + 2y = 4$).
7. **Elimination Method:** A method of solving simultaneous equations by adding or subtracting equations to eliminate one variable and solve for the other.
8. **Substitution Method:** A method where one variable is expressed in terms of the other using one equation and then substituted into the second equation.
9. **Cross Multiplication Method:** A technique used to solve two linear equations in two variables by applying a specific formula involving determinants.
10. **Simultaneous Equations in Three Variables:** A system of three linear equations involving three unknowns (x, y, z). Solved by eliminating one variable to reduce it to two equations in two variables.

2.11 Examination Oriented Questions

- Q1. Given: $C = 102 + 0.7Y$, $I = 150 - 100i$, $M_s = 300$, $M_t = 0.25Y$, and $M_z = 124 - 200i$. Find (a) the equilibrium level of income and the equilibrium rate of interest and (b) the level of C , I , M_t , and M_z when the economy is in equilibrium.
- Q2. The demand and supply laws are given by the equations $2p + 3q = 32$ and $3p - q = 26$. Find the equilibrium price and quantity before and after a reduction of $\frac{8}{3}$ per unit is allowed to the supplier.
- Q3. Solve for x :
- $$\frac{xy}{x+y} - \frac{2}{3} \frac{yz}{y+z} - \frac{4}{3} \frac{zx}{z+x} = \frac{4}{5}$$
- Q4. Solve the following:
- i) $x + y = 5xy$, $y + z = 6yz$, $z + x = 7zx$

$$\begin{aligned} \text{i)} \quad & 2x - 3y = 1 \text{ and } 5x + 4y = 14 \\ & \frac{2}{x} + \frac{3}{y} = 12 \text{ and } \frac{4}{x} - \frac{5}{y} = 2 \\ \text{ii)} \quad & \end{aligned}$$

Q4. Find

- (a) Equilibrium income level and interest rate and
- (b) Levels of C , I , M_t , and M_z in equilibrium

$$\begin{aligned} \text{when } C &= 89 + 0.6Y, \quad I = 120 - 150i, \quad M_s = 275 \\ M_t &= 0.1Y, \quad M_z = 240 - 250i. \end{aligned}$$

2.12 Hints to Check Your Progress (CYP)

CYP-I

Q1 Refer Section 2.6.2

CYP-II

Q1 Refer Section 2.8 (i)

Q2 Refer Section 2.8 (iii)

Q3 Refer Section 2.7

2.13 Suggested Readings

- Aggarwal, C.S. & Joshi, R.C. (2020). Mathematics for Students of Economics. New Academic Publishing Co.
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	Unit I

MATRIX OPERATIONS WITH MATRICES SOLUTION, RANK OF THE MATRIX AND MATRIX INVERSION

STRUCTURE

- 3.0 Objectives
- 3.1 Learning outcomes
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- 3.9 Adjoint of a Square Matrix
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 - 3.10.1 Finding Inverse of a Matrix through Adjoint
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3.0 Objectives

The main objectives of this lesson are to enable you to:

- ☐ Understand the definition, structure, and types of matrices.
- ☐ Perform fundamental operations on matrices including addition, multiplication, transpose, adjoint, and inverse.
- ☐ Compute the determinant of a matrix and interpret its significance.
- ☐ Apply matrix methods (Cramer's Rule, Matrix Inverse, and Gauss Elimination) to solve systems of linear equations.
- ☐ Determine the rank of a matrix and understand its implications.

- Apply matrix algebra to key economic models such as equilibrium price systems, national income determination, IS-LM models, and constrained optimization problems.

3.1 Learning outcomes

After going through this lesson, you will be able to:

- Define and classify various types of matrices and their properties.
- Use matrix operations to manipulate and simplify algebraic structures.
- Solve simultaneous equations using different matrix techniques.
- Demonstrate the application of matrix algebra in economic analysis.
- Analyze consistency in systems of linear equations.
- Solve constrained optimization problems using matrix-based methods like the Lagrangian multiplier technique.

3.2 Introduction

The study of matrices and determinants holds a foundational place in mathematical economics. It provides powerful algebraic tools for solving complex systems of linear equations, enabling economists and business analysts to model and analyze economic behavior quantitatively. From input-output analysis and equilibrium modeling to constrained optimization and IS-LM analysis, matrices offer precision, structure, and clarity in economic problem-solving. This lesson aims to introduce the core concepts of matrices and determinants, their algebraic properties, and applications in solving real-world economic models.

3.3 Matrices: Definition

Definition of Matrix: A matrix is a rectangular array of numbers. In other words, a set of mn numbers arranged in the form of rectangular array of m rows and n column is called $m \times n$ matrix (to be read a ‘ m ’ by ‘ n ’ matrix). The general form of a matrix of order $m \times n$ is usually written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

The numbers $a_{11}, a_{12}, a_{22}, \dots, a_{ij}$ etc. are called the elements of the matrix A . m is the number of rows and n is the number of columns.

$$\text{For example: } A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & 5 & 5 & 5 \\ 2 & 4 & 6 & 6 \end{bmatrix}_{3 \times 4} \text{ is a matrix of order } 3 \times 4$$

3.4 Notations of Matrices

Matrices are denoted by capital letters A, B, C or X, Y, Z etc. and its elements by small letters a, b, c . The elements of the matrix are enclosed by any of the brackets i.e. $[], (), \{\}$. The position of the elements of a matrix is indicated by the subscripts attached to the element. For example, a_{12} indicates that the element lies in the first row and second column i.e., first subscript denotes row and second denotes column.

3.5 Order of a Matrix

The number of rows and columns of a matrix determines the order of the matrix. Thus, a matrix having m rows and n column is said to be of the order $m \times n$ (reads as ' m ' by ' n '). In particular, a matrix having 3 rows and 4 columns is of the order 3×4 and it is called a 3×4 matrix.

$$\text{For example: } A = \begin{pmatrix} 3 & 5 & 6 \\ 7 & 4 & 8 \end{pmatrix}_{2 \times 3}$$

Is a matrix of order 2×3 since there are two rows and three columns.

3.6 Types of Matrices

- 1) **Rectangular Matrix:** A matrix in which the number of rows and columns are not equal is called rectangular matrix. For example,

$$A = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 4 & 1 \end{bmatrix}_{2 \times 3} \text{ is a rectangular matrix of order } 2 \times 3.$$

- 2) **Square Matrix:** A matrix in which the number of rows equal to the number of columns is called a square matrix. For example,

$$A = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 2 & 4 \\ 3 & 2 & 6 \end{pmatrix} \text{ is a square matrix of order } 3 \times 3 \text{ i.e. of order } 3.$$

Note: The elements 2, 2, 6 in this matrix are called diagonal elements and the line along which they lie is called the principal diagonal.

- 3) **Diagonal Matrix:** A square matrix in which all elements except the diagonal elements are zeroes is called a diagonal matrix. For example,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}_{3 \times 3} \text{ is a diagonal matrix of order } 3 \times 3.$$

- 4) **Scalar Matrix:** A diagonal matrix in which all diagonal elements are equal (but not equal to 1), is called a scalar matrix. For example,

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ is a scalar matrix of order } 3 \times 3.$$

- 5) **Identity (or Unit) Matrix:** A square matrix whose each diagonal elements is unity and all other elements are zeros is called an identity (or unit) matrix. An identity matrix of order 3 is denoted by I_3 or simply by I . For example,

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ is a unit matrix of order } 3.$$

- 6) **Null (or Zero) Matrix:** A matrix of any order (rectangular or square) whose each of its element is zero is called a null matrix (or a zero matrix) and is denoted by O. For example,

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are null matrices of order 2×3 respectively.

- 7) **Row Matrix:** A matrix having only one row and any number of columns is called a row matrix (or a row vector). For example,

$$A = [1, 2, 3] \text{ is a row matrix of order } 1 \times 3$$

- 8) **Column Matrix:** A matrix having only one columns and any number of rows is called a column matrix (or a column vector). For example,

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ is a column a matrix of order } 3 \times 1$$

- 9) **Upper Triangular and Lower Triangular Matrix:** A square matrix is called an upper triangular matrix if all the elements below the principal diagonal are zero and it is said to be lower triangular matrix if all the elements above the principal diagonal are zero. For example,

$$A = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 4 \\ 0 & 0 & 8 \end{pmatrix} \text{ is an upper triangular matrix and}$$

$$B = \begin{pmatrix} 7 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 3 \end{pmatrix} \text{ is a lower triangular matrix}$$

- 10) **Sub Matrix:** A matrix obtained by deleting some rows or column or both of a given matrix is called its sub matrix. For example,

$$\begin{pmatrix} 4 & 5 \\ 5 & 2 \end{pmatrix}_{2 \times 2} \text{ is a sub matrix of } \begin{pmatrix} 4 & 5 & 6 \\ 3 & 3 & 6 \\ 5 & 2 & 1 \end{pmatrix}_{3 \times 3}$$

The sub matrix of the order 2×2 is obtained by deleting 2nd row and 3rd column of matrix A.

- 11) **Trace of a Matrix:** In a square matrix A, the sum of all the diagonal elements is called trace of A. It is denoted by $\text{tr } A$. For example,

$$\text{If } A = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 2 & 3 \end{pmatrix}_{3 \times 3} \text{ then } \text{tr } A = 2 + 1 + 3 = 6$$

$$\text{In general, } \text{tr } (A) = a_{11} + a_{22} + a_{33} + \dots + a_{nn} = \sum_{i=1}^n a_{ii}$$

- 12) **Equality of Two Matrices:** Two matrices A and B are said to be equal if and only if they are of the same order and each element of A is equal to the corresponding element of B. For example,

$$\text{If } A = \begin{pmatrix} 3 & 4 \\ 0 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 4 \\ 0 & -1 \end{pmatrix}, \text{ then } A = B$$

Since the matrix A is of the order 2×2 and the matrix B is of the order 2×2 and also the corresponding elements of A and B are equal.

- 13) **Equivalent Matrices:** Two matrices A and B are said to be equivalent if they are of the same order.

If $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then A and B are equivalent.

Example: Construct 2×3 matrix whose elements are given by $a_{ij} = \frac{i-j}{2}$

Solution: We are to find the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ of order 2×3 , where a_{ij}

$$= \frac{i-j}{2}$$

$$a_{11} = \frac{1-1}{2} = 0, a_{12} = \frac{1-2}{2} = -\frac{1}{2}, a_{13} = \frac{1-3}{2} = -1$$

$$a_{21} = \frac{2-1}{2} = \frac{1}{2}, a_{22} = \frac{2-2}{2} = 0, a_{23} = \frac{2-3}{2} = -\frac{1}{2}$$

$$\therefore A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & -1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

3.7 Operations on Matrices (or Matrix Operations)

3.7.1 Addition (or Subtraction) of Matrices

The addition (or subtraction) of two or more matrices is possible only if these matrices have the same dimension i.e. the matrices must have the same number of rows and same number of columns. The sum (or difference) of matrices is obtained by adding (or subtracting) the corresponding elements of the given matrices. For example

$$\text{If } A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 4 & 8 \\ 3 & -2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 5 \\ 0 & 1 & 6 \end{bmatrix}$$

$$\text{Then, } A + B = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 4 & 8 \\ 3 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 5 \\ 0 & 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4 & 3+1 & 4+0 \\ -2+1 & 4+3 & 8+5 \\ 3+0 & -2+1 & -1+6 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ -1 & 7 & 13 \\ 3 & -1 & 5 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 4 & 8 \\ 3 & -2 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 5 \\ 0 & 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1-4 & 3-1 & 4-0 \\ -2-1 & 4-3 & 8-5 \\ 3-0 & -2-1 & -1-6 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 4 \\ -3 & 1 & 3 \\ 3 & -3 & -7 \end{bmatrix}$$

3.7.2. Multiplication of a Matrix by a Scalar (or Scalar Multiplication)

The multiplication of a matrix by a scalar K implies multiplication of every element of the matrix A by K .

If $A = [a_{ij}]$ is any matrix of dimension $m \times n$ and K is any scalar, then the multiplication of the matrix A by the scalar K is denoted by KA is obtained by multiplying each element of A by the scalar K , i.e.,

$$KA = K [a_{ij}]_{m \times n} = [K a_{ij}]_{m \times n}$$

Example. If $A = \begin{pmatrix} 3 & 8 & 11 \\ 6 & -3 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -6 & 15 \\ 3 & 8 & 17 \end{pmatrix}$. Compute $7A + 5B$

Solution: We have $A = \begin{pmatrix} 3 & 8 & 11 \\ 6 & -3 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -6 & 15 \\ 3 & 8 & 17 \end{pmatrix}$

$$\therefore 7A = \begin{pmatrix} 7 \times 3 & 7 \times 8 & 7 \times 11 \\ 7 \times 6 & 7 \times -3 & 7 \times 8 \end{pmatrix} = \begin{pmatrix} 21 & 56 & 77 \\ 42 & -21 & 56 \end{pmatrix}$$

$$5B = \begin{pmatrix} 5 \times 1 & 5 \times -6 & 5 \times 15 \\ 5 \times 3 & 5 \times 8 & 5 \times 17 \end{pmatrix} = \begin{pmatrix} 5 & -30 & 75 \\ 15 & 40 & 85 \end{pmatrix}$$

$$\begin{aligned} \therefore 7A + 5B &= \begin{pmatrix} 21 & 56 & 77 \\ 42 & -21 & 56 \end{pmatrix} + \begin{pmatrix} 5 & -30 & 75 \\ 15 & 40 & 85 \end{pmatrix} \\ &= \begin{pmatrix} 21+5 & 56-30 & 77+75 \\ 42+15 & -21+40 & 56+85 \end{pmatrix} = \begin{pmatrix} 26 & 26 & 152 \\ 57 & 19 & 141 \end{pmatrix} \end{aligned}$$

3.7.3 Properties of Matrix Addition

If A, B and C are any three matrices of same dimension, then

- 1) Matrix addition is commutative i.e.

$$A + B = B + A$$

- 2) Matrix addition is associative i.e.

$$(A + B) + C = A + (B + C)$$

- 3) Matrix addition is distributive w.r.t. scalar K, i.e.

$$K(A + B) = KA + KB$$

3.7.4 Multiplication of Matrices

Multiplication (or product) of two matrices A and B is possible if and only if the number of columns A is equal to the number of rows of B.

$$\text{Let } A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 5 & 1 \end{pmatrix}_{3 \times 2}, B = \begin{pmatrix} 4 & 0 \\ 3 & 8 \end{pmatrix}_{2 \times 2}$$

The product AB is possible because the number of columns of Matrix A is equal to the number of row of the matrix B . The dimension of the product AB will be 3×2 .

Definition: If $A = (a_{ij})$ be a $m \times n$ matrix and $B = (b_{jk})$ be an $n \times p$ matrix, then the product AB is defined as the matrix $C = (c_{ik})$ of the type $m \times p$ where

$$C_{ik} = \sum_{j=1}^n a_{ij} \times b_{jk}$$

Rule of Multiplication of Two Matrices: The rule of the multiplication of the matrices is row-column wise ($\rightarrow \downarrow$). The first row of AB is obtained by multiplying the 1st row with 1st, 2nd and 3rd column of B . Similarly, second row of AB is obtained by multiplying the 2nd row of A with 1st, 2nd and 3rd column of B respectively and so on.

Example. If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

Compute AB and BA and show that $AB \neq BA$.

Solution: Since A is 2×3 and B is 3×2 , then AB is defined.

$$\begin{aligned} \therefore AB &= \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 2 + (-2) \times 4 + 3 \times 2 & 1 \times 3 + (-2) \times 5 + 3 \times 1 \\ (-4) \times 2 + 2 \times 4 + 5 \times 2 & (-4) \times 3 + 2 \times 5 + 5 \times 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

Since B is 3×2 and A is 2×3 , then BA is also defined.

$$\begin{aligned} \therefore BA &= \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2 \times 1 + 3 \times (-4) & 2 \times (-2) + 3 \times 2 & 2 \times 3 + 3 \times 5 \\ 4 \times 1 + 5 \times (-4) & 4 \times (-2) + 5 \times 2 & 4 \times 3 + 5 \times 5 \\ 2 \times 1 + 1 \times (-4) & 2 \times (-2) + 1 \times 2 & 2 \times 3 + 1 \times 5 \end{bmatrix} \\ &= \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix} \end{aligned}$$

Hence, $AB \neq BA$

3.7.5 Properties of Matrix Multiplication

- i) Matrix multiplication, in general, is not commutative, i.e.

$$AB \neq BA$$

- ii) Matrix multiplication is associative i.e.

$$A(BC) = (AB)C$$

Where A, B, C are any three matrices of dimension $m \times n$, $n \times p$, $p \times q$ respectively.

- iii) Matrix multiplication is distributive i.e.

$$A(B + C) = AB + AC$$

Where A, B, C are any three matrices of dimension $m \times n$, $n \times p$ and $n \times p$ respectively

3.8 Transpose of a Matrix

Let A be any matrix. The matrix obtained from A by interchanging rows and columns of A is called to transpose of A and is denoted by A' or A^T (read as A transpose). Thus if $A = (a_{ij})$ is an $m \times n$ matrix, this $A' = [a_{ji}]$ will be a $n \times m$ matrix.

$$\text{For example, If } A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}_{2 \times 3}, \text{ then } A' = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}_{3 \times 2}$$

3.8.1 Properties of Transpose of Matrices

- i) The transpose of a transposed matrix is equal to the matrix itself, i.e.,

$$(A')' = A$$

- i) The transpose of the sum of the two matrices is equal to the sum of the transpose of the matrices, i.e.,

$$(A + B)' = A' + B'$$

- ii) The transpose of the product of two matrices is equal to the product of their transpose in the reverse order, i.e.,

$$(AB)' = B'A'$$

Example. If $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}$

Show that $(AB)' = B'A'$.

Solution: $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$, $A' = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}$

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{pmatrix} \quad B' = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1+2-2 & 2+0+2 \\ 2+2+0 & 4+0+0 \end{pmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix} \quad \dots(i)$$

$$B'A' = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1+2-2 & 2+2-0 \\ 2+0+0 & 4+0+0 \end{pmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix} \quad \dots(ii)$$

From (i) and (ii), $(AB)' = B'A'$.

Some Special Types of Matrices Based on Transpose

1) Symmetric Matrix: A square matrix A is said to be symmetric matrix if the transpose of A is equal to A, i.e., $A' = A$. For example,

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 5 & 0 & 7 \\ 4 & 7 & 3 \end{bmatrix} \text{ is a symmetric matrix as in this case } A' = A \text{ or } A = A'.$$

2) Skew-Symmetric Matrix: A square matrix A is said to be skew symmetric if the transpose of A is equal to $-A$, i.e., $A' = -A$. For example,

$$A = \begin{bmatrix} 0 & 3 & 5 \\ -3 & 0 & 4 \\ -5 & -4 & 0 \end{bmatrix} \text{ is a skew-symmetric matrix as in this case, } A' = -A \text{ or } A =$$

$-A'$.

Note: In a skew-symmetric matrix, all diagonal elements are zero.

3) Orthogonal Matrix: A square matrix A is said to be orthogonal if $AA' = A'A = I_3$

3.9 Adjoint of a Square Matrix

Definition: If A is any square matrix, then adjoint of A is defined as the transpose of the matrix obtained by replacing the element of A by their corresponding cofactors. In other words, the adjoint of a matrix A is the transpose of its cofactor matrix. It is denoted by $\text{adj. } A$

$$\text{Thus if } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad C(A) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

Then, $\text{adj. } A = \text{Transpose of the cofactor matrix}$

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

Steps to Compute Adjoint of a Matrix

- i) Check that the given matrix A is a square matrix.
- i) Compute Cofactor Matrix C (A). It is obtained by replacing each element in the matrix by its corresponding cofactor.
- ii) Take the transpose of the Cofactor Matrix. The result would give the adjoint of the matrix.

Example. Find the adjoint of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

Solution. Here $A = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix}$

The cofactors of the elements of the 1st Row in $|A|$ are:

$$+ \begin{vmatrix} 4 & 5 \\ -6 & -7 \end{vmatrix}, - \begin{vmatrix} 3 & 5 \\ 0 & -7 \end{vmatrix}, + \begin{vmatrix} 3 & 4 \\ 0 & -6 \end{vmatrix}$$

i.e. $(-28 + 30), -(-21 - 0), +(-18 - 0)$

i.e. $2, 21, -18$

The cofactors of the elements of the 2nd Row in $|A|$ are:

$$- \begin{vmatrix} 0 & -1 \\ -6 & -7 \end{vmatrix}, + \begin{vmatrix} 1 & -1 \\ 0 & -7 \end{vmatrix}, - \begin{vmatrix} 1 & 0 \\ 0 & -6 \end{vmatrix}$$

i.e. $-(0 - 6), +(-7 - 0), -(-6 - 0)$

i.e. $6, -7, 6$

The cofactors of the elements of the 3rd Row in $|A|$ are:

$$+ \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix}, - \begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix}, + \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix}$$

i.e. $(0 + 4), -(5 + 3), +(4 - 0)$

i.e. $4, -8, 4$

adj. $A =$ transpose of the cofactor matrix $=$ transpose of $\begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}$

$$\therefore \text{adj. } A = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

Important Theorem: If A is a square matrix of order n , then $A (\text{adj. } A) = (\text{adj. } A) A = |A| \cdot I$ where I is the identity matrix.

3.10 Inverse of A Square Matrix

Let A be any square matrix of order n . The n -square matrix B of the same order is called the inverse of A if $AB = BA = I$.

The inverse of the matrix A is denoted by A^{-1} so $B = A^{-1}$.

Note: For the inverse of the matrix, the matrix must be **non-singular**. **A matrix A is said to be non-singular if $|A| \neq 0$. Whereas a matrix is said to be singular if its determinant is equal to zero, i.e., $|A| = 0$.**

3.10.1 Finding Inverse of a Matrix through Adjoint

The inverse of A is calculated by the following formula:

$$A^{-1} = \frac{\text{adj. } A}{|A|} \text{ provided } |A| \neq 0.$$

Steps to Find the Inverse of A Matrix

- i) Find the value of $|A|$ if $|A| \neq 0$, go to step 2. If $|A| = 0$, we cannot find the inverse of the matrix.
- i) We shall then find the adjoint of the matrix.
- ii) Apply the following to calculate the inverse of A .

$$A^{-1} = \frac{\text{adj. } A}{|A|} \text{ provided } |A| \neq 0.$$

Note: The necessary condition for finding the inverse of a matrix A is that (i) the given matrix is a square matrix and (ii) the determinant of the matrix is not equal to zero i.e. $|A| \neq 0$.

Example. Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution. Here $A = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$

$$\text{Then, } |A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 0(2 - 3) - (1 - 9) + 2(1 - 6)$$

$$= 0 + 8 - 10 = -2 \neq 0$$

$\therefore A$ is non-singular and hence A^{-1} exists.

Now we proceed to find adj. A

Cofactors of the Elements of the 1st Row in $|A|$ are:

$$+ \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, - \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$\text{i.e., } (2 - 3), \quad -(1 - 9), \quad +(1 - 6)$$

$$\text{i.e., } -1, \quad 8, \quad -5$$

Cofactors of the Elements of the 2nd Row in |A|:

$$-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, +\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$$

$$\text{i.e., } -(1 - 2), \quad + (0 - 6), \quad - (0 - 3)$$

$$\text{i.e., } 1, \quad - 6, \quad 3$$

Cofactors of the Elements of the 3rd Row in |A| are:

$$+\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}, +\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$$

$$\text{i.e., } +(3 - 4), \quad - (0 - 2), \quad + (0 - 1)$$

$$\text{i.e., } - 1, \quad 2, \quad - 1$$

$\therefore$ adj. A = transpose of the cofactor matrix

$$= \text{transpose of } \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$\therefore \text{adj. A} = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{\text{adj. A}}{A}$$

$$= \frac{1}{-2} \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

Some Important Theorems on Inverse Matrices

- a) If A and B are non-singular matrices of the same order, then
 $(AB)^{-1} = B^{-1} \cdot A^{-1}$
- b) The inverse of the transpose of a matrix is equal to the transpose of the inverse of that matrix *i.e.*
 $(A')^{-1} = (A^{-1})'$
- c) The inverse of the inverse of a matrix is the matrix itself *i.e.*
 $(A^{-1})^{-1} = A.$

3.11 Minor of A Matrix

We know that Submatrix of a given matrix A is either the matrix itself or a matrix obtained by deleting some rows or columns or both.

The determinants of every square submatrix of the matrix A is called the **minor of A**.

3.12 Rank of A Matrix

Definition: The number 'r' is called the rank of the matrix A if

- i) There exists at least one minor of order r of A which is non zero and
- i) Every minor of order (r + 1) of A is zero

The rank of a matrix is denoted by $\rho(A)$

Steps to Find Rank of A Matrix

- i) The given matrix should be a square matrix. If it is not so, the matrix should be made a square matrix by deleting the extra row or the column.

- i) Find the determinant of the square matrix given or obtained after deleting extra row or column.
- ii) If determinant of the sub matrix is zero, then take the sub-matrices of the given matrix. If the determinant of any one of the sub-matrices is not zero, then the order of that sub matrix would be the rank of the given matrix

Note: From the above definition of the rank of the matrix, we observe that:

- i) The rank of null matrix is zero *i.e.* $\rho(O) = 0$ where O is a zero matrix.
- i) The rank of a non-singular matrix of order n is n .
- ii) The rank of an **identity matrix** of order n is n .
- iv) The rank of a 3×4 matrix has rank not more than 3.
- v) The rank is always found for the square matrix.

Example. Find the rank of the matrix:

$$\begin{bmatrix} 3 & 2 & 1 \\ 3 & 4 & 3 \\ -1 & 2 & 2 \end{bmatrix}$$

Solution: Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 4 & 3 \\ -1 & 2 & 2 \end{bmatrix}$

$$\begin{aligned} \therefore |A| &= 3(8 - 6) - 2(6 + 3) + 1(6 + 4) \\ &= 6 - 18 + 10 = -2 \neq 0 \end{aligned}$$

Thus, minor of order 3 of A is not zero.

$\therefore$ The rank of the matrix is 3 *i.e.*

$$\rho(A) = 3$$

Check Your Progress-I

Based on your understanding of the previous sections, solve these questions:

Q1. Find a,b,c, d where
$$\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 15 & -2 \\ 6 & 3 \end{bmatrix}$$

Q2. Find the value of x such that
$$\begin{vmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & x & 0 \\ 0 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{vmatrix} = 0$$

Q3. Define: Symmetric matrix, Skew-Symmetric matrix and Orthogonal matrix.

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}$$

Q4. Show that $(AB)' = B'A'$ if $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Q5. Find the adjoint of:

Q6. Find a matrix B such that $AB = I$ if $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

3.13 Solution of System of Simultaneous Linear Equations

Consider a system of three linear equations:

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

in three unknowns x, y and z

A set of values x, y and z which simultaneously satisfy all the three equations is called the solution the above system of equations.

3.13.1 Methods of Solving System of Linear Equations

There are three methods of solving a system of linear equations. These methods are:

- 1) **Cramer's Rule**
- 2) **Matrix Inverse Method**
- 3) **Gauss Elimination Method**

- 1) **Cramer's Rule:** Gabriel Cramer, a Swiss Mathematician has developed a method, popularly known as **Cramer's Rule** to solve a system of linear equation **with the help of determinants**.

Let us consider a system of three linear equations:

$$a_1x + b_1y + c_1z = d_1 \quad \dots (i)$$

$$a_2x + b_2y + c_2z = d_2 \quad \dots (ii)$$

$$a_3x + b_3y + c_3z = d_3 \quad \dots (iii)$$

The solution of the equations using Cramer's Rule is given by:

$$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta} \text{ provided } \Delta \neq 0$$

Where Δ = determinant formed by the coefficients of x, y and z

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Δ_1 = determinant obtained from Δ by replacing the first column of Δ by constants on the R.H.S. of given equation

$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Δ_2 = determinant obtained from Δ by replacing the second column of Δ by constants on the R.H.S. of given equation

$$= \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

Δ_3 = determinant obtained from Δ by replacing the third column of Δ by constants on the R.H.S. of given equation

$$= \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

- Note:** i) If $\Delta \neq 0$ the above system of linear equations are consistent and have a unique solution.
- ii) If $\Delta = 0$ and at least $\Delta_1, \Delta_2, \Delta_3$ is non-zero, then the given system is inconsistent and thus has no unique solution.

The following examples illustrate the Cramer's rule for solving a system of linear equation.

Example. Solve the following system of equation by Cramer's Rule:

Solution:

$$\begin{aligned} \text{Here } \Delta &= \begin{vmatrix} 3 & 1 & -1 \\ 1 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 3 \begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} \\ &= 3(4 - 6) - (1 - 4) - (3 - 8) \\ &= -6 + 3 + 5 = 2 \neq 0 \end{aligned}$$

Since $\Delta \neq 0$ the given system of equations has a unique solution.

$$\begin{aligned}\Delta_1 &= \begin{vmatrix} 5 & 1 & -1 \\ 6 & 4 & 2 \\ 4 & 3 & 1 \end{vmatrix} \\ &= 5 \begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} - 1 \begin{vmatrix} 6 & 2 \\ 4 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 6 & 4 \\ 4 & 3 \end{vmatrix} \\ &= 5 (4 - 6) - 1 (6 - 8) - 1 (18 - 16) \\ &= -10 + 2 - 2 = -10\end{aligned}$$

$$\begin{aligned}\Delta_2 &= \begin{vmatrix} 3 & 5 & -1 \\ 1 & 6 & 2 \\ 2 & 4 & 1 \end{vmatrix} \\ &= 3 \begin{vmatrix} 6 & 2 \\ 4 & 1 \end{vmatrix} - 5 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix} \\ &= 3 (6 - 8) - 5 (1 - 4) - 1 (4 - 12) \\ &= -6 + 5 + 8 = 17\end{aligned}$$

$$\begin{aligned}\Delta_3 &= \begin{vmatrix} 3 & 1 & 5 \\ 1 & 4 & 6 \\ 2 & 3 & 4 \end{vmatrix} \\ &= 3 \begin{vmatrix} 4 & 6 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix} + 5 \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} \\ &= 3 (16 - 18) - 1 (4 - 12) + 5 (3 - 8) \\ &= -6 + 8 - 25 = -23\end{aligned}$$

By Cramer's Rule we, have

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{-10}{2} = -5$$

$$\therefore y = \frac{\Delta_2}{\Delta} = \frac{17}{2} = 8.5$$

$$\therefore z = \frac{\Delta_3}{\Delta} = \frac{-23}{2} = -11.5$$

2) Matrix Inverse Method: Under the method, we use inverse of the matrix to solve a system of equations.

Consider a system of three linear equations:

$$a_1 x + b_1 y + c_1 z = d_1 \quad \dots (i)$$

$$a_2 x + b_2 y + c_2 z = d_2 \quad \dots (ii)$$

$$a_3 x + b_3 y + c_3 z = d_3 \quad \dots (iii)$$

These equations can be written in the matrix form as:

$$AX = B$$

$$\text{when } A = \text{coefficient matrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

If $|A| \neq 0$, then the unique solution of the equations using matrix inverse method is given by:

$$X = A^{-1} B$$

$$\text{where } X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, A^{-1} = \text{inverse of the coefficient matrix}, B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Note: 1. If $|A| \neq 0$ the above system of linear equations are consistent and have a unique solution.

2. If $|A| \neq 0$ the above system has either no solution or an infinite number of solution.

The following examples illustrate the matrix inverse method for solving system of solutions.

Example. Solve the following system equations by matrix method:

$$2x + y + z = 1$$

$$x - 2y - z = 1.5$$

$$3y - 5z = 9$$

Solution: The given equations are rewritten as:

$$2x + y + z = 1$$

$$x - 2y - z = 1.5$$

$$0x + 3y - 5z = 9$$

Putting the equations in the matrix form as:

$$AX = B$$

$$\text{Where } A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1.5 \\ 9 \end{pmatrix}$$

$$\text{Now } |A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{vmatrix}$$

$$= 2(10 + 3) - 1(-5 - 3) + 0(-1 + 2)$$

$$= 26 + 8 + 0 = 34 \neq 0$$

Since $|A| \neq 0$, the system has a unique solutions given by:

$$X = A^{-1} B$$

Now we find A^{-1}

The Cofactors of the Elements of the 1st Row of $|A|$ are:

$$+ \begin{vmatrix} -2 & -1 \\ 3 & -5 \end{vmatrix}, - \begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix}, + \begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix}$$

$$\text{i.e.,} \quad (10 + 3), \quad - (-5 - 0), \quad + (3 - 0)$$

$$\text{or} \quad 13, \quad 5, \quad 3$$

The Cofactors of the Elements of the 2nd Row of $|A|$ are:

$$- \begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix}, + \begin{vmatrix} 2 & 1 \\ 0 & -5 \end{vmatrix}, - \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix}$$

$$\text{i.e.,} \quad - (-5 - 3), \quad + (-10 - 0), \quad - (6 - 0)$$

$$\text{or} \quad 8, \quad -10, \quad -6$$

The Cofactors of the Elements of the 3rd Row of $|A|$ are:

$$+ \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix}, - \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}, + \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$$

$$\text{i.e.,} \quad (-1 + 2), \quad - (-2 - 1), \quad + (-4 - 1)$$

$$\text{or} \quad 1, \quad 3, \quad -5$$

$$C(A) = \text{Cofactor Matrix} = \begin{pmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{pmatrix}$$

$$\text{adj. } A = [C(A)]' = \begin{pmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{pmatrix}$$

$$\therefore \text{adj. } A = \begin{pmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{34} \begin{pmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{pmatrix}$$

From (1) we have $X = A^{-1} B$

$$\text{i.e. } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{34} \begin{pmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 1.5 \\ 9 \end{pmatrix} = \frac{1}{34} \begin{pmatrix} 37 \\ 17 \\ -51 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ -1.5 \end{pmatrix}$$

$$\therefore x = 1, y = 0.5, z = -1.5$$

3) Gauss Elimination Method: Under this method, we use elementary row operations to solve a system of linear equations. Let us consider the system of linear equations in 3 unknown in matrix forms as:

$$AX = B$$

$$\text{where } A = \begin{pmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

In Gauss elimination method, we first form an augmented (A/B) and then augmented matrix [A/B] is reduced to an upper triangular matrix by applying elementary row operations. Then we obtain the solution by back substitution method.

The following examples illustrates the Gauss elimination method:

Example: Solve the following system of equations by Gauss Elimination Method:

$$2x + y - z = 9$$

$$x + y + z = 9$$

$$3x - y - z = -1$$

Solution: Write down the given equations in the matrix forms:

$$AX = B$$

$$\text{where } A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 9 \\ 9 \\ -1 \end{pmatrix}$$

Here $|A| = 8 \neq 0$, the system of equations are consistent and has a unique solution.

We apply the Gauss elimination method

The augmented matrix is

$$[A / B] = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 9 \\ 1 & 1 & 1 & 9 \\ 3 & -1 & -1 & -1 \end{array} \right]$$

Applying row operation $R_1 \leftrightarrow R_2$ we get

$$[A / B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 1 & -1 & 9 \\ 3 & -1 & -1 & -1 \end{array} \right]$$

Applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$ we get

$$[A / B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -9 \\ 0 & -4 & -4 & -28 \end{array} \right]$$

Applying $R_2 \rightarrow R_2 (-1)$, we get

$$[A / B] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 3 & 9 \\ 0 & -4 & -4 & -28 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 + 4R_2$, we get

$$[A / B] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 8 & 8 \end{bmatrix}$$

$$\text{which } \Rightarrow \quad x + y + z = 9 \quad \dots (i)$$

$$0x + y + 3z = 9 \quad \dots (ii)$$

$$0x + 0y + 8z = 8 \quad \dots (iii)$$

From (iii), we obtain $-8z = 8 \Rightarrow z = -1$

Substituting $z = -1$ in (ii) we get $y = 6$ and

on substituting $y = 6$ and $z = -1$ in (i) we get $x = 2$

$$\therefore \quad x = 2, y = 6, \quad z = -1$$

$$\textbf{Check:} \quad \text{L.H.S } 2x + y - z = 2(2) + 6 - (-1)$$

$$= 4 + 6 - (-1) = 9 = \text{R.H.S}$$

3.14 Consistent and Inconsistent Equations

A system of linear equations may have a unique solution, infinite number of solution or no solution at all.

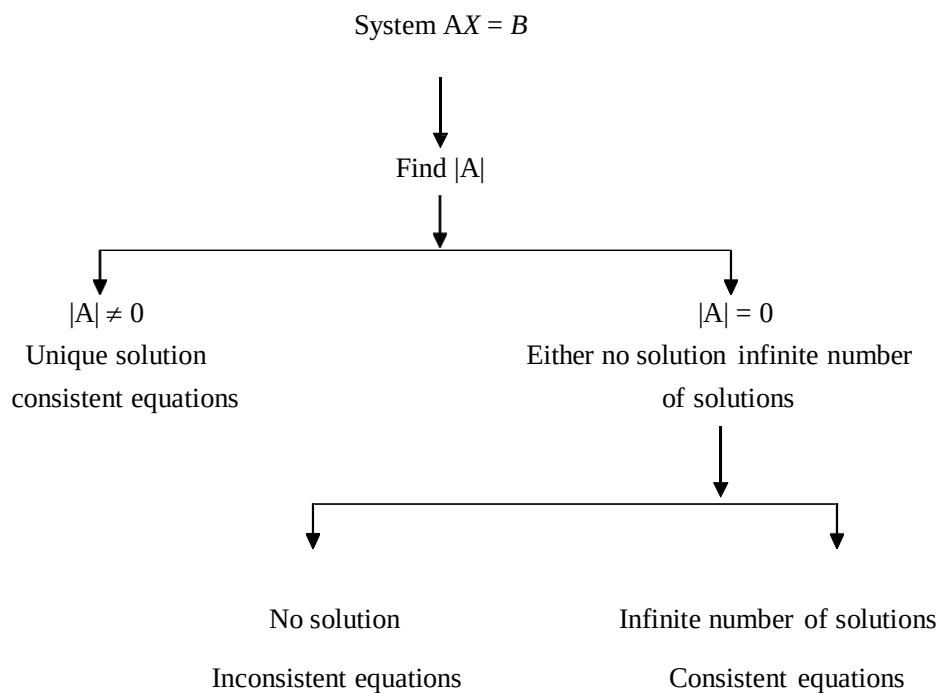
If it has a solution (whether a unique or infinite number of solutions), the equations are consistent and if it has no solution, then the equations are inconsistent.

Condition for Consistency:

For a system of linear equations written in matrix form $AX = B$, the condition for consistency is:

- i) If $|A| \neq 0$, the given system of equations has a unique solutions and the equations are consistent.
- ii) If $|A| = 0$, then the given system of equations has either no solution or infinite number of solutions. If it has infinite number of solutions, then the equations are consistent and if it has no solution, then the equations are inconsistent.

The following chart summaries the consistency of equations:



Note 1: In case $|A| \neq 0$ unique solution exists and, we can use either Cramer's rule, Matrix Inverse method or Gauss elimination method.

Note 2: In case $|A| = 0$ i.e. either no solution or infinite number of solutions. **We only apply Gaussian elimination method.** We cannot use Cramer's rule or Matrix Inverse method. This is shown below:

Cramer's rule $\left(x = \frac{|A_1|}{|A|}, y = \frac{|A_2|}{|A|}, z = \frac{|A_3|}{|A|} \right)$ cannot be applied because $|A| = 0$

Matrix Inverse Method $\left(x = A^{-1} B, A^{-1} = \frac{adj.A}{|A|} \right)$ cannot be applied because

$|A| = 0$

3.15 Economic Applications of Matrices

1. Equilibrium Prices

Matrices are used to solve systems of linear equations representing demand and supply conditions for multiple goods. By expressing these equations in matrix form ($AX = B$), one can efficiently determine the equilibrium prices using methods like matrix inversion or Cramer's rule.

Example. The equilibrium condition for the related markets is given by

$$11p_1 - p_2 - p_3 = 31$$

$$-p_1 + 6p_2 - 2p_3 = 26$$

$$-p_1 - 2p_2 + 7p_3 = 24$$

Find the equilibrium price for each market using Cramer's rule

Solution: Given

$$11p_1 - p_2 - p_3 = 31$$

$$-p_1 + 6p_2 - 2p_3 = 26$$

$$-p_1 - 2p_2 + 7p_3 = 24$$

$$\begin{aligned}\text{Here, } \Delta &= \begin{vmatrix} 11 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 7 \end{vmatrix} \\ &= 11 \begin{vmatrix} 6 & -2 \\ -2 & 7 \end{vmatrix} - (-1) \begin{vmatrix} -1 & -2 \\ -1 & 7 \end{vmatrix} + (-1) \begin{vmatrix} -1 & 6 \\ -1 & -2 \end{vmatrix} \\ &= 11 (42 - 4) + 1 (-7 - 2) - (2 + 6) = 11(38) - 9 - 8 \\ &= 418 - 9 - 8 = 401\end{aligned}$$

$$\begin{aligned}\Delta_1 &= \begin{vmatrix} 31 & -1 & -1 \\ 26 & 6 & -2 \\ 24 & -2 & 7 \end{vmatrix} \\ &= 31 \begin{vmatrix} 6 & -2 \\ -2 & 7 \end{vmatrix} - (-1) \begin{vmatrix} 26 & -2 \\ 24 & 7 \end{vmatrix} + (-1) \begin{vmatrix} 26 & 6 \\ 24 & -2 \end{vmatrix} \\ &= 31 (42 - 4) + 1 (182 + 48) - (-52 - 144) \\ &= 1178 + 230 + 196 = 1604\end{aligned}$$

$$\begin{aligned}\Delta_2 &= \begin{vmatrix} 11 & 31 & -1 \\ -1 & 26 & -2 \\ -1 & 24 & 7 \end{vmatrix} \\ &= 11 \begin{vmatrix} 26 & -2 \\ 24 & 7 \end{vmatrix} - 31 \begin{vmatrix} -1 & -2 \\ -1 & 7 \end{vmatrix} + (-1) \begin{vmatrix} -1 & 26 \\ -1 & 24 \end{vmatrix} \\ &= 11 (182 + 48) - 31 (-7 - 2) - (-24 + 26) \\ &= 11 \times 230 - 31 (-9) - 1 (+2) \\ &= 2530 + 279 - 2 = 2807\end{aligned}$$

$$\begin{aligned}\Delta_3 &= \begin{vmatrix} 11 & -1 & 31 \\ -1 & 6 & 26 \\ -1 & -2 & 24 \end{vmatrix} \\ &= 11 \begin{vmatrix} 6 & 26 \\ -2 & 24 \end{vmatrix} - (-1) \begin{vmatrix} -1 & 26 \\ -1 & 24 \end{vmatrix} + 31 \begin{vmatrix} -1 & 6 \\ -1 & -2 \end{vmatrix} \\ &= 11 (144 + 52) + 1 (-24 + 26) + 31 (2 + 6) \\ &= 2156 + 2 + 248 = 2406\end{aligned}$$

Using Cramer's rule,

$$p_1 = \frac{\Delta_1}{\Delta} = \frac{1604}{401} = 4$$

$$p_2 = \frac{\Delta_2}{\Delta} = \frac{2807}{401} = 7$$

$$p_3 = \frac{\Delta_3}{\Delta} = \frac{2406}{401} = 6$$

Thus, $p_1 = 4$, $p_2 = 7$ and $p_3 = 6$

2. National Income Determination Model

In Keynesian models, matrices help represent relationships between income, consumption, investment, and government spending. The national income can be solved as a system of equations using matrix algebra to find the equilibrium output level.

Example. Consider the following national income determination model:

$$Y = C + I + G, C = a + b(Y - T), T = d + ty$$

Solved the model for Y, C and T using Cramer's rule

Solution: We can rewrite the given system of equations as:

$$Y - C + 0T = I + G$$

$$-bY + C + bT = a$$

$$-tY + 0C + T = d$$

The equations in the matrix form is

$$AX = B$$

$$\text{where } A = \begin{pmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{pmatrix}, X = \begin{pmatrix} Y \\ C \\ T \end{pmatrix}, B = \begin{pmatrix} I + G \\ a \\ d \end{pmatrix}$$

$$\text{Now } |A| = \begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1) \begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + 0 \begin{vmatrix} -b & 1 \\ -t & 0 \end{vmatrix}$$

$$= 1(1 - 0) + 1(-b + bt) + 0(0 + t)$$

$$= 1 - b + bt$$

$$|A_1| = \begin{vmatrix} I + G & -1 & 0 \\ a & 1 & b \\ d & 0 & 1 \end{vmatrix}$$

$$= (I + G) \begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1) \begin{vmatrix} a & b \\ d & 1 \end{vmatrix} + 0 \begin{vmatrix} a & 1 \\ d & 0 \end{vmatrix}$$

$$= (I + G)(1 - 0) + 1(a - bd) + 0(0 - d)$$

$$= I + G + a - bd$$

$$\begin{aligned}
 |A_2| &= \begin{vmatrix} 1 & I+G & 0 \\ b & a & b \\ -t & d & 1 \end{vmatrix} \\
 &= 1 \begin{vmatrix} a & b \\ d & 1 \end{vmatrix} - (I+G) \begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + 0 \begin{vmatrix} -b & a \\ -t & d \end{vmatrix} \\
 &= 1 (a - bd) - (I+G) (-b + bt) + 0 (-bd + at) \\
 &= (a - bd) - (I+G) (bt - b)
 \end{aligned}$$

$$\begin{aligned}
 |A_3| &= \begin{vmatrix} 1 & -1 & I+G \\ -b & 1 & a \\ t & 0 & d \end{vmatrix} \\
 &= 1 \begin{vmatrix} 1 & a \\ 0 & d \end{vmatrix} - (-1) \begin{vmatrix} -b & a \\ -t & d \end{vmatrix} + (I+G) \begin{vmatrix} -b & 1 \\ t & 0 \end{vmatrix} \\
 &= 1 (d - 0) + 1 (-bd + at) + (I+G) (0 + t) \\
 &= d + (at - bd) + (I+G) t
 \end{aligned}$$

Using Cramer's Rule

$$Y = \frac{|A_1|}{|A|} = \frac{I+G+a-bd}{1-b+bt}$$

$$C = \frac{|A_2|}{|A|} = \frac{(a-bd)-(I+G)(bt-b)}{1-b+bt}$$

$$T = \frac{|A_3|}{|A|} = \frac{d+(at-bd)+(I+G)t}{1-b+bt}$$

3. IS-LM Model (Equilibrium Interest Rate and Income)

Matrices simplify the simultaneous solution of the IS and LM equations, which represent equilibrium in the goods and money markets. By expressing these equations in matrix form, we can calculate the equilibrium values of income (Y) and interest rate (i).

Example Given: the IS equation $0.3Y + 100i - 252 = 0$ and the LM equation $0.25Y - 200i - 176 = 0$. Find the equilibrium level of income and interest rate.

Solution The IS and LM equations can be reduced to the form:

$0.3Y + 100i = 252$ and $0.25Y - 200i = 176$, expressing into matrix form where

$$A = \begin{bmatrix} 0.3 & 100 \\ 0.25 & -200 \end{bmatrix} X = \begin{bmatrix} Y \\ i \end{bmatrix} B = \begin{bmatrix} 252 \\ 176 \end{bmatrix}$$

$$|A| = 0.3(-200) - 100(0.25) = -85$$

$$C = \begin{bmatrix} 200 & -0.25 \\ -100 & 0.3 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 200 & -100 \\ -0.25 & 0.3 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{185} \begin{bmatrix} 200 & -100 \\ -0.25 & 0.3 \end{bmatrix} = \begin{bmatrix} \frac{40}{17} & \frac{20}{17} \\ \frac{0.05}{17} & \frac{-0.06}{17} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{40}{17} & \frac{20}{17} \\ \frac{0.05}{17} & \frac{-0.06}{17} \end{bmatrix} \begin{bmatrix} 252 \\ 176 \end{bmatrix} = \begin{bmatrix} \frac{10080 + 3520}{17} \\ \frac{12.6 - 10.56}{17} \end{bmatrix} = \begin{bmatrix} 800 \\ 0.12 \end{bmatrix}$$

Thus, $Y=800$ and $i=0.12$

4. Constrained Optimization for Producers and Consumers

Matrices are essential in solving constrained optimization problems—where consumers aim to maximize utility and producers aim to minimize cost or maximize profit, subject to constraints like budgets or resource limits. Matrices help in solving utility maximization problems under budget constraints using methods like the Lagrangian multiplier. The system of first-order conditions can be compactly represented and solved using matrix techniques. Producers often minimize cost or maximize output subject to production constraints (like fixed inputs or budgets). Matrix algebra helps solve the resulting system of equations efficiently, especially when dealing with multiple inputs and constraints.

Example: Use crammers rule to solve for x and y when the cost function $C = 8x^2 - xy + 12y^2$ and the firm is bound by λ contract to provide a minimum combination of goods totaling 42 that is, subject to the constraint $x + y = 42$.

Check your Progress-II

Based on your understanding of the previous section, solve these questions:

- Q1. Solve the equations: $x + y + z = 1$, $x + 2y + 3z = 6$ and $x + 3y + 4z = 6$.

- Q2. The equilibrium condition for two goods is given by $5p_1 - 2p_2 = 15$ and $-p_1 + 8p_2 = 16$. Find the equilibrium price using Cramer's rule.

Q3. Solve and check whether the equations are consistent: $7x + 7y + 7z = 2$, $-x + 11y + 7z = 1$, $11x + 5y + 7z = 0$.

Solution: Considering the lagrangian function

$$C = 8x^2 - xy + 12y^2 + \lambda(42 - x - y)$$

$$\frac{\partial TC}{\partial x} = 16x - y - \lambda = 0$$

$$\frac{\partial TC}{\partial y} = -x + 24y - \lambda = 0$$

Rearranging the equations,

$$16x - y - \lambda = 0$$

$$-x + 24y - \lambda = 0$$

$$-x - y = -42$$

And setting into matrix form,

$$\begin{array}{cccc|c} 16 & -1 & -1 & x & 0 \\ -1 & 24 & -1 & y & 0 \\ -1 & -1 & 0 & \lambda & -42 \end{array}$$

Expanding along the third column,

$$|A| = (-1)(1 + 24) - (-1)(-16 - 1) + 0 = -42$$

$$A_1 = \begin{array}{ccc} 0 & -1 & -1 \\ 0 & 24 & -1 \\ -42 & -1 & 0 \end{array}$$

Expanding along the first column,

$$|A_1| = -42(1 + 24) = -1050$$

$$A_2 = \begin{vmatrix} 16 & 0 & -1 \\ -1 & 0 & -1 \\ -1 & -42 & 0 \end{vmatrix}$$

Expanding along the second column,

$$|A_2| = -(-42)(-16 - 1) = -714$$

$$A_3 = \begin{vmatrix} 16 & -1 & 0 \\ -1 & 24 & 0 \\ -1 & -1 & -42 \end{vmatrix}$$

Expanding along the third column,

$$|A_3| = -42(384 - 1) = -16086$$

Thus,

$$x = \frac{|A_1|}{|A|} = \frac{-1050}{-42} = 25, \quad y = \frac{|A_2|}{|A|} = \frac{714}{-42} = 17,$$

$$\lambda = \frac{|A_3|}{|A|} = \frac{-16086}{-42} = 383$$

3.16 Let us Sum Up

This chapter provided a comprehensive introduction to matrix algebra, covering different types of matrices along with concepts like the inverse, adjoint, and rank. It emphasized the use of matrices in solving systems of simultaneous equations efficiently. We also explored key economic applications, such as determining equilibrium prices, analyzing national income, and solving the IS-LM model for income and interest rate equilibrium. Additionally, the chapter highlighted the role of matrices in constrained optimization problems for both consumers and producers.

3.17 Keywords

1. **Matrix** – A rectangular array of numbers arranged in rows and columns,

used to represent data or systems of equations. Denoted usually by capital letters such as A, B, etc.

2. **Order of a Matrix** – The dimensions of a matrix given as the number of rows $\times$ number of columns ($m \times n$).
3. **Square Matrix** – A matrix where the number of rows equals the number of columns (e.g., 3×3 , 4×4).
4. **Rectangular Matrix** – A matrix with unequal number of rows and columns.
5. **Row Matrix** – A matrix with only one row and any number of columns.
6. **Column Matrix** – A matrix with only one column and any number of rows.
7. **Zero (Null) Matrix** – A matrix in which all elements are zero.
8. **Diagonal Matrix** – A square matrix where all non-diagonal elements are zero.
9. **Scalar Matrix** – A diagonal matrix in which all diagonal elements are equal (but not necessarily 1).
10. **Identity (Unit) Matrix** – A square matrix with 1s on the main diagonal and 0s elsewhere. Denoted as I or I_n .
11. **Transpose of a Matrix** – A matrix formed by interchanging the rows and columns of a given matrix. Denoted by A^T .
12. **Symmetric Matrix** – A square matrix where $A = A^T$.
13. **Skew-Symmetric Matrix** – A square matrix where $A = -A^T$ and the diagonal elements are zero.
14. **Orthogonal Matrix** – A square matrix whose transpose is equal to its inverse, i.e., $A^T = A^{-1}$.

15. **Determinant** – A scalar value associated with a square matrix that helps determine its invertibility. Denoted by $|A|$.
16. **Minor of a Matrix** – The determinant of a square submatrix obtained by deleting one or more rows and/or columns from the original matrix.
17. **Cofactor** – The signed minor of an element in a matrix used in computing determinants and adjoints.
18. **Adjoint (Adjugate) Matrix** – The transpose of the cofactor matrix of a square matrix.
19. **Inverse of a Matrix** – For a non-singular square matrix A , the matrix $A^{-1}A = I$.
20. **Rank of a Matrix** – The maximum order of a non-zero minor in the matrix; indicates the dimension of the row or column space.
21. **Matrix Addition** – An operation where corresponding elements of matrices of the same order are added together.
22. **Matrix Multiplication** – An operation where the element in the i th row and j th column of the product matrix is obtained by multiplying the i th row of the first matrix with the j th column of the second.
23. **Scalar Multiplication** – Multiplying each element of a matrix by a scalar (real number).
24. **Cramer's Rule** – A method for solving a system of linear equations using determinants, valid when the determinant of the coefficient matrix is non-zero.
25. **Matrix Inverse Method** – Solving a system of linear equations using the inverse of the coefficient matrix: $X = A^{-1}B$.
26. **Gauss Elimination Method** – A method for solving systems of

equations by converting the matrix to row-echelon form using elementary row operations and solving through back-substitution.

27. **Consistent System** – A system of equations that has at least one solution (unique or infinite).
28. **Inconsistent System** – A system of equations that has no solution.
29. **Input-Output Analysis** – An economic model that uses matrices to describe the interdependencies between different sectors of an economy.
30. **IS-LM Model** – A macroeconomic model that shows equilibrium in the goods (IS) and money (LM) markets using matrix equations.
31. **Lagrangian Multiplier Method** – A method of constrained optimization used to maximize or minimize an objective function subject to equality constraints, often solved using matrix algebra.

3.18 Examination Oriented Questions

- Q1. The equilibrium condition for three related markets are given by the equations: $3p_1 - p_2 + p_3 = 2$, $-15p_1 + 6p_2 - 5p_3 = 5$ and $5p_1 - 2p_2 + 2p_3 = 1$. Find the equilibrium prices.
- Q2. Given the following National Income Model and Solve for Y, C and I using Cramer's Rule. $Y = C + I$; $I = d + ey$ ($d > 0$; $0 < e < 1$; $C = a + by$ ($a > 0$; $0 < b < 1$))
- Q3. What is the rank of a 5×5 matrix whose inverse i) exists and ii) does not exist?
- Q4. Solve the equations using Gauss Elimination method: $x - 2y + 3z = 6$, $3x + y - 4z = -7$, $5x - 3y + 2z = 5$. Also, solve the equations by Matrix inverse method. Do you get the same answer? Why or Why not? Explain.

Q5. Obtain the inverse of the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}$ and hence solve: $x + 3y = 4$, $2x - 2y = 9$.

Q6. Find A^{-1} and Verify that $A^{-1}A = AA^{-1} = I$ if $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

Q7. For what value of x rank of $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 4 & 2 & x \end{bmatrix}$ is one?

Q8. Given: the IS equation $0.4Y + 150i = 209$ and the LM equation $0.1Y - 250i + 35$. Find the equilibrium level of income and rate of interest.

3.19 Hints to Check your Progress (CYP)

CYP- I

- Q1. Refer Section 3.7.1
- Q2. Refer Section 3.7.2 and 3.7.4
- Q3. Refer Section 3.8.1
- Q4. Refer Section 3.8
- Q5. Refer Section 3.9
- Q6. Refer Section 3.7.4, 3.9 and 3.10.1

CYP – II

- Q1. Refer Section 3.13.1
- Q2. Refer Section 3.15 (1)
- Q3. Refer Section 3.14

3.20 Suggested Readings

- Aggarwal, C.S., & Joshi, R.C. (2015). Mathematics for Students of Economics. New Academic Publishing Co.

- Allen, R.G.D. (1971). Mathematical Analysis for Economists. Macmillan.
- Anthony, M., & Biggs, N. (1996). Mathematics for Economics and Finance: Methods and Modelling. Cambridge University Press.
- Black, J., & Bradley, J.F. (2005). Essential Mathematics for Economists. Wiley.
- Dowling, E.T. (2003). Schaum's Outline of Introduction to Mathematical Economics (3rd ed.). McGraw-Hill.
- Chiang, A.C., & Wainwright, K. (2005). Fundamental Methods of Mathematical Economics (4th ed.). McGraw-Hill.

M.A. Economics	Lesson No. 4
C.No. : Eco-103	Semester - 1st
	Unit I

QUADRATIC EQUATIONS: EIGEN ROOTS AND VECTORS

STRUCTURE

- 4.0 Objectives
- 4.1 Learning outcomes
- 4.2 Introduction
- 4.3 Quadratic Equation: Definition
- 4.4 Roots of a Quadratic Equation
- 4.5 Methods To Solve a Quadratic Equation
- 4.6 Simultaneous Quadratic Equations
- 4.7 Economic Applications of Quadratic Equations
- 4.8 Matrix Notation of a Quadratic Form
- 4.9 Types of Quadratic Forms
- 4.10 Working Rule to Find the Nature of the Quadratic Form $Q = X' AX$
- 4.11 Characteristic Equation, Eigen Roots and Eigen Vectors
- 4.12 Let us Sum Up
- 4.13 Keywords
- 4.14 Examination Oriented Questions
- 4.15 Hints to Check Your Progress

4.16. Suggested Readings

4.0 Objectives

The main objectives of this lesson are to enable you to:

- ☐ Define and identify quadratic equations and quadratic forms.
- ☐ Represent quadratic forms in matrix notation.
- ☐ Classify quadratic forms as positive definite, negative definite, semi-definite, or indefinite.
- ☐ Understand and derive characteristic equations of square matrices.
- ☐ Calculate eigenvalues (eigen roots) and corresponding eigenvectors of matrices.
- ☐ Apply these concepts to solve economic problems such as equilibrium analysis and marginal cost scenarios.

4.1 Learning outcomes

After going through this lesson, you will be able to:

- ☐ Express quadratic forms using matrix representations and determine their nature using principal minors.
- ☐ Derive and solve characteristic equations of given matrices.
- ☐ Find and interpret eigenvalues and eigenvectors in the context of linear transformations.
- ☐ Apply knowledge of quadratic forms and eigenvalues to economic examples such as cost functions, demand-supply equilibrium, and optimization problems.
- ☐ Distinguish between different types of quadratic forms based on the sign of the function $Q = X - AX$.

4.2 Introduction

Quadratic forms and eigen concepts are powerful mathematical tools with

extensive applications in economics, optimization theory, and matrix algebra. This lesson introduces students to quadratic forms, their matrix representation, and classification based on definiteness. It also explores the concepts of characteristic equations, eigenvalues (or eigen roots), and eigenvectors, which play a pivotal role in systems of linear equations, differential equations, and economic modeling. Understanding these topics is essential for analyzing equilibrium states in economic systems, solving constrained optimization problems, and evaluating system stability. This lesson provides a bridge between abstract algebraic constructs and real-world economic applications.

4.3 Quadratic Equation: Definition

An equation of the form $ax^2 + bx + c = 0$ ($a \neq 0$) where a , b , c are constants, x is unknown quantity, is called a quadratic equation.

Thus $3x^2 + 5x + 3 = 0$, $5x^2 + 7 = 0$, $x^2 + 4x + 1 = 0$ etc. are all quadratic equations.

Remark: The equation $ax^2 + bx + c = 0$ is called a standard quadratic equation. Here a is the coefficient of x^2 , b the coefficient of x and c the constant term.

4.4 Roots of a Quadratic Equation

The values of the unknown quantities which satisfy the given equation are called the roots of that equation.

For example, the equation $x^2 - 4x + 3 = 0$ is satisfied by $x = 1$ and $x = 3$ separately

$\therefore x = 1$ and 3 are the roots of the equation $x^2 - 4x + 3 = 0$.

4.5 Methods to Solve a Quadratic Equation

There are two methods of solving such equations:

- a) Method of Factors
- b) Method of Completing the Square

a) Method of Factors

This can be done with the help of the following steps:

- i) Write the equation in the standard form

$$ax^2 + bx + c = 0$$
- i) Factorise the expression on LHS.
- ii) Put each factor involving x equal to zero.
- iv) Solve each factor to get the roots of the given equation.

Example Solve the equation $5x^2 - 19x + 12 = 0$

Solution The given equation is

$$5x^2 - 19x + 12 = 0$$

$$\text{or } 5x^2 - 15x - 4x + 12 = 0$$

$$\text{or } 5x(x - 3) - 4(x - 3) = 0$$

$$\text{or } (5x - 4)(x - 3) = 0$$

$$\therefore \text{either } x - 3 = 0 \text{ or } 5x - 4 = 0$$

$$\therefore \text{either } x - 3 = 0 \text{ or } x = 4/5$$

$$\therefore x = 3, \frac{4}{5}$$

b) Method of Completing the Square

It is not always easy to factorise every quadratic equation. To avoid this difficulty, a method (known as completing the square) of solving the general quadratic equation has been derived. This method involves the following steps:

- i) Transpose the constant term to RHS.
- i) Divide both sides by the coefficient of x^2 .

ii) Add the square of half of the coefficient of x to both sides so as to make the LHS a perfect square.

iv) Take the square root of both sides and solve for x.

To solve the Quadratic Equation;

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

Proof: The given equation is $ax^2 + bx + c = 0 \quad (a \neq 0)$

Transposing c to the other side, we have

$$ax^2 + bx = -c$$

Dividing both sides by a, the coefficient of x^2

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Adding $\left(\frac{b}{2a}\right)^2$ to both sides to make L.H.S. a perfect square

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\therefore \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \quad \text{or} \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking the square root of both sides, we get

$$\left(x + \frac{b}{2a}\right) = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\text{or } x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Hence } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Thus, } x = \frac{-\text{coeff. of } x \pm \sqrt{(\text{coeff. of } x)^2 - 4(\text{coeff. of } x^2)(\text{constant term})}}{2 \cdot \text{coeff. of } x^2}$$

The roots of the equation $ax^2 + bx + c = 0$ are evidently

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example Solve $6x^2 - 17x + 12 = 0$, using the formula for quadratic equation.

Solution The given equation is

$$6x^2 - 17x + 12 = 0$$

Comparing it with $ax^2 + bx + c = 0$, we have

$$a = 6, b = -17, c = 12$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Putting the values of a, b and c, we have

$$x = \frac{-(-17) \pm \sqrt{(-17)^2 - 4(6)(12)}}{2 \times 6}$$

$$= \frac{17 \pm \sqrt{289 - 288}}{12} = \frac{17 \pm 1}{12} = \frac{3}{2}, \frac{4}{3}$$

$$\text{Hence } x = \frac{3}{2}, \frac{4}{3}$$

4.6 Simultaneous Quadratic Equations

Now we shall discuss simultaneous quadratic equations containing two variables. The different methods of solving such equations are illustrated below:

Example Solve the equation: $x^2 + y^2 = 185$, $x + y = 19$

Solution The given equations are:

$$x^2 + y^2 = 185 \quad \text{..... (i)}$$

$$x + y = 19 \quad \text{..... (ii)}$$

$$\text{From (ii), } y = 19 - x$$

Putting this value of y in (i), we get

$$x^2 + (19 - x)^2 = 185$$

$$\text{or } x^2 + 361 - 38x + x^2 = 185$$

$$\text{or } 2x^2 - 38x + 176 = 0$$

$$\text{Cancelling 2, } x^2 - 19x + 88 = 0$$

$$\text{or } (x - 11)(x - 8) = 0$$

$$\therefore x = 11, 8$$

$$\text{When } x = 11, y = 19 - 11 = 8$$

$$\text{When } x = 8, y = 19 - 8 = 11$$

$$\text{Hence } \left. \begin{matrix} x = 11 \\ y = 8 \end{matrix} \right\} , \left. \begin{matrix} x = 8 \\ y = 11 \end{matrix} \right\}$$

Example Solve the equations:

$$\frac{1}{x} + \frac{1}{y} = \frac{5}{6} \text{ and } 3x + 4y = 18$$

Solution

The given equations are:

$$\frac{1}{x} + \frac{1}{y} = \frac{5}{6} \quad \dots\dots\dots (i)$$

$$3x + 4y = 18 \quad \dots\dots\dots (ii)$$

$$\text{From (ii) } 4y = 18 - 3x \Rightarrow y = \frac{18 - 3x}{4}$$

Putting this value of y in (i), we get

$$\frac{1}{x} + \frac{4}{18 - 3x} = \frac{5}{6}$$

$$\frac{18 - 3x + 4x}{x(18 - 3x)} = \frac{5}{6}$$

$$\text{or } \frac{18 + x}{3x(6 - x)} = \frac{5}{6}$$

$$\text{or } 2(18 + x) = 5x(6 - x)$$

$$\text{or } 36 + 2x = 30x - 5x^2$$

$$\text{or } 5x^2 - 28x + 36 = 0$$

$$x = \frac{28 \pm \sqrt{784 - 720}}{10} = \frac{28 \pm 8}{10} = \frac{18}{5}, 2$$

$$\text{When } x = \frac{18}{5}, y = \frac{18 - 3x}{4} = \frac{18 - 3\left(\frac{18}{5}\right)}{4} = \frac{36 - 9}{20} = \frac{9}{5}$$

$$\text{When } x = 2, y = \frac{18 - 3(2)}{4} = \frac{12}{4} = 3$$

$$\text{Hence } \left. \begin{array}{l} x = \frac{18}{5} \\ y = \frac{1}{5} \end{array} \right\} \quad \left. \begin{array}{l} x = 2 \\ y = 3 \end{array} \right\}$$

4.7 Economic Applications of Quadratic Equations

Quadratic equations have the following applications in economics:

Example The demand and supply relations are:

$$p^2 + q^2 = 20 \quad \text{..... (i)}$$

$$2p + q = 8 \quad \text{..... (ii)}$$

To find the equilibrium price and quantity, we have to find positive value of p and q which satisfy with (i) and (ii)

$$\text{From (ii), } q = 8 - 2p$$

$$\text{Putting the value of q in (i), } p^2 + (8 - 2p)^2 = 20$$

$$\text{or } p^2 + 64 - 32p + 4p^2 = 20 \text{ or } 5p^2 - 32p + 44 = 0$$

$$\therefore p = \frac{32 \pm \sqrt{1024 - 880}}{10} = \frac{32 \pm 12}{10} = \frac{22}{5} \text{ or } 2$$

$$\text{when } p = \frac{22}{5}, q = 8 - 2 \left(\frac{22}{5} \right) = -\frac{4}{5}$$

Reject this negative value of q and corresponding value of p

$$\text{When } p = 2, q = 8 - 2(2) = 4$$

Hence equilibrium price $p = 2$, equilibrium quantity $q = 4$

Example The demand for goods of an industry is given by the equation $pq = 100$, where p stands for price and q for quantity demanded. Supply is given by the equation $20 + 3p = q$. What is the equilibrium price and quantity demanded?

Solution The demand and supply equations are given by

$$pq = 100 \quad \text{..... (i)}$$

$$20 + 3p = q \quad \text{..... (ii)}$$

Put value of q from (ii) in (i), we get

$$p(20 + 3p) = 100 \Rightarrow 3p^2 + 20p = 100$$

$$\text{or } 3p^2 + 20p - 100 = 0$$

Solving it using quadratic formula, we have

$$p = \frac{-20 \pm \sqrt{400 - 4 \times 3 \times (-100)}}{2 \times 3}$$

$$= \frac{-20 \pm 40}{6} = \frac{-20 + 40}{6}, \frac{-20 - 40}{6} = \frac{10}{3}, -10$$

$$\text{Reject the negative value of } p, \therefore p = \frac{10}{3}$$

$$\text{Put } p = \frac{10}{3} \text{ in (i) } \frac{10}{3} \times q = 100 \therefore q = 30$$

$$\therefore \text{Equilibrium price} = 10/3$$

$$\text{Equilibrium quantity} = 30$$

Example The marginal cost curve of a firm under perfect competition is given as

$$MC = q^2 - 8q - 1$$

If the market price is fixed at Rs. 19 per unit, find the equilibrium level of output.

Solution It is given that $MC = q^2 - 8q - 1$, Also $p = 19$

We know that under perfect competition a firm is in equilibrium when $MC = p$.

$$\therefore q^2 - 8q - 1 = 19 \text{ or } q^2 - 8q - 20 = 0$$

$$\therefore q = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(-20)}}{2(1)} = \frac{8 \pm \sqrt{64 + 80}}{8}$$

$$= \frac{8 \pm \sqrt{144}}{8} = \frac{8 \pm 12}{8} = 10, -2$$

Reject the negative value of q. $\therefore q = 10$

$\therefore$ Equilibrium level of output = 10

4.8 Matrix Notation of a Quadratic Form

Consider a quadratic form $Q = \sum_{i=1}^3 \sum_{j=1}^3 a_{ij} x_i x_j$ where $a_{ij} = a_{ji}$

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ so that } X' = (x_1, x_2, x_3)$$

It will be easily verified

$$Q = \sum_{i=1}^3 \sum_{j=1}^3 a_{ij} x_i x_j = X' A X$$

Thus, given a symmetric matrix A, we can write down quadratic form $X' A X$ as a product of matrices.

The symmetric matrix A is called the matrix of quadratic form.

For example, the expression $ax^2 + 2bxy + cy^2$ is a quadratic form.

In matrix notation, it can be written as $X' A X$ where

$$X' A X = ax^2 + 2bxy + cy^2 = [x, y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } X' = [x, y]$$

Note: Method to write the symmetric matrix of quadratic form

$$X'AX = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$$

In first row, write in order

$$\text{Coeff. of } x_1^2, \frac{1}{2} \text{ coeff. of } x_1x_2, \frac{1}{2} \text{ coeff. of } x_1x_3$$

In second row, write in order

$$\frac{1}{2} \text{ coeff. of } x_2x_1, \text{coeff. of } x_2^2, \frac{1}{2} \text{ coeff. of } x_2x_3$$

In third row, write in order

$$\frac{1}{2} \text{ coeff. of } x_3x_1, \frac{1}{2} \text{ coeff. of } x_3x_2, \text{coeff. of } x_3^2$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Example. Write the quadratic form corresponding to the symmetric matrix

$$\begin{bmatrix} 2 & 5 & 3 \\ 5 & 0 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

Solution. Here $A = \begin{bmatrix} 2 & 5 & 3 \\ 5 & 0 & -2 \\ 3 & -2 & 1 \end{bmatrix}$

$$\text{Let } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ so that } X' = [x_1, x_2, x_3]$$

The quadratic form corresponding to the matrix A is

$$X'AX = [x_1, x_2, x_3] \begin{bmatrix} 2 & 5 & 3 \\ 5 & 0 & -2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (\text{On matrix multiplication})$$

$$= [x_1, x_2, x_3] \begin{bmatrix} 2x_1 + 5x_2 + 3x_3 \\ 5x_1 + 0x_2 - 2x_3 \\ 3x_1 - 2x_2 + x_3 \end{bmatrix}$$

$$= x_1(2x_1 + 5x_2 + 3x_3) + x_2(5x_1 + 0x_2 - 2x_3) + x_3(3x_1 - 2x_2 + x_3)$$

$$= 2x_1^2 + 5x_1x_2 + 3x_1x_3 + 5x_2x_1 + 0x_2^2 - 2x_2x_3 + 3x_3x_1 - 2x_3x_2 + x_3^2$$

$$= 2x_1^2 + x_3^2 + 10x_1x_2 + 6x_1x_3 - 4x_2x_3$$

Example. Obtain the matrix corresponding to quadratic form:

$$X'AX = 2x_1^2 - x_2^2 + 6x_3^2 + 4x_1x_2 - 5x_2x_3 + 4x_3x_1$$

Solution. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ so that $X' = [x_1, x_2, x_3]$

$$X'AX = [x_1, x_2, x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + 2a_{12} x_1 x_2 + 2a_{23} x_2 x_3 + 2a_{31} x_3 x_1$$

Comparing (i) with given expression

$$a_{11} = 2, a_{22} = -1, a_{33} = 6, a_{12} \text{ (or } a_{21}) = 2, a_{23} \text{ (or } a_{32}) = -5/2, a_{31} \text{ (or } a_{13}) = 2$$

$$\therefore A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & -1 & -5/2 \\ 2 & -5/2 & 6 \end{bmatrix}$$

$$\text{Aliter: Given } X'AX = 2x_1^2 - x_2^2 + 6x_3^2 + 4x_1x_2 - 5x_2x_3 + 4x_3x_1$$

$$A = \begin{bmatrix} \text{coeff. of } x_1^2 & \frac{1}{2} \text{coeff. of } x_1x_2 & \frac{1}{2} \text{coeff. of } x_1x_3 \\ \frac{1}{2} \text{coeff. of } x_2x_1 & \text{coeff. of } x_2^2 & \frac{1}{2} \text{coeff. of } x_2x_3 \\ \frac{1}{2} \text{coeff. of } x_3x_1 & \frac{1}{2} \text{coeff. of } x_3x_2 & \text{coeff. of } x_3^2 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & -1 & -5/2 \\ 2 & -5/2 & 6 \end{bmatrix}$$

4.9 Types of Quadratic Forms

Quadratic forms are classified into the following categories:

A) Definite Quadratic Forms

- i) Positive definite (pd)
- i) Negative definite (nd)

B) Semi-Definite Quadratic Form

- ii) Positive Semi-Definite (psd)

iv) Negative Semi-Definite (nsd)

C) Indefinite Quadratic Forms

v) Indefinite (ind)

depending on the value of the function $\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$ for all vectors $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

Thus, the quadratic form $Q = X' A X$ is said to be

- i) pd if $Q > 0$ for all $x \neq 0$
- ii) nd if $Q < 0$ for all $x \neq 0$
- iii) psd if $Q = 0$ for at least one $x \neq 0$ and $Q > 0$ for remaining values of $x \neq 0$
- iv) nsd if $Q = 0$ for at least one $x \neq 0$ and $Q < 0$ for remaining values of $x \neq 0$
- v) ind, if $Q > 0$ for some values of x and $Q < 0$ for other values of x .

4.10 Working Rule to Find the Nature of the Quadratic Form $Q = X' A X$

Step 1: Find the associated symmetric matrix A of the given quadratic form $Q = X' A X$

Step 2: Find the principal minors m_1, m_2, m_3 of A

Step 3: A quadratic form Q is said to be:

- a) **Positive definite** if all principal minors m_1, m_2 and m_3 are positive i.e., $m_1 > 0, m_2 > 0$, and $m_3 > 0$.

- b) **Negative definite** if all principal minors m_1, m_2 and m_3 are of alternate sign i.e., $m_1 < 0, m_2 > 0$, and $m_3 < 0$.
- c) **Positive semi-definite** if $m_i \geq 0$ for $i = 1, 2, 3$ and $m_i = 0$ for at least one value of i i.e. $m_1 > 0, m_2 > 0$ and $m_3 = 0$.
- d) **Negative semi-definite** if $m_1 < 0, m_2 < 0$ and $m_3 = 0$.
- e) **Indefinite**: In all other cases the quadratic form is said to be indefinite.

Note: if $Q = X'AX$ is positive, negative or indefinite, then the associated matrix is positive, negative or indefinite.

Example. Find the nature of the Q.F. $Q = X'AX = 2x^2 - 4xy + 3xz + 6y^2 + 6yz + 8z^2$

Solution. The given quadratic form is

$$Q = X'AX = 2x^2 - 4xy + 3xz + 6y^2 + 6yz + 8z^2$$

The associated symmetric matrix is:

$$A = \begin{bmatrix} 2 & -2 & 3/2 \\ -2 & 6 & 3 \\ 3/2 & 3 & 8 \end{bmatrix}$$

We now consider the principal minors of all orders

$$m_1 = |2| = 2 > 0$$

$$m_2 = \begin{vmatrix} 2 & -2 \\ -2 & 6 \end{vmatrix} = (12 - 4) = 8 > 0$$

$$m_3 = \begin{vmatrix} 2 & -2 & 3/2 \\ -2 & 6 & 3 \\ 3/2 & 3 & 8 \end{vmatrix} = \frac{29}{2} > 0$$

Since $m_1 > 0, m_2 > 0$ and $m_3 > 0$, the matrix A is positive definite. Therefore, the quadratic form is positive definite.

Example. Identify the nature of the quadratic form:

$$Q = X'AX = -3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$$

Solution. The given quadratic form is

$$Q = -3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$$

The associated symmetric matrix is:

$$A = \begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

We now consider the principal minors of all orders

$$m_1 = |-3| = -3 < 0$$

$$m_2 = \begin{vmatrix} -3 & -1 \\ -1 & -3 \end{vmatrix} = 9 - 1 = 8 > 0$$

$$m_3 = \begin{vmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & 3 \end{vmatrix} = -16 < 0$$

Since $m_1 < 0$, $m_2 > 0$ and $m_3 < 0$, the matrix is negative definite.
Therefore the quadratic form is negative definite.

Check your Progress-I

Based on your understanding of the previous section, solve these questions:

Q1. What is a quadratic form? Explain its various types.

Q2. Solve the equations:

i) $x^2 + xy + y^2 = 19$, $x^2 - xy + y^2 = 7$

i) $2x - y = 3$, $xy = 2$

ii) $x + y = 19$, $x^2 + y^2 = 185$

Q3. Given the demand law $p = 85 - 4q - q^2$. Find the amount demanded (q) when $p = 40$. If the price rises to 64, how much will the demand contract?

Q4. The marginal cost curve of a firm under perfect competition is $MC = q^2 - 5q - 9$, where q represents units of output. If the given price is 5, find the equilibrium level of output.

Q5. Write the quadratic form corresponding to symmetric matrices:

i)
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{pmatrix}$$

i)
$$\begin{pmatrix} 0 & 5 & 3 \\ 5 & 5 & -2 \\ 3 & -2 & 0 \end{pmatrix}$$

ii)
$$\begin{pmatrix} 2 & 3 & 5 \\ 3 & 3 & -4 \\ 5 & -4 & 1 \end{pmatrix}$$

Q6. Write a symmetric matrix of the following quadratic forms:

i) $Q = x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 + 5x_2x_3 + 6x_3x_1$

ii) $Q = x_1^2 + 3x_2^2 - 2x_3^2 - 4x_1x_2 - 2x_2x_3 + 8x_3x_1$

iii) $Q = ax^2 + 2hxy + by^2$

iv) $Q = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$

Q7. Determine the nature of the following quadratic form:

i) $Q = 2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_1x_3 - 4x_2x_3$

i) $Q = 6x^2 + 3y^2 + 3z^2 - 4yz - 4zx - 2xy$

ii) $Q = x_1^2 + 4x_2^2 + x_3^2 - 2x_1x_2 + 4x_3x_1 - 4x_2x_3$

iv) $Q = 2x_1^2 + 3x_2^2 + 4x_3^2 + 2x_1x_2$

4.11 Characteristic Equation, Eigen Roots and Eigen Vectors

- 1) **Characteristic Equation:** If A is a square matrix of order 3, we can form the matrix $(A - \lambda I)$, where I is a unit matrix of order three.

The determinant of this matrix equated to zero i.e.

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{33} & a_{33} - \lambda \end{vmatrix} = 0$$

is called the characteristic equation of A. On expanding the determinant, the characteristic equation takes the form:

$$(-1)^3 \lambda^3 + k_1 \lambda^2 + k_2 \lambda + k_3 = 0$$

Where k is an expression in terms of the elements a_{ij} .

- 2) **Characteristic Roots (or Eigen Roots):** The roots of the characteristic equation $|A - \lambda I| = 0$ which are say $\lambda_1, \lambda_2, \lambda_3$ are called characteristic roots of matrix A. They are also referred to as Eigen Roots or Eigen values.

- 3) **Characteristic Vectors (or Eigen Vectors):** Corresponding to each characteristic root λ , we have a corresponding non-zero vector X which satisfies the equation $[A - \lambda I] X = 0$. These non-zero vectors X are called characteristic vectors or Eigen Vectors.

Example. Find the characteristic equation of the matrix:

$$A = \begin{bmatrix} -5 & 1 & 2 \\ 0 & -2 & 0 \\ 4 & 2 & 3 \end{bmatrix}$$

Solution. The characteristic equation of A is $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} -5-\lambda & 1 & 2 \\ 0 & -2-\lambda & 0 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$$

Expanding along the second row

$$|A - \lambda I| = (-2 - \lambda) [(-5 - \lambda)(3 - \lambda) - 8] = 0$$

$$\text{Or } (-2 - \lambda) (\lambda^2 + 2\lambda - 23) = 0$$

$$\text{Or } -\lambda^3 - 4\lambda^2 + 19\lambda + 46 = 0$$

Example. Find the eigen roots and eigen vectors for the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Solution. The characteristic equation of the matrix A is $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

On expanding, we get

$$\text{Or } (1 - \lambda)(3 - \lambda) - 8 = 0$$

$$\text{Or } \lambda^2 - 4\lambda - 5 = 0$$

Solving the quadratic equation

$$(\lambda - 5)(\lambda + 1) = 0$$

$$\therefore \lambda = 5, -1$$

Hence the eigen roots of A are $\lambda_1 = 5$ and $\lambda_2 = -1$

Now, in order to obtain the eigen vector associated with each root, we consider

$$(A - \lambda I) \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\lambda_1 = 5$ eigen vector is given by:

$$(A - \lambda I) X = 0 \text{ or } \begin{pmatrix} 1-5 & 4 \\ 2 & 3-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{Or } \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

By solving we get

$$-4x_1 + 4x_2 = 0 \text{ or } x_1 - x_2 = 0$$

$$2x_1 - 2x_2 = 0$$

Choosing $x_1 = k \therefore x_2 = k$

$$\therefore X = (k, k)^T$$

For $\lambda_2 = -1$, characteristic vector is given by

$$(A - \lambda I) X = 0 \Rightarrow \begin{pmatrix} 1+1 & 4 \\ 2 & 3+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{Or } \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Which gives

$$2x_1 + 4x_2 = 0$$

$$2x_1 + 4x_2 = 0 \text{ or } x_1 + 2x_2 = 0$$

Choosing $x_2 = k \therefore x_1 = -2k$

$$\therefore X = (-2k, k)'$$

For $\lambda_1 = 5$, eigen vector is given by:

$$(A - \lambda I) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \text{ or } \begin{pmatrix} 1-5 & 4 \\ 2 & 3-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{Or } \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

On solving, we get $-4x_1 + 4x_2 = 0$ and $2x_1 - 2x_2 = 0$

This reduces the equation $x_1 - x_2 = 0$

$$\therefore x_1 = x_2$$

Put this value into $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Let $x_2 = k$, then

$$\therefore x = \begin{pmatrix} k \\ k \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda = -1$, eigen vector is given by

$$\begin{pmatrix} 1+1 & 4 \\ 2 & 3+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

On solving we get

$$2x_1 + 4x_2 = 0 \text{ and } 2x_1 + 4x_2 = 0$$

This reduces the equation

$$2x_1 + 4x_2 = 0 \Rightarrow x_1 = -2x_2$$

Put this value into $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$x = \begin{pmatrix} -2x_2 \\ x_2 \end{pmatrix}$$

Let $x_2 = k$, then

$$\therefore x = \begin{pmatrix} -2k \\ k \end{pmatrix} = k \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Check your Progress-II

Based on your understanding of the previous section, solve these questions:

Q1. Find the characteristic equation of the matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Q2. Find the Eigen roots and Eigen vectors of the matrix:

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

4.12 Let us Sum Up

In this chapter, we studied the matrix representation of quadratic forms and explored various types such as positive definite, negative definite, and indefinite forms. We learned the rules to determine the nature of a quadratic form using leading principal minors. Additionally, the chapter covered the concept of characteristic equations along with methods to find the eigenvalues (eigen roots) and eigenvectors of a matrix.

4.13 Keywords

1. **Quadratic Form:** A homogeneous polynomial of degree two in one or more variables, typically expressed as $Q = \sum_{i,j} a_{ij} x_i x_j$ or in matrix form as $Q = X'AX$, where A is a symmetric matrix.
2. **Symmetric Matrix:** A square matrix A is symmetric if $a_{ij} = a_{ji}$ for all i, j . Quadratic forms are always associated with symmetric matrices.
3. **Matrix Notation of Quadratic Form:** Quadratic forms can be written in the compact form $Q = X'AX$, where X is a column vector of variables and A is a symmetric matrix.
4. **Principal Minors:** The determinants of the leading principal submatrices of matrix A . Used to determine the nature (definiteness) of a quadratic form.

5. **Positive Definite Form:** A quadratic form is positive definite if $Q > 0$ for all non-zero vectors X . This occurs when all principal minors are positive.
6. **Negative Definite Form:** A quadratic form is negative definite if $Q < 0$ for all non-zero vectors X , identified when principal minors alternate in sign starting with negative.
7. **Positive Semi-Definite Form:** The form is non-negative for all vectors, i.e., $Q \geq 0$, and zero for at least one non-zero vector.
8. **Negative Semi-Definite Form:** The form is non-positive for all vectors, i.e., $Q \leq 0$, and zero for at least one non-zero vector.
9. **Indefinite Quadratic Form:** A form which takes both positive and negative values for different inputs of X .
10. **Characteristic Equation:** An equation derived from $|A - \lambda I| = 0$, where λ is a scalar and I is the identity matrix. The solutions are the eigenvalues of A .
11. **Eigenvalues (Eigen Roots):** The scalar roots λ obtained from the characteristic equation. They represent the scaling factor by which the eigenvector is stretched.
12. **Eigenvectors (Characteristic Vectors):** Non-zero vectors X that satisfy $(A - \lambda I) X = 0$ for a given eigenvalue λ . Each eigenvalue has a corresponding eigenvector.
13. **Definiteness of a Matrix:** A matrix is said to be positive definite, negative definite, or indefinite based on the sign of its associated quadratic form $X'AX$.

4.14 Examination Oriented Questions

1. Define characteristic equation, Eigen roots and Eigen vectors.
2. Find the characteristic equation of the matrix:

$$\begin{bmatrix} 6 & 3 \\ 3 & -2 \end{bmatrix}$$

3. Find the Eigen roots and the corresponding Eigen vectors for matrix:

$$\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

4. Find the Eigen roots and Eigen vectors of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

5. Obtain the symmetric matrix of the quadratic form:
 $X A X = a x^2 + 2 h x y + b y^2$

6. Find the nature of the quadratic form:

$$Q = X A X = 2x^2 + 4y^2 + 4z^2 + 2xy - 2zx + 6yz$$

7. Find the nature of the quadratic form:

$$Q = X A X = 2x^2 + 4xy + y^2 + 3yz + 4z^2.$$

8. Solve the equations:

i) $\frac{\bar{X}}{y} - \frac{\bar{y}}{X} = \frac{10}{3}, X + y = 10$

i) $X + \frac{1}{y} = 3, y + \frac{2}{X} = 2$

ii) $X + y = 5, \frac{2}{X} + \frac{3}{y} = 2$

9. The total cost $C(x)$ of producing x units of a commodity is given by $C(x) = x^2 + 5x + 7$ and total revenue is given by $R(x) = 2x^2 + 11x$

- 3. Find the number of units produced and sold if total profit is Rs. 30.
(Hint: Profit = $R(x) - C(x) = 30$]

10. Find the equilibrium price and quantity for the following market: $D = 5 - p^2$, $S = 2p - 3$.

4.15 Hints to Check your Progress (CYP)

CYP- I

- Q1. Refer Section 4.3 and 4.9
Q2. Refer Section 4.7
Q3. Refer Examples of Section 4.7
Q4. Refer Examples of Section 4.7
Q5. Refer Section 4.8
Q6. Refer Section 4.8
Q7. Refer Section 4.10

CYP – II

- Q1. Refer Characteristic equation and based examples of Section 4.11
Q2. Refer Section 4.11

4.16 Suggested Readings

- Chiang, A. C., & Wainwright, K. (2005). Fundamental Methods of Mathematical Economics (4th ed.). McGraw-Hill Education.
- Simon, C. P., & Blume, L. (1994). Mathematics for Economists. W. W. Norton & Company.
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- Apostol, T. M. (1967). Calculus, Volume II: Multi-Variable Calculus and Linear Algebra with Applications to Differential Equations and Probability. Wiley.
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M.A. Economics	Lesson No. 5
C.No. : Eco-103	Semester - 1st
	Unit I

FUNCTIONS: LINEAR, NON – LINEAR, CONVEX AND CONCAVE

STRUCTURE

- 5.0 Objectives
- 5.1 Learning outcomes
- 5.2 Introduction
- 5.3 Concept of Function
- 5.4 Domain and Range of Function
- 5.5 Types of functions
- 5.6 Linear and Non Linear Functions
- 5.7 Value of a Function at a Point
- 5.8 Increasing and Decreasing Functions
- 5.9 Concavity and Convexity
- 5.10 Relative Extrema
- 5.11 Inflection Points
- 5.12 Let Us Sum Up
- 5.13 Keywords
- 5.14 Examination Oriented Questions

5.15 Hints to Check your Progress

5.16 Suggested Readings

5.0 Objectives

The main objectives of this lesson are to enable you to:

- ☐ Define and explain the concept of a function, including domain and range.
- ☐ Classify different types of functions: explicit, implicit, single-valued, multi-valued, polynomial, rational, exponential, and logarithmic.
- ☐ Distinguish between linear and non-linear functions using mathematical and graphical representations.
- ☐ Evaluate the value of a function at a given point.
- ☐ Analyze the behavior of functions through first and second derivatives to determine increasing/decreasing trends and concavity/convexity.
- ☐ Identify and classify relative extrema and points of inflection.
- ☐ Apply function concepts in solving business and economic problems.

5.1 Learning outcomes

After going through this lesson, you will be able to:

- ☐ Accurately define and identify different types of functions and their characteristics.
- ☐ Compute function values for given input variables and apply domain-range concepts.
- ☐ Graphically interpret and algebraically analyze functions for monotonicity and curvature.

- Evaluate critical points and distinguish between relative maxima and minima using the first and second derivative tests.
- Demonstrate understanding of function behavior through real-world examples in economics and business contexts.
- Solve conceptual and application-based problems involving linear and nonlinear functions.

5.2 Introduction

The concept of a function is one of the foundational pillars of mathematics, particularly relevant in the study of calculus and its applications in economics and business. Alongside functions, the ideas of limits and continuity form the bedrock for understanding derivatives, optimization, and modeling real-world phenomena. This lesson delves into the definitions, classifications, and applications of functions—linear and nonlinear—while exploring how these functions behave through increasing/decreasing trends, concavity, convexity, and critical points. Understanding these concepts equips students with the analytical tools necessary for interpreting mathematical relationships in economics and other disciplines.

5.3 Concept of Function

The term ‘function’ refers to the relationship between the variables. When change in one variable is observed due to change in the other variable, then it is said to have fundamental relationship. In other words, a variable y is said to be a function of another variable x when to every value of x , there corresponds a definite value of y . It is written as:

$$y = f(x)$$

Here, x is called independent variable and y is called dependent variable.

5.4 Domain and Range of Function

The set of values which an independent variable can assume is called the domain of the function and the set of corresponding values of y (i.e. dependent variable) is called the range of the function.

5.5 Types of functions

Functions are classified into the following categories:

- 1) **Explicit and Implicit Function:** A function $y = f(x)$ is said to be an explicit function of x if the value of y is directly given in terms of x . It is written as:

$$y = f(x)$$

But if the variables y and x are mixed in the functional relationship, then the function is said to be an implicit function. It is written as:

$$f(x, y) = 0$$

For example, $y = 4x^2 + 3$ is an explicit function whereas $x^2 + y^2 = a^2$ is an implicit function.

- 2) **Single-valued and Multi-valued Functions:** A function is said to be a single valued function of x if to every value of x , there corresponds one and only one value of y . For example, $y = x^2 + 1$, $y = \log x$ are all single valued functions of x . And a function $y = f(x)$ is said to be many valued function of x if a value of x gives rise to more than one value of y . For example, $y^2 = x$ is an example of many valued function.

- 3) **Even and Odd Functions:** A function $f(x)$ is said to be an even function of x if $f(-x) = f(x)$. For example, $f(x) = x^2$ is an even function. Similarly a function $f(x)$ is said to be an odd function of x if $f(-x) = -f(x)$. For example $f(x) = x^3$ is an odd function.

- 4) **Polynomial Functions:** A function of the form $y = f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, where n is a positive integer and $a_i \neq 0$, is called a polynomial function of degree n .

i) If $n = 0$, we have $y = a$, a constant function.

i) If $n = 1$, we have $y = a_0 + a_1x$, a linear function.

ii) If $n = 2$, we have $y = a_0 + a_1x + a_2x^2$, a quadratic or parabolic function.

- iv) If $n = 3$, we have $y = a_0 + a_1x + a_2x^2 + a_3x^3$, a cubic function and so on.
- v) If $a_0 = a_1 = \dots = a_{n-1} = 0$, we have $y = ax^n$ (take $a_n = a$), a power function.

- 5) **Rational Function:** A rational function is defined as the ratio of two polynomials. For example, $y = \frac{f(x)}{g(x)}$ is a rational function where $f(x)$ and $g(x)$ are polynomials in x and $g(x) \neq 0$.

For example $f(x) = \frac{x^2 + 2x + 1}{x + 5}$, $x \neq -5$ is a rational function.

- 6) **Composite Function:** If y is a function of u and u is a function of x , then y is said to be composite function of x . For example, if $y = f(u)$ and $u = g(x)$, then $y = f[g(x)]$ is a composite function of x .
- 7) **Exponential Function:** An exponential function is a function of the form $y = a^x$. When $a > 1$, the exponential function is increasing and when $0 < a < 1$, the exponential function is decreasing.
- 8) **Logarithmic Function:** Logarithmic functions are inverse to exponential functions. The function $f(x) = \log x$ is called logarithmic function.

5.6 Linear and Non Linear Functions

A. Linear Function

A linear function is a mathematical function that represents a constant rate of change in the dependent variable with respect to one or more independent variables. It is defined as a first-degree polynomial function, meaning that the variables in the function appear only to the first power, are not multiplied by each other, and are each multiplied by a constant coefficient. The function may also include a constant or intercept term.

In its general form, a linear function of n variables can be expressed as:

$$f(x_1, x_2, \dots, x_n) = a_0 + a_1x_1 + a_2x_2 + \dots + a_nx_n$$

Where:

- $x_1, x_2, \dots, x_n$ are independent (input) variables,
- a_0 is a constant term (intercept),
- $a_1, a_2, \dots, a_n$ are constant real-number coefficients.

Key Features of Linear Functions:

1. **Linearity in Parameters:** Each variable contributes linearly to the output. That is, the change in the dependent variable is proportional to the change in any of the independent variables, holding others constant.
2. **No Interactions or Higher Orders:** There are no squared terms (like x^2), cross-products (like xy), or non-algebraic expressions (like $\ln x$, e^x , etc.).
3. **Graphical Representation:** The graph of a linear function in two variables is a straight line; in three variables, it is a plane; and in higher dimensions, it is a hyperplane.
4. **Constant Marginal Effects:** The partial derivative of the function with respect to any variable is a constant, implying a constant marginal contribution of that variable.

Examples: Linear **demand functions:** $Q_d = a - bP$

Linear **cost functions:** $C(Q) = F + cQ$

B. Non Linear Function

Non-linear function is a function in which the relationship between the dependent and independent variables is not linear. Specifically, one or more of the following conditions hold:

- A variable is raised to a power other than one (e.g., squared, cubic),

- Variables appear in product or interaction terms (e.g., xy),
- The function includes non-polynomial elements (e.g., exponential, logarithmic, or trigonometric terms).

A general non-linear function may take a form such as:

$$f(x, y) = ax^2 + bxy + \ln y + e^x$$

which contains polynomial and non-polynomial expressions, illustrating its deviation from linearity.

Key Features of Non-Linear Functions:

1. **Variable Marginal Effects:** The rate of change of the dependent variable with respect to any independent variable depends on the value of that variable (i.e., partial derivatives are not constants).
2. **Complex Interactions:** Non-linear functions often involve variables that interact with each other, creating curved surfaces in their graphical representations.
3. **Graphical Behavior:** The graph of a non-linear function is not a straight line or flat surface but may be curved, exhibit inflection points, or show increasing or decreasing curvature.
4. **Non-Additivity:** The effect of one variable may depend on the level of another variable, which means the superposition principle does not hold.

Examples:

Cobb-Douglas Production Function: $Q = AL^\alpha K^\beta$

Log-Utility Function: $U = \ln x + \ln y$

Exponential Growth Functions: $y = Ae^{rt}$

Quadratic Cost Functions: $C(Q) = a + bQ + cQ^2$

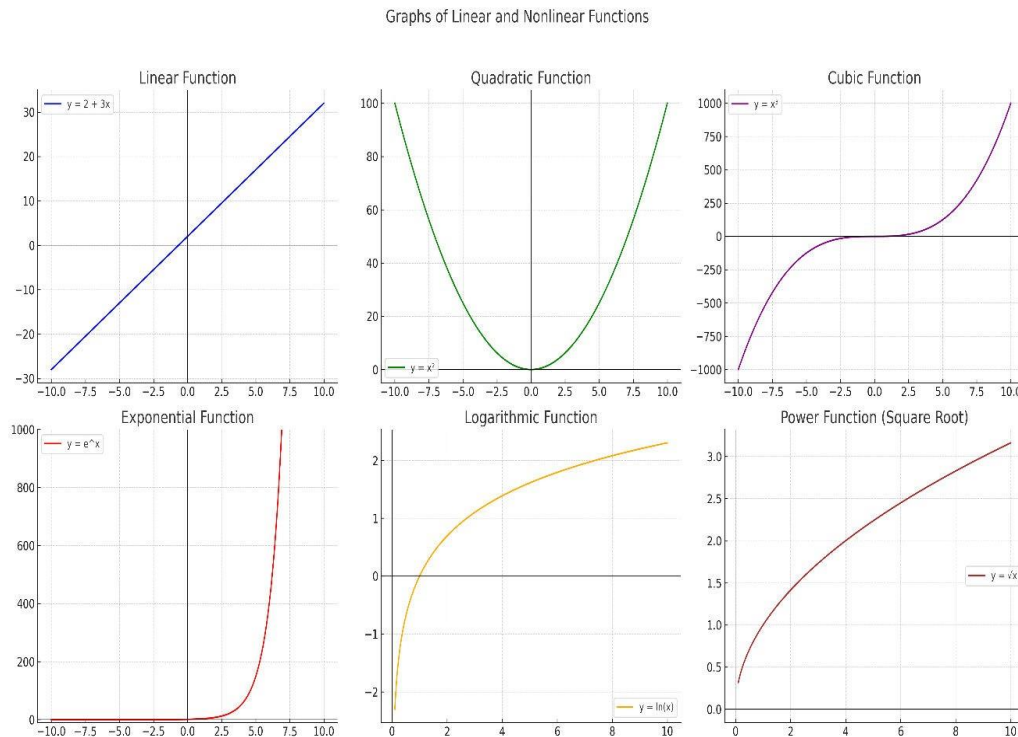


Figure 5.1: Graphical Representation of Linear and Non Linear Functions

5.7 Value of a Function at a Point

Let $f(x)$ be any function of x , then $f(a)$ is the value of the function at $x = a$. This value is obtained by putting $x = a$ in the given function $f(x)$. For example,

If $f(x) = x + 4$, then

$$f(0) = 0 + 4 = 4, f(1) = 1 + 4 = 5$$

Example If $f(x) = x^2 - 5x + 6$, find (i) $f(0)$, (ii) $f(a)$, (iii) $f(-2)$

Solution If $f(x) = x^2 - 5x + 6$

(i) Put $x = 0$ in the given function, we have

$$\begin{aligned} f(0) &= 0^2 - 5(0) + 6 \\ &= 6 \end{aligned}$$

(ii) Put $x = a$ in the given function, we have

$$f(a) = a^2 - 5a + 6$$

(ii) Put $x = -2$ in the given function, we have

$$f(-2) = (-2)^2 - 5(-2) + 6$$

$$= 4 + 10 + 6$$

$$= 20$$

Example

(a) For what value of x ,

$f(x) = x^2 + 3x - 2$ satisfy the equation

$$f(x) = f(2x)?$$

(b) If $f(x) = \log x$, show that

$$(i) f(ab) = f(a) + f(b) \text{ and } (ii) f\left(\frac{a}{b}\right) = f(a) - f(b)$$

(c) If $f(x) = \frac{1}{1-x}$, show that $f[f\{f(x)\}] = x$

Solution

(a) Given: $f(x) = x^2 + 3x - 2$

$$f(2x) = (2x)^2 + 3(2x) - 2$$

$$f(2x) = 4x^2 + 6x - 2$$

By the given condition

$$f(x) = f(2x)$$

$$\therefore x^2 + 3x - 2 = 4x^2 + 6x - 2$$

$$\text{Or } 3x^2 + 3x = 0$$

$$\therefore x = 0, -1$$

(b) (i) Given: $f(x) = \log x$

$$f(a) = \log a; f(b) = \log b$$

$$f(ab) = \log (ab)$$

$$= \log a + \log b$$

$$= f(a) + f(b)$$

$$\therefore f(ab) = f(a) + f(b)$$

$$\text{i) } f\left(\frac{a}{b}\right) = \log\left(\frac{a}{b}\right)$$

$$= \log a - \log b$$

$$= f(a) - f(b)$$

(c) Given $f(x) = \frac{1}{1-x}$
Put $x = f(x)$, we get

$$f[f(x)] = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{1-x-1} = \frac{1-x}{-x} = \frac{x-1}{x}$$

$$\text{Put } x = f[f\{f(x)\}] = \frac{1}{1-\frac{x-1}{x}}$$

$$\therefore f[f\{f(x)\}] = \frac{x}{x-x+1} = \frac{x}{1} = x$$

Example If $f(x) = \frac{x}{1+x}$ find $f[f\{f(2)\}]$

Solution Given $f(x) = \frac{x}{1+x}$, putting $x = 2$, we have

$$f(2) = \frac{2}{1+2} = \frac{2}{3}$$

$$[f\{f(f(2))\}] = \frac{\frac{2}{3}}{1+\frac{2}{3}} = \frac{\frac{2}{3}}{\frac{5}{3}} = \frac{2}{5}$$

$$\therefore [f\{f(f(2))\}] = \frac{\frac{2}{5}}{1+\frac{2}{5}} = \frac{\frac{2}{5}}{\frac{7}{5}} = \frac{2}{7}$$

Check your Progress-I

Based on your understanding of the previous section, solve these questions

1. (a) What is a function? Discuss the different types of functions.

- (b) Distinguish between:

- (i) Explicit and Implicit functions
- (ii) Single valued and Multi-valued functions
- (ii) Exponential and Logarithmic functions
- (iv) Composite function and Rational function

2. If $f(x) = \frac{5x^2 - 3x + 2}{x - 1}$, find $f(3)$, $f(-2)$ and $f(0)$.

3. (i) If $f(x) = \frac{3x}{x + 2}$, find $f[f(f(5))]$

(ii) If $f(x) = \frac{x}{x + 1}$, show that $f[f(f(5))] = \frac{5}{16}$

5.8 Increasing and Decreasing Functions

A function $f(x)$ is said to be increasing (decreasing) at $x = a$ if in the immediate vicinity of the point $[a, f(a)]$ the graph of the function rises (falls) as it moves from left to right. Since the first derivative measures the rate of change and slope of a function,

a positive first derivative at $x = a$ indicates the function is increasing at s ; a negative first derivative indicates it is decreasing. In short, as seen in the below figure 5.2:



Figure 5.2 – Increasing and Decreasing Functions

Thus,

$f'(a) > 0$: increasing function at $x = a$

$f'(a) < 0$: decreasing function at $x = a$

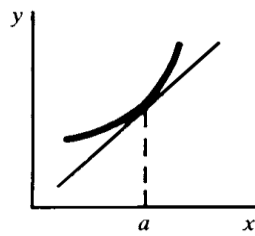
A function that increases (or decreases) over its entire domain is called a monotonic function. It is said to increase (decrease) monotonically.

5.9 Concavity and Convexity

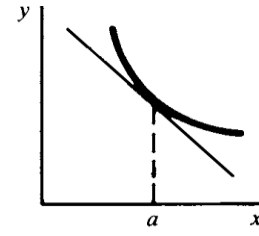
A function $f(x)$ is concave at $x = a$ if in some small region close to the point $[a, f(a)]$ the graph of the function lies completely below its tangent line. A function is convex at $x = a$ if in an area very close to $[a, f(a)]$ the graph of the function lies completely above its tangent line. A positive second derivative at $x = a$ denotes the function is convex at $x = a$; a negative second derivative at $x = a$ denotes the function is concave at a . The sign of the first derivative is irrelevant for concavity. In brief, as seen in Fig. 5.2

$f''(a) > 0$: $f(x)$ is convex at $x = a$

$f''(a) < 0$: $f(x)$ is concave at $x = a$

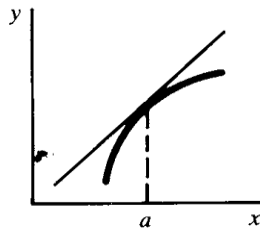


(a) $f'(a) > 0$
 $f''(a) > 0$

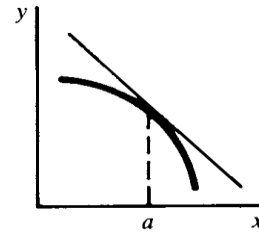


(b) $f'(a) < 0$
 $f''(a) > 0$

Convex at $x = a$



(c) $f'(a) > 0$
 $f''(a) < 0$



(d) $f'(a) < 0$
 $f''(a) < 0$

Concave at $x = a$

Figure 5.3 - Concave and Convex Function

Thus,

If $f''(x) > 0$ for all x in the domain, $f(x)$ is strictly convex.

If $f''(x) < 0$ for all x in the domain, $f(x)$ is strictly concave.

Example From the graphs in Fig. 5.4, indicate which graphs

- (1) are increasing for all x ,
- (2) are decreasing for all x ,
- (3) are convex for all x ,
- (4) are concave for all x ,

(5) have relative maxima or minima,

(6) have inflection points.

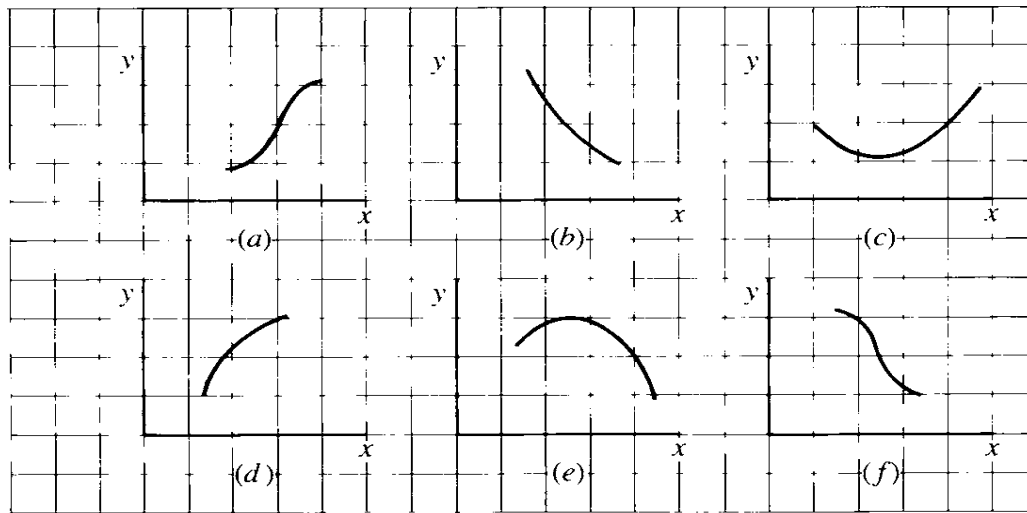


Figure 5.4

Solution

1. a, d: increasing for all x .
2. b, f: decreasing for all x .
3. b, c: convex for all x .
4. d, e: concave for all x .
5. c, e: exhibit a relative maximum or minimum.
6. a, f: have an inflection point

Example

Test to see whether the following functions are increasing, decreasing, or stationary at $x = 4$.

Solution

1. $y = 3x^2 - 14x + 5$
 $y' = 6x - 14$

$$y'(4) = 6(4) - 14 = 10 > 0, \text{ Function is increasing.}$$

$$2. \quad y = x^3 - 7x^2 + 6x - 2$$

$$y' = 3x^2 - 14x + 6$$

$$y'(4) = 3(4)^2 - 14(4) + 6 = -2 < 0, \text{ Function is decreasing.}$$

$$3. \quad y = x^4 - 6x^3 + 4x^2 - 13$$

$$y' = 4x^3 - 18x^2 + 8x$$

$$y'(4) = 4(4)^3 - 18(4)^2 + 8(4) = 0, \text{ Function is stationary.}$$

Example Test to see if the following functions are concave or convex at $x = 3$.

Solution

$$a) \quad y = -2x^3 + 4x^2 + 9x - 15$$

$$y' = -6x^2 + 8x + 9$$

$$y'' = 12x + 8$$

$$y''(3) = -12(3) + 8 = -28 < 0, \text{ concave}$$

$$b) \quad y = (5x^2 - 8)^2$$

$$y' = 2(5x^2 - 8)(10x) = 20x(5x^2 - 8) = 100x^3 - 160x$$

$$y'' = 300x^2 - 160$$

$$y''(3) = 300(3)^2 - 160 = 2540 > 0 \text{ convex}$$

5.10 Relative Extrema

A relative extremum is a point at which a function is at a relative maximum or minimum. To be at a relative maximum or minimum at a point a , the function must be at a relative plateau, i.e., neither increasing nor decreasing at a . If the function is neither increasing nor decreasing at a , the first derivative of the function at a must equal zero or be undefined. A point in the domain of a function where the derivative equals zero or is undefined is called a critical point or value.

To distinguish mathematically between a relative maximum and minimum, the second-derivative test is used. Assuming $f'(a) = 0$,

1. If $f''(a) > 0$, indicating that the function is convex and the graph of the function lies completely above its tangent line at $x = a$, the function is at a relative minimum at $x = a$.
2. If $f''(a) < 0$, denoting that the function is concave and the graph of the function lies completely below its tangent line at $x = a$, the function is at a relative maximum at $x = a$.
3. If $f''(a) = 0$, the test is inconclusive.

For functions which are differentiable at all values of x , called differentiable or smooth functions, one need only consider cases where $f'(x) = 0$ in looking for critical points. To summarize,

$f'(a) = 0$ $f''(a) > 0$: relative minimum at $x = a$ (Refer Figure 5.4)

$f'(a) = 0$ $f''(a) < 0$: relative maximum at $x = a$

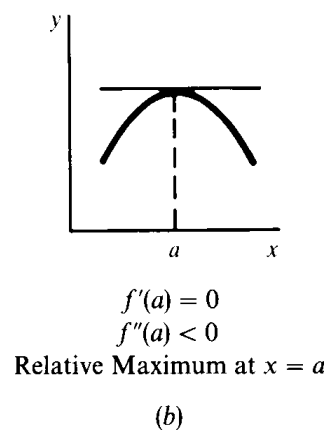
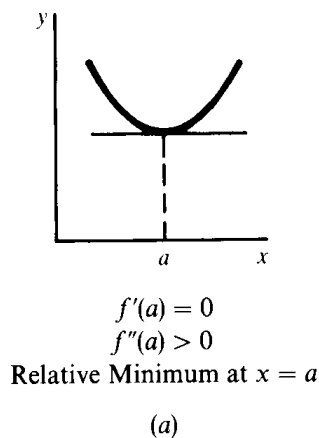


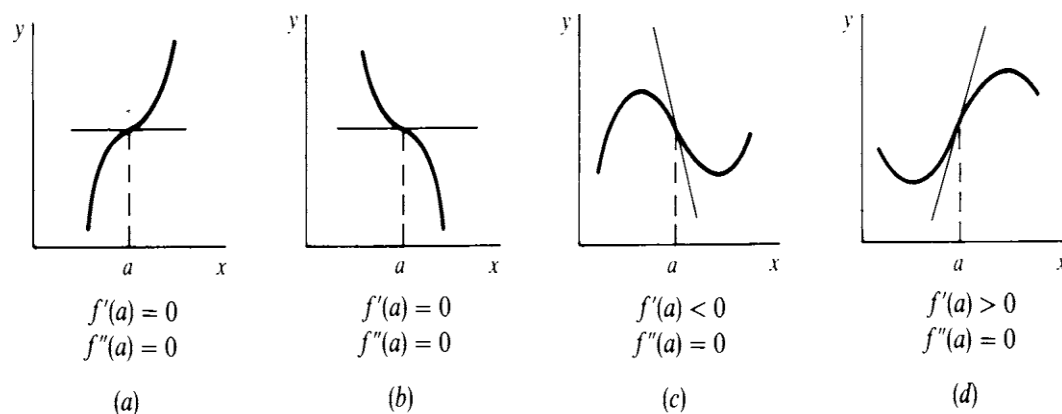
Figure 5.5- Relative Minimum and Maximum

5.11 Inflection Points

An inflection point is a point on the graph where the function crosses its tangent

line and changes from concave to convex or vice versa. Inflection points occur only where the second derivative equals zero or is undefined. The sign of the first derivative is immaterial. In sum, for an inflection point at a , as seen in Fig. 5.6

1. $f''(a) = 0$ or is undefined.
2. Concavity changes at $x = a$.
3. Graph crosses its tangent line at $x = a$.



Inflection Points at $x = a$

Figure 5.6- Inflection Points

Example Find the relative extrema for the following function by

- 1) Finding the critical value(s)
- 2) Determining if at the critical value(s) the function is at a relative maximum or minimum.

Solution $f(x) = -7x^2 + 126x - 23$

1. Take the first derivative, set it equal to zero, and solve for x to find the critical value(s).

$$f'(x) = -14x + 126 = 0$$

$$x = 9 \text{ critical value}$$

2. Take the second derivative, evaluate it at the critical value(s), and check for concavity to distinguish between a relative maximum and minimum.

$$f''(x) = -14$$

$$f''(9) = -14 < 0 \quad \text{concave, relative maximum}$$

Check your Progress-II

Based on your understanding of the previous section, solve these questions:

- Q1. Determine whether the function is increasing, decreasing, or stationary:

i) $y = 2x^2 - 5x + 1$ at $x = 2$

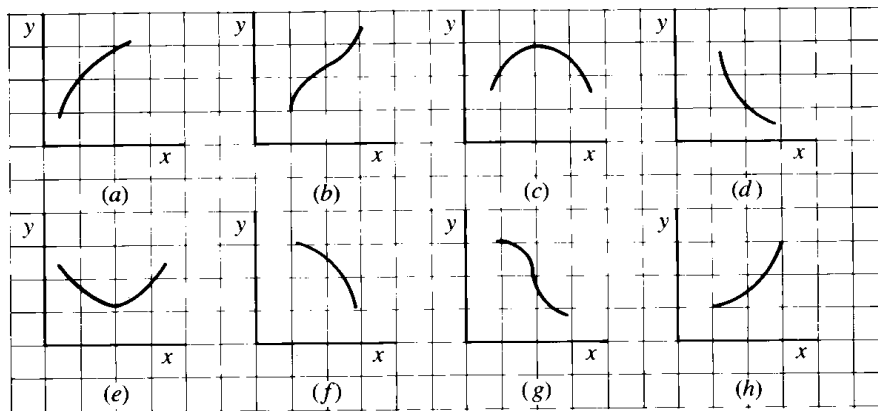
i) $y = x^3 - 3x^2 + 2x$ at $x = 1$

- Q2. Test to see if the following function is concave or convex at $x = 2$
 $y = 3x^3 - 9x^2 + 2x + 7$

- Q3. Indicate with respect to the graphs in following figure which functions have

- (1) Positive first derivatives for all x ,
- (2) Negative first derivatives for all x ,
- (3) Positive second derivatives for all x ,

- (4) Negative second derivatives for all x ,
- (5) First derivatives equal to zero or undefined at some point,
- (6) Second derivatives equal to zero or undefined at some point.



5.12 Let Us Sum Up

In this chapter, we studied various fundamental concepts related to functions. The discussion encompassed the types of functions, with particular emphasis on linear and nonlinear functions. We examined how to determine the value of a function at a given point, along with a detailed exploration of the domain and range of functions. Furthermore, we analyzed increasing and decreasing functions and discussed the necessary conditions for a function to be classified as concave or convex.

5.13 Keywords

1. **Function:** A relationship between two variables where each input (x) has exactly one output (y), written as $y = f(x)$.

2. **Independent Variable:** The variable (usually x) that can be freely chosen.
3. **Dependent Variable:** The variable (usually y) whose value depends on the value of x .
4. **Domain:** The set of all possible input values (x -values) for which the function is defined.
5. **Range:** The set of all output values (y -values) that a function can produce.
6. **Value of a Function at a Point:** The output value when a specific input is substituted into the function, $f(a)$.
7. **Increasing Function:** A function where $f'(x) > 0$; the graph rises from left to right.
8. **Decreasing Function:** A function where $f'(x) < 0$; the graph falls from left to right.
9. **Monotonic Function:** A function that is entirely increasing or decreasing over its domain.
10. **Concave Function:** A function where $f''(x) < 0$; the graph bends downward.
11. **Convex Function:** A function where $f''(x) > 0$; the graph bends upward.
12. **Critical Point:** A point where the first derivative is zero or undefined; potential maximum or minimum.
13. **Relative Maximum:** A point where the function reaches a peak locally; $f'(x) = 0$ and $f''(x) < 0$.
14. **Relative Minimum:** A point where the function reaches a trough locally; $f'(x) = 0$ and $f''(x) > 0$.
15. **Inflection Point:** A point where the graph changes concavity; $f''(x) = 0$ and concavity changes.

5.14 Examination Oriented Questions

Q1. Determine whether the function is increasing, decreasing, or stationary:

i) $y = x^4 - 4x^3 + 6x^2 - 3$ at $x = 3$

ii) $y = x^3 + x^2$ at $x = 0$

Q2. Test to see if the following function is concave or convex

i) $y = -x^3 + 6x^2 - 4x + 5$ at $x = 1$

ii) $y = 4x^4 - 5x^2 + 1$ at $x = 0$

iii) $y = (3x^2 + 1)^2$ at $x = 2$

Q3. Find the relative extrema for the following functions by (a) finding the critical value(s) and (b) determining if at the critical value(s) the function is at a relative maximum or minimum.

i) $y = 3x^3 - 36x^2 + 135x - 13$

ii) $y = 2x^4 - 16x^3 + 32x^2 + 5$

iii) $y = -(x - 8)^4$

iv) $y = (5 - x)^3$

v) $y = -2(x - 6)^6$

Q4. Distinguish between linear and nonlinear functions with the help of examples.

Q5. If $f(x) = 2^x$, show that $f(x+3) - f(x-1) = (15/2) f(x)$

Q6. If $f(x) = \log \frac{1+x}{1-x}$ show that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$

Q7. Find the value of $f(1/2)$ and $f(1)$ $f(x) = 3x^2 - 2x + 5 + \frac{2}{x}$ when

5.15 Hints to Check your Progress (CYP)

CYP- I

Q1. Refer Section 5.3 and Section 5.5

Q2. Refer Section 5.7

Q3. Refer Section 5.7

CYP – II

Q1. Refer Section 5.8

Q2. Refer Section 5.9

Q3. Refer Section 5.9

5.16 Suggested Readings

- Chiang, A.C., & Wainwright, K. (2005). Fundamental Methods of Mathematical Economics (4th ed.). McGraw-Hill Education.
- Sydsaeter, K., Hammond, P., Seierstad, A., & Strom, A. (2016). Further Mathematics for Economic Analysis (2nd ed.). Pearson Education.
- Simon, C. P., & Blume, L. (1994). Mathematics for Economists. W.W. Norton & Company.
- Schaum's Outline Series. Introduction to Mathematical Economics by Edward T. Dowling.
- Haeussler, E. F., Paul, R. S., & Wood, R. J. (2009). Introductory Mathematical Analysis for Business, Economics, and the Life and Social Sciences. Pearson Education.

M.A. Economics	Lesson No. 6
C.No. : Eco-103	Semester - 1st
	Unit I

LIMITS, CONTINUITY, AND SEQUENCES, LIMITS OF SEQUENCE, ECONOMIC EXAMPLES AND APPLICATIONS

STRUCTURE

- 6.0 Objectives
- 6.1 Learning outcomes
- 6.2 Introduction
- 6.3 Limit of a Function: Definition
- 6.4 Existence of Limit
- 6.5 Some Fundamental Theorems on Limits
- 6.6 Methods of Finding the Limit of a Function
- 6.7 Evaluation of Limit when $x \rightarrow \infty$
- 6.8 Some Important Limits (without proofs)
- 6.9 Continuity of Function: Definition
- 6.10 Discontinuity of Function
- 6.11 Let Us Sum Up
- 6.12 Keywords
- 6.13 Examination Oriented Questions
- 6.14 Hints to Check Your Progress

6.15 Suggested Readings

6.0 Objectives

The main objectives of this lesson are to enable you to:

- ☐ Define the limit of a function and explain right-hand and left-hand limits.
- ☐ Determine the existence of a limit using fundamental theorems.
- ☐ Apply various methods—such as factorization, substitution, and rationalization—to evaluate limits.
- ☐ Understand the concept of continuity at a point and over intervals.
- ☐ Identify and classify discontinuities in functions.
- ☐ Solve practical problems related to limits and continuity for deeper conceptual understanding.

6.1 Learning outcomes

After going through this lesson, you will be able to:

- ☐ Accurately compute the limit of algebraic functions using different techniques.
- ☐ Distinguish between right-hand, left-hand, and bilateral limits.
- ☐ Analyze and determine whether a function is continuous at a given point.
- ☐ Identify points of discontinuity and explain the nature of discontinuity.
- ☐ Utilize learned techniques to evaluate limits that involve infinity and indeterminate forms.

6.2 Introduction

Calculus forms the foundation of higher mathematics, and two of its most critical concepts are limit and continuity. These concepts help us understand and describe the behavior of functions as their input values approach specific points or extend toward

infinity. The idea of a limit is essential for defining instantaneous rates of change and is the basis for derivatives and integrals. On the other hand, continuity ensures that functions behave predictably without abrupt jumps, breaks, or gaps—an essential requirement in both theoretical and applied contexts.

This lesson introduces the formal definitions of limits and continuity, explores graphical and algebraic interpretations, and provides the necessary tools and techniques for evaluating limits and determining the continuity of a function at a point. A solid grasp of these concepts is indispensable for students planning to advance in mathematics, economics, engineering, computer science, or any field where dynamic systems are studied and modeled. The lesson also includes a variety of solved examples, exercises, and real-life contexts to enhance conceptual clarity and application skills.

6.3 Limit of a Function: Definition

If a function of x , say $f(x)$ approaches a fixed number ' l ' when x approaches ' a ' from either side of ' a ', then ' l ' is called the limit of $f(x)$ and is written as:

$$\lim_{x \rightarrow a} f(x) = l$$

Here ' x ' tends to ' a ' ($x \rightarrow a$) means x is sufficiently close to ' a ' but never equal to ' a '.

Example Consider the $\lim_{x \rightarrow a} f(x)$ where $f(x) = x + 2$

When $x \rightarrow 2^+$ i.e., x tends to 2 from the right side, the values assumed by x are given as:

$x \rightarrow 2^+$	2.1	2.01	2.001	2.0001	2.00001	
$f(x) = x + 2$	4.1	4.01	4.001	4.0001	4.00001	

From this table, we see that as $x \rightarrow 2^+$, $f(x) \rightarrow 4$. Here, 4 is called the right hand limit of $f(x)$ as $x \rightarrow 2^+$, and we write $\lim_{x \rightarrow 2^+} f(x) = 4$.

When $x \rightarrow 2^-$ i.e., x tends to 2 from the left side, the values assumed by x are given as:

$x \rightarrow 2^-$	1.9	1.99	1.999	1.9999	1.99999	
$f(x) = x + 2$	3.9	3.99	3.999	3.9999	3.99999	

From this table, we see that as $x \rightarrow 2^-$ (from the left), $f(x) \rightarrow 4$. Here 4 is called the left hand limit of $f(x)$ as $x \rightarrow 2^-$, and we write $\lim_{x \rightarrow 2^-} f(x) = 4$.

Thus, as $x \rightarrow 2$ from either side of 2, $f(x) \rightarrow 4$.

$$\text{Hence, } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow 2} (x + 2) = 4$$

Note: When $x \rightarrow 2$ and $x > 2$, we write, $x \rightarrow 2^+$ and when $x \rightarrow 2$ and $x < 2$ and we write $x \rightarrow 2^-$

Right Hand and Left Hand Limits

i) Right Hand Limit

If $f(x) \rightarrow l_1$ as $x \rightarrow a^+$, then l_1 is called the right hand limit of $f(x)$ and is written as:

$$\lim_{x \rightarrow a^+} f(x) = l_1$$

ii) Left Hand Limit

If $f(x) \rightarrow l_2$ as $x \rightarrow a^-$, then l_2 is called the left hand limit of $f(x)$ and is written as:

$$\lim_{x \rightarrow a^-} f(x) = l_2$$

6.4 Existence of Limit

$\lim_{x \rightarrow a} f(x)$ is said to exist if $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow a^-} f(x)$ both exist and are equal.

Symbolically, $\lim_{x \rightarrow a} f(x) = 1$

if and only if $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = l_1 = l_2 = 1$

$\lim_{x \rightarrow a} f(x)$ does not exist if either the right hand limit or left hand limit do not exist or they exist but are not equal.

6.5 Some Fundamental Theorems on Limits

Suppose there are two functions $f(x)$ and $g(x)$ having $\lim_{x \rightarrow a} f(x) = l_1$ and $\lim_{x \rightarrow a} g(x) = l_2$ where l_1 and l_2 are finite quantities, then

- i) The limit of sum (or difference) of two functions is equal to the sum (or difference) of the limits of the two functions separately, i.e.,

$$\lim_{x \rightarrow a} \{f(x) \pm g(x)\} = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = l_1 \pm l_2$$

- i) The limit of the product of two functions is equal to the product of limit of two functions separately, i.e.,

$$\lim_{x \rightarrow a} \{f(x) \cdot g(x)\} = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = l_1 \cdot l_2$$

- ii) The limit of the quotient of the two functions is equal to the quotient of the limits of the separate functions provided the limit of the divisor is not zero, i.e.,

$$\lim_{x \rightarrow a} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{\lim_{x \rightarrow a} f(x) l_1}{\lim_{x \rightarrow a} g(x) l_2} = \frac{l_1}{l_2} \quad \text{Provided } l_2 \neq 0$$

- iv) The limit of a constant is equal to the constant itself, i.e., if $f(x) = k$, a constant, then $\lim_{x \rightarrow a} f(x) = k$, where k is a constant.

- v) The limit of the n th power of any function is equal to the n th power of the limit of the function i.e. $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n = (l_1)^n$

6.6 Methods of Finding the Limit of a Function

Following are the main methods to find the limit of a function:

- 1) Method of Factors

2) Method of Substitution

3) Method of Rationalisation

Let us discuss them briefly

1) Method of Factors

Factorise the numerator and denominator of $f(x)$ and cancel the common factors and then put the value of x .

Example Evaluate: $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$

Solution

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x + 1)(x - 1)}{(x - 1)} \quad [\square \quad a^2 - b^2 = (a + b)(a - b)]$$

$$= \lim_{x \rightarrow 1} (x + 1)$$

$$= 1 + 1 = 2$$

Example Evaluate $\lim_{x \rightarrow 2} \frac{x^2 - 5x + 6}{x^2 - 4}$

Solution

$$\lim_{x \rightarrow 2} \frac{x^2 - 5x + 6}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{(x - 2)(x - 3)}{(x + 2)(x - 2)}$$

$$= \lim_{x \rightarrow 2} \frac{x - 3}{x + 2}$$

$$= \frac{2 - 3}{2 + 2} = -\frac{1}{4}$$

2) Method of Substitution

Put $x = a + h$, so that $h \rightarrow 0$ as $x \rightarrow a$

$$\lim_{x \rightarrow a} f(x) = \lim_{h \rightarrow 0} f(x + h)$$

Then simplify the numerator and denominator and put $h = 0$ to get the required limit.

Example Evaluate: $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$ by the method of substitution.

Solution Given $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$

Put $x = 2 + h$ so that as $x \rightarrow 2$, $h \rightarrow 0$

$$\therefore = \lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2} = \lim_{h \rightarrow 0} \frac{(2 + h)^3 - 8}{(2 + h) - 2}$$

Factorising [$\therefore (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$]

$$= \lim_{h \rightarrow 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h[h^2 + 12 + 6h]}{h}$$

$$= \lim_{h \rightarrow 0} (h^2 + 12 + 6h)$$

$$= 0 + 12 + 0 = 12$$

Use of Binomial Theorem

Binomial theorem can also be used to evaluate the limit of a function. Binomial theorem for any index is given by

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$\text{Similarly, } \left(1 + \frac{h}{a}\right)^n = 1 + n \frac{h}{a} + \frac{n(n-1)}{2!} \left(\frac{h}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{h}{a}\right)^3 + \dots$$

Example Evaluate: $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a}$

Solution Put $x = a + h$ so that as $x \rightarrow a$, $h \rightarrow 0$

$$\begin{aligned} \therefore \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \lim_{h \rightarrow 0} \frac{(a+h)^n - a^n}{a+h-a} \\ &= \lim_{h \rightarrow 0} \frac{\left[a^n \left(1 + \frac{h}{a} \right)^n \right] - a^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^n \left[\left(1 + \frac{h}{a} \right)^n - 1 \right]}{h} \end{aligned}$$

Using Binomial Theorem, we have

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{a^n \left[\left\{ 1 + n \frac{h}{a} + \frac{n(n-1)}{2!} \left(\frac{h}{a} \right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{h}{a} \right)^3 + \dots \right\} - 1 \right]}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^n \cdot \left[\frac{nh}{a} + \frac{n(n-1)}{2!} \left(\frac{h}{a} \right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{h}{a} \right)^3 + \dots \right]}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^n \cdot n \cdot \left[1 + \frac{n-1}{2!} \frac{h}{a} + \frac{n(n-2)}{3!} \left(\frac{h}{a} \right)^2 + \dots \right]}{h} \end{aligned}$$

Cancelling h and putting $h \rightarrow 0$, we have

$$= a^n \cdot \frac{n}{a} [1 + 0 + 0 + \dots]$$

$$= na^{n-1}$$

$$\therefore \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

3) Method of Rationalisation

When the given function $f(x)$ contains square roots, then limit of such type of function can be evaluated by rationalising the denominator or numerator.

Example Evaluate: $\lim_{x \rightarrow 0} \frac{\sqrt{9+2x} - 3}{x}$

Solution Given $\lim_{x \rightarrow 0} \frac{\sqrt{9+2x} - 3}{x}$

Rationalising the numerator, we get

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{9+2x} - 3}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{9+2x} - 3}{x} \times \frac{\sqrt{9+2x} + 3}{\sqrt{9+2x} + 3} \\ &= \lim_{x \rightarrow 0} \frac{9+2x-9}{x(\sqrt{9+2x}+3)} \quad [\square (a-b)(a+b) = a^2 - b^2] \\ &= \lim_{x \rightarrow 0} \frac{2x}{x(\sqrt{9+2x}+3)} = \lim_{x \rightarrow 0} \frac{2}{(\sqrt{9+2x}+3)} = \frac{2}{\sqrt{9+0}+3} \\ &= \frac{2}{3+3} = \frac{2}{6} = \frac{1}{3} \end{aligned}$$

Example Show that: $\lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a} = \frac{1}{2\sqrt{a}}$

Solution Given: $\lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$

Rationalising the numerator, we get

$$\begin{aligned}
 &= \lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a} = \lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \\
 &= \lim_{x \rightarrow a} \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} \quad [\square (a - b)(a + b) = a^2 - b^2] \\
 &= \lim_{x \rightarrow a} \frac{1}{\sqrt{x} + \sqrt{a}} \\
 &= \lim_{x \rightarrow a} \frac{1}{\sqrt{x} + \sqrt{a}} \\
 &= \frac{1}{\sqrt{a} + \sqrt{a}} = \frac{1}{2\sqrt{a}}
 \end{aligned}$$

6.7 Evaluation of Limit when $x \rightarrow \infty$

For finding the limit of the function $f(x)$ as $x \rightarrow \infty$, we divide both the numerator and denominator by the highest power of x which occurs either in denominator or

numerator and then use $\frac{1}{2}, \frac{1}{x^2}, \dots$ etc. $\rightarrow 0$ as $x \rightarrow \infty$

Example Evaluate $\lim_{x \rightarrow \infty} \frac{2x^2 + 5x + 3}{3x^2 + 10x + 8}$

Solution $\lim_{x \rightarrow \infty} \frac{2x^2 + 5x + 3}{3x^2 + 10x + 8}$

Since the highest power of x is x^2 , we divide the numerator and denominator by x^2 , we get

$$\begin{aligned}
 & \lim_{x \rightarrow \infty} \frac{2x^2 + 5x + 3}{3x^2 + 10x + 8} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{2x^2}{x^2} + \frac{5x}{x^2} + \frac{3}{x^2}}{\frac{3x^2}{x^2} + \frac{10x}{x^2} + \frac{8}{x^2}} = \lim_{x \rightarrow \infty} \frac{2 + \frac{5}{x} + \frac{3}{x^2}}{3 + \frac{10}{x} + \frac{8}{x^2}}
 \end{aligned}$$

$$\text{As } x \rightarrow \infty, \frac{5}{x}, \frac{3}{x^2}, \frac{10}{x}, \frac{8}{x^2} \rightarrow 0$$

$$= \frac{2+0+0}{3+0+0} = \frac{2}{3}$$

6.8 Some Important Limits (without proofs)

The following are some important limits without proof which are widely used:

$$\begin{aligned}
 \text{i)} \quad & \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e & \text{iv)} \quad & \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \\
 \text{ii)} \quad & \lim_{x \rightarrow 0} (1+x)^{1/x} = e & \text{v)} \quad & \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1 \\
 \text{iii)} \quad & \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log a
 \end{aligned}$$

Example Show that: $\lim_{x \rightarrow 0} (1+ax)^{1/x} = e^a$

Solution $\lim_{x \rightarrow 0} (1+ax)^{1/x} = \lim_{x \rightarrow 0} (1+ax)^{a/ax}$

$$\lim_{x \rightarrow 0} [1+ax]^{(1/ax)a} \quad \text{[Since as } x \rightarrow 0, ax \rightarrow 0]$$

$$= (e)^a = e^a$$

Example Show that $\lim_{x \rightarrow 1} \frac{\log x}{x-1} = 1$

Solution Let $x = 1 + h$, so that when $x \rightarrow 1$, $h \rightarrow 0$

$$\therefore \lim_{x \rightarrow 1} \frac{\log x}{x-1} = \lim_{h \rightarrow 0} \frac{\log(1+h)}{1+h-1} = \lim_{h \rightarrow 0} \frac{1}{h} \cdot \log(1+h)$$

$$= \lim_{h \rightarrow 0} \log(1+h)^{1/h} \quad [\text{Since } n \log m = \log m^n]$$

$$= \log e = 1$$

Example Evaluate $\lim_{x \rightarrow 0} \frac{a^x - b^x}{x}$

Solution
$$\lim_{x \rightarrow 0} \frac{a^x - b^x}{x} = \lim_{x \rightarrow 0} \frac{(a^x - 1) - (b^x - 1)}{x}$$

$$= \lim_{x \rightarrow 0} \left[\frac{a^x - 1}{x} - \frac{b^x - 1}{x} \right]$$

$$= \lim_{x \rightarrow 0} \frac{a^x - 1}{x} - \lim_{x \rightarrow 0} \frac{(b^x - 1)}{x}$$

$$= \log a - \log b = \log \frac{a}{b}$$

Check your Progress - I

Based on your understanding of the previous sections, solve these questions:

Q1. Define limit of a function.

Q2. Evaluate the following limits:

i) $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$

ii) $\lim_{x \rightarrow 1} \frac{2x^2 - 3x + 1}{x - 1}$

iii) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$

Q3. Evaluate: $\lim_{x \rightarrow \infty} \frac{7x^2 + x - 2}{3x^2 - 4x + 1}$

Q4. Evaluate: $\lim_{x \rightarrow a} [(x^5 - a^5)/(x - a)]$

6.9 Continuity of Function: Definition

A function $f(x)$ is said to be continuous at point $x = a$ if

i) $f(a)$ exists, i.e., $f(x)$ is defined at $x = a$ and must have finite value.

i) $\lim_{x \rightarrow a} f(x)$ exists

i.e., $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} f(x)$ limit exist and are equal.

ii) $\lim_{x \rightarrow a} f(x) = f(a)$, i.e., the value of the function = limit of function.

Graphically, a function is continuous at a point if the graph of the function has no gap at that point.

Example Show that $f(x) = \frac{x^2+1}{x-1}$ is continuous at $x = 2$

Solution Given: $f(x) = \frac{x^2+1}{x-1}$

i) Value of the Function

$$\text{At } x = 2, f(2) = \frac{2^2+1}{2-1} = \frac{4+1}{1} = 5$$

i) Right hand limit

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \left(\frac{x^2+1}{x-1} \right) \quad \text{Put } x = 2 + h \text{ so that as}$$

$$x \rightarrow 2, h \rightarrow 0$$

$$= \lim_{h \rightarrow 0} \left(\frac{(2+h)^2+1}{2+h-1} \right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{(4+4h+h^2)+1}{2+h-1} \right)$$

$$\begin{aligned}
 & \left[\frac{(5 + 4h + h^2)}{1 + h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{5 + 4h + h^2}{1 + h} \right] \\
 &= \frac{5 + 0 + 0}{1 + 0} = 5
 \end{aligned}$$

ii) Left hand limit

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \left(\frac{x^2 + 1}{x - 1} \right) \quad \text{Put } x = 2 - h \text{ so that as}$$

$$x \rightarrow 2, h \rightarrow 0$$

$$\begin{aligned}
 & \left[\frac{(2 - h)^2 + 1}{2 - h - 1} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{(2 - h)^2 + 1}{2 - h - 1} \right] \\
 & \left[\frac{(4 - 4h + h^2) + 1}{2 - h - 1} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{5 - 4h + h^2}{1 - h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{5 - 4h + h^2}{1 - h} \right] \\
 &= \frac{5 - 0 + 0}{1 - 0} = 5
 \end{aligned}$$

$$\text{As } \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) = f(2) = 5$$

Hence, the function $f(x)$ is continuous at $x = 2$

Example Examine the following function for continuity:

$$f(x) = \begin{cases} \frac{x^3 - 8}{x^2 - 4} & \text{When } x \neq 2 \\ 3 & \text{When } x = 2 \end{cases}$$

Solution Given that $f(2) = 3$

$$\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 4} = \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 4)}{(x-2)(x+2)}$$

$$= \lim_{h \rightarrow 2} \frac{x^2 + 2x + 4}{x + 2}$$

$$= \frac{4 + 4 + 4}{2 + 2} = \frac{12}{4} = 3$$

Since $f(2) = \lim_{x \rightarrow 2} f(x)$, $f(x)$ is continuous at $x = 2$

Example Examine the continuity of the function at $x = 2$

$$f(x) = \begin{cases} 1+x; & x \leq 2 \\ 5-x; & x > 2 \end{cases}$$

Solution At $x = 2$, $f(2) = 1 + 2 = 3$ [$\square f(x) = 1 + x$ when $x = 2$]

Right hand limit:

$$\begin{aligned} \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (5 - x) \text{ Put } x = 2 + h \text{ so that as } x \rightarrow 2, h \rightarrow 0 \\ &= \lim_{h \rightarrow 0} \{5 - (2 + h)\} \\ &= [5 - (2 + 0)] = 5 - 2 = 3 \end{aligned}$$

Left hand limit:

$$\begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} (1 + x) \text{ Put } x = 2 - h \text{ so that as } x \rightarrow 2, h \rightarrow 0 \\ &= \lim [1 + (2 - h)] \\ &= 1 + 2 - 0 = 3 \end{aligned}$$

Since $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$, so $f(x)$ is a continuous at $x = 2$.

Example Find the value of k so that the given $f(x)$ is continuous at $x = 3$.

$$f(x) = \begin{cases} 2x - 1; & x < 3 \\ k; & x = 3 \\ x + 2; & x > 3 \end{cases}$$

Solution Right hand limit

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (x + 2) \quad \text{Put } x = 3 + h \text{ so that as } x \rightarrow 3, h \rightarrow 0$$

$$= \lim_{h \rightarrow 0} (3 + h + 2)$$

$$= 3 + 0 + 2 = 5$$

Left hand limit

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (2x - 1) \quad \text{Put } x = 3 - h \text{ so that as } x \rightarrow 3, h \rightarrow 0$$

$$= \lim_{h \rightarrow 0} [2(3 - h) - 1]$$

$$= [2(3 - 0) - 1] = 6 - 1 = 5$$

$$\text{Also } f(3) = k$$

For continuity of function

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3) = k$$

$$\text{So } f(3) = k = 5$$

$\therefore k = 5$ at which the given function is continuous at $x = 3$

6.10 Discontinuity of Function

A function $f(x)$ that is not continuous at $x = a$ is said to be discontinuous at $x = a$. In other words, under discontinuity, the conditions of continuity are not fulfilled. It means:

- i) Either $f(x)$ is not defined at $x = a$.

i) or $\lim_{x \rightarrow a^-} f(x), \lim_{x \rightarrow a^+} f(x)$ either do not exist or if exist, then not equal.

If the two limits are equal, then limits are not equal to $f(a)$.

Example Show that the function $f(x) = \frac{x^2 + 9}{x - 3}$ is discontinuous at $x = 3$.

Solution

$$\begin{aligned} \text{LHL} &= \lim_{x \rightarrow 3^-} f(x) = \lim_{h \rightarrow 0} \frac{(3-h)^2 + 9}{3+h-3} \\ &= \lim_{h \rightarrow 0} \frac{9 + h^2 - 6h + 9}{h} = \lim_{h \rightarrow 0} \frac{18 + h^2 - 6h}{h} \\ &= \lim_{h \rightarrow 0} \frac{18}{h} + \lim_{h \rightarrow 0} \frac{h^2 - 6h}{h} \\ &= \lim_{h \rightarrow 0} \frac{18}{h} + \lim_{h \rightarrow 0} (h - 6) \\ &= \infty + 0 - 6 \\ &= \infty \end{aligned}$$

Hence LHL does not exist.

$$\begin{aligned} \text{RHL} &= \lim_{x \rightarrow 3^+} f(x) = \lim_{h \rightarrow 0} \frac{x^2 + 9}{x - 3} \\ &= \lim_{h \rightarrow 0} \frac{(3+h)^2 + 9}{3+h-3} = \lim_{h \rightarrow 0} \frac{9 + h^2 + 6h + 9}{h} \\ &= \lim_{h \rightarrow 0} \frac{18 + h^2 + 6h}{h} \\ &= \lim_{h \rightarrow 0} \frac{18}{h} + \lim_{h \rightarrow 0} 6h \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{18}{h} + \lim_{h \rightarrow 0} \frac{(h+6)}{h} \\
 &= \infty + 0 + 6 \\
 &= \infty
 \end{aligned}$$

So, RHL does not exist.

Hence, the function is discontinuous at $x = 3$.

Example Show that $f(x) = \begin{cases} x^2 & \text{when } x \neq 1 \\ 2 & \text{when } x = 1 \end{cases}$ is discontinuous at $x = 1$.

$$f(1) = 2 \quad [\square \quad f(x) = 2, \text{ when } x = 1]$$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} x^2 \quad \text{Put } x = 1 + h$$

$$= \lim_{h \rightarrow 0} (1 + h)^2$$

$$= (1 + 0)^2 = 1$$

$$\text{Since } f(1) \neq \lim_{x \rightarrow 1} f(x)$$

$\therefore$ The function is discontinuous at $x = 1$.

Check your Progress - II

Based on your understanding of the previous sections, solve these questions:

Q1. Explain the discontinuity of a function.

Q2. Show that the $f(x) = \frac{x^2 + 4}{x - 12}$ is continuous at $x = -2$.

Q3. Show that the function $f(x) = \frac{1}{x - 2}$ is continuous for values of x from $x = -2$ to -1 .

Q4. Show that the following function is $f(x) = \frac{x^2 - 16}{x - 4}$, $x \neq 4$; $f(4) = 8$ continuous at $x = 4$.

6.11 Let Us Sum Up

In this chapter, we explored the fundamental concepts of limit and continuity of a function, which are essential in understanding the behavior of functions at specific points and over intervals. We examined how a function approaches a particular value

as the input approaches a certain point and discussed the conditions under which a function is considered continuous. Furthermore, we analyzed various types of discontinuities, illustrating situations where a function fails to be continuous

6.12 Keywords

$$1. \quad \lim_{x \rightarrow a} \{f(x) \pm g(x)\} = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = l_1 + l_2$$

$$2. \quad \lim_{x \rightarrow a} \{f(x) \cdot g(x)\} = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = l_1 + l_2$$

$$3. \quad \lim_{x \rightarrow a} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{l_1}{l_2} \text{ provided } l_2 \neq 0$$

4. The limit of a constant is equal to the constant itself, i.e., if $f(x) = k$, a constant, then $\lim_{x \rightarrow a} f(x) = k$, where k is a constant.

5. The limit of the n th power of any function is equal to the n th power of the limit of the function i.e. $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n = (l_1)^n$

$$6. \quad \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e$$

$$7. \quad \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$8. \quad \lim_{x \rightarrow 0} (1 + x)^{1/x} = e$$

$$9. \quad \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$10. \quad \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log a$$

11. A function $f(x)$ is said to be continuous at point $x = a$ if
- ☐ $f(a)$ exists, i.e., $f(x)$ is defined at $x = a$ and must have finite value.
 - $\lim_{x \rightarrow a} f(x)$ exists i.e., $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ limit exist and are equal.
 - ☐ $\lim_{x \rightarrow a} f(x) = f(a)$, i.e., the value of the function = limit of function.
12. A function $f(x)$ that is not continuous at $x = a$ is said to be discontinuous at $x = a$ if
- ☐ Either $f(x)$ is not defined at $x = a$. or
 - ☐ $\lim_{x \rightarrow a^-} f(x)$, $\lim_{x \rightarrow a^+} f(x)$ either do not exist or if exist, then not equal.

6.13 Examination Oriented Questions:

Q1. Evaluate the following:

i)
$$\lim_{x \rightarrow 1} \frac{x^3 - 3x + 2}{x^3 + x^2 - 5x + 3}$$

i)
$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$$

ii)
$$\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x}$$

Q2. Evaluate:
$$\lim_{x \rightarrow \infty} \frac{(x+1)(2x+1)}{(x+2)(3x+4)}$$

Q3. Show that:
$$\lim_{x \rightarrow a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$$

Q4. Test the continuity of the function at $x = 2$

$$f(x) = \frac{2x^2 - x - 6}{4x^2 - 7x - 2}, x \leq -2; x^2, x > 0$$

- Q5. Let $f(x) = 2p + 3$ if $x \leq 1$ and $f(x) = 1 - px^2$ if $x > 1$. For what value of p does $\lim_{x \rightarrow 1} f(x)$ exist?

6.14 Hints to Check your Progress (CYP)

CYP- I

- Q1. Refer Section 6.3
 Q2. Refer Part (1), (2) and (3) of Section 6.6
 Q3. Refer Section 6.7
 Q4. Refer the use of Binomial Theorem mentioned in Section 6.6

CYP – II

- Q1. Refer Section 6.10
 Q2. Refer Section 6.9
 Q3. Refer Section 6.9
 Q4. Refer Section 6.9

6.15 Suggested Readings

- Chiang, A.C. and Wainwright, K. (2005), Fundamental Methods of Mathematical Economics, 4th Edition, McGraw-Hill Education.
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M.A. Economics	Lesson No. 7
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	Unit I

PRINCIPLES OF DIFFERENTIATION EXPANSION BY TAYLOR SERIES AND ECONOMIC APPLICATIONS

STRUCTURE

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7.0 Objectives

The main objectives of this lesson are to enable you to:

- ☐ To understand the meaning and concept of derivatives in calculus.
- ☐ To learn and apply various rules of differentiation.
- ☐ To differentiate algebraic, logarithmic, exponential, and implicit functions.
- ☐ To understand economic applications such as elasticity, revenue, and cost functions.
- ☐ To compute marginal concepts like marginal revenue and marginal cost using differentiation.

7.1 Learning outcomes

After going through this lesson, you will be able to:

- ☐ Students will be able to define and explain the concept of a derivative.
- ☐ Students will perform differentiation using both the first principle and standard rules.
- ☐ Students will apply differentiation techniques to economic problems.
- ☐ Students will compute and interpret elasticity of demand, supply, and income.

- Students will derive average and marginal cost/revenue functions from total functions

7.2 Introduction

Differential calculus is a fundamental branch of mathematics that focuses on analyzing how functions change in response to variations in their variables. In economics, this mathematical tool becomes especially valuable for understanding the behavior of key economic indicators such as cost, revenue, demand, and supply.

This lesson introduces the concept of simple differentiation, beginning with the basic definition of a derivative and extending to standard rules and techniques of differentiation. It emphasizes not only the computational aspects but also the interpretation of derivatives in real-world economic contexts. By linking mathematics with economics, this lesson equips learners with the analytical skills needed to evaluate marginal changes, optimize economic functions, and derive relationships among variables.

Ultimately, this foundational knowledge in differentiation helps in making informed economic decisions by studying how small changes in inputs or external factors affect outputs such as profit, utility, and production efficiency.

7.3 Introduction to Differential Calculus

Differential calculus is a branch of mathematics which is concerned with determining the rate of change of a given function due to a unit change in the independent variables. In other words, it is the study of the changes that occurs in one variable when other variables on which it depends changes. Before taking up the detailed study of differential calculus, it is appropriate to give the definition of certain terms which are essential for the study of it.

1) Increment

A small change in the value of variable x is called increment in x and is denoted by Δx or δx (read as delta x). Thus, Δx denotes increment or small change in the value of x . Similarly, Δy denotes increment or small change in the value of y .

2) Average Rate of Change

The average rate of change of a function f over an interval x to $x + \Delta x$ is defined by ratio $\Delta y / \Delta x$.

Let us consider the function $y = f(x)$... (1)

Let Δx be an increment in x and Δy the corresponding increment in y .

$$\therefore y + \Delta y = f(x + \Delta x)$$

By subtracting, we get

$$\Delta y = f(x + \Delta x) - f(x)$$

Dividing both sides by Δx , we get

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$\frac{\Delta y}{\Delta x}$ is called the average rate of change of y w.r.t. x . It is also called incremental ratio.

3) Instantaneous Rate of Change

The limiting value of average rate of change of function $f(x)$ as $\Delta x \rightarrow 0$ known as instantaneous rate of change of function $f(x)$ w.r.t. x . It is expressed in the following equation:

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

7.4 Derivative of a Function: Definition

Let $y = f(x)$ be a single valued function x .

Let Δx be the increment in the value of x and Δy the corresponding increment in the value of y . Then.

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If it exists, is called the derivative or differential coefficient of y w.r.t. x and is

denoted by $\frac{dy}{dx}$ or $\frac{d}{dx}[f(x)]$ or $f'(x)$.

$$\text{Thus, } \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

7.4.1 Interpretation of Derivative

The derivative of a function denotes the instantaneous rate of change of a function. It shows how the dependent variable changes for each unit change in the independent variable.

75 Differentiation and its Techniques

The process of finding the derivative of a function is called differentiation. When a function has a derivative, it is said to be differentiable.

The derivative of a function can be obtained in two ways:

- (1) By using the first principle or abinitio or from definition
- (2) By using some standard rules of differentiation.

76 Method for finding the derivative from first principle or abinitio or from definition

When derivatives are obtained without making use of the standard theorems/ rules of differentiation, the technique of doing it is called differentiation from definition or from first principal or from abinitio. It involves the following five steps:

Step I: Put the given function equal to y i.e.,

$$y = f(x) \quad \dots (i)$$

Step II: Let Δx be the increment in x and Δy the corresponding increment in y

$$\therefore y + \Delta y = f(x + \Delta x) \quad \dots (ii)$$

Step III: Subtract (i) and (ii), we get

$$\begin{aligned} (y + \Delta y) - y &= f(x + \Delta x) - f(x) \\ \text{or } \Delta y &= f(x + \Delta x) - f(x) \quad \dots (iii) \end{aligned}$$

Step IV: Dividing both sides of (iii) by Δx , we get

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Step V: Proceeding to the limit $\Delta x \rightarrow 0$ we get

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Example Find the derivatives of $3x^2 - 4x + 5$ w.r.t. x from definition.

Solution: I. $y = 3x^2 - 4x + 5 \quad \dots (i)$

II. Let Δx be an increment in x and Δy the corresponding increment in y .

$$\therefore y + \Delta y = 3(x + \Delta x)^2 - 4(x + \Delta x) + 5 \quad \dots (ii)$$

III. Subtracting (i) from (ii), we get

$$(y + \Delta y) - y = 3(x + \Delta x)^2 - 4(x + \Delta x) + 5 - 3x^2 + 4x - 5$$

$$\text{or } \Delta y = 6x \cdot \Delta x + 3(\Delta x)^2 - 4\Delta x$$

IV. Dividing both sides by Δx we get

$$\frac{\Delta y}{\Delta x} = \frac{6x \cdot \Delta x + (\Delta x)^2 - 4 \cdot \Delta x}{\Delta x}$$

$$\text{or } \frac{\Delta y}{\Delta x} = 6x + 3 \cdot \Delta x - 4$$

V. Proceeding to the limit as $\Delta x \rightarrow 0$, we get

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} (6x + 3\Delta x - 4) = 6x - 4$$

Hence, $\frac{dy}{dx} = 6x - 4$

7.7 Standard Rules of Differentiation (Without Proofs)

The procedure for obtaining derivatives from the first principle or from definition or abinitio is very laborious and cumbersome especially for more complicated functions. In practice, derivatives are obtained using standard rules or formulae. Here, we give some rules of differentiation without proofs.

Rule 1: Derivative of a Power Function-Power Function Rule

A power function is a function composed of a variable base and a constant exponent (or power) such as x^n , x^5 , x^2 etc.

Let $y = x^n$, then

$$\frac{dy}{dx} = \frac{d}{dx} (x^n) = nx^{n-1} \text{ for any positive integer.}$$

That is, the derivative of a power function x^n equals the product of the exponent n and the variables x raised to the power one less exponent, i.e., $n - 1$.

Rule: $\frac{d}{dx} [x^{\text{Power}}] = [\text{Power}] \cdot (x)^{\text{Power}-1}$

Example Find $\frac{dy}{dx}$ of the following functions:

i) $y = x^2$

ii) $y = x^{1/4}$

Solution: i) $y = x^2$, then $\frac{dy}{dx} = 2x^{2-1} = 2x$

$$\text{i) } y = x^{1/4}, \text{ then } \frac{dy}{dx} = \frac{1}{4}x^{1/4-1} = \frac{1}{4}x^{-3/4}$$

Rule 2: Derivative of Extended Form of Power Function

In the extended form of the power function, the base of the function is the first degree polynomial in x such as $(ax + b)^n$.

Let $y = (ax + b)^n$, then

$$\frac{dy}{dx} = \frac{d}{dx} (ax + b)^n = n (ax + b)^{n-1} \cdot a$$

Rule: $\frac{d}{dx} (ax + b)^{\text{Power}} = \text{Power} \times (ax + b)^{\text{Power}-1} \times \text{coefficient of } x.$

Example Find $\frac{dy}{dx}$ of the following functions:

$$\text{i) } y = (2x - 5)^3 \qquad \text{ii) } u = \left(\frac{3}{2}x - 3 \right)^{1/2}$$

Solution: i) $y = (2x - 5)^3, \frac{dy}{dx} = 3 (2x - 5)^{3-1} \cdot (2) = 6 (2x - 5)^2$

$$\text{ii) } y = \left(\frac{3}{2}x - 3 \right)^{1/2}, \frac{dy}{dx} = \frac{1}{2} \left(\frac{3}{2}x - 3 \right)^{\frac{1}{2}-1} \cdot \left(\frac{3}{2} \right) = \frac{3}{4} \left(\frac{3}{2}x - 3 \right)^{-1/2}$$

Rule 3: Derivatives of a Constant-Constant Rule

A constant does not change as x is changed and its derivative is always zero.

Let $y = c$ where c is a constant, then

$$\frac{dy}{dx} = \frac{d(c)}{dx} = 0$$

Example Find the derivative of:

(i) $y = 3$

(ii) $y = 6$

(ii) $y = a$

Solution: (i) $y = 3, \frac{dy}{dx} = 0$

(ii) $y = 6, \frac{dy}{dx} = 0$

(ii) $y = a, \frac{dy}{dx} = 0$

Rule 4: Derivative of a Multiplicative Constant-Multiplicative Constant

Rule

Differentiation does not affect a multiplicative constant.

Let $y = kx^n$, where the coefficient k and exponent n are constant, then

$$\frac{dy}{dx} = \frac{d}{dx}(kx^n) = k \cdot \frac{d}{dx}(x^n) = k n x^{n-1}$$

That is, the derivative of a constant multiplied by a function is the product of the constant and the derivative of the function.

Example Find $\frac{dy}{dx}$ of the following:

(i) $y = 10x^4$ (ii) $y = 10x$ (iii) $y = 3x^2$ (iv) $y = \frac{3}{7x^2}$

Solution: (i) $y = 10x^4$, then $\frac{dy}{dx} = 10 \cdot 4x^{4-1} = 40x^3$

(ii) $y = 10x$, then $\frac{dy}{dx} = 10 \cdot 1x^{1-1} = 10x^0 = 10$

$$(ii) \quad y = 3x^2, \text{ then } \frac{dy}{dx} = 3 \cdot 2x^{2-1} = 6x^1 = 6x$$

$$(iv) \quad y = \frac{3}{7x^2} = \frac{3}{7}x^{-2}, \text{ then } \frac{dy}{dx} = \frac{3}{7}(-2)x^{-2-1} = \frac{-6}{7}x^{-3} = \frac{-6}{7} \cdot \frac{1}{x^3}$$

Rule 5: Derivative of sum (or a Difference) of Two (or More) Functions- Sum (or Difference) Rule

Let $y = u \pm v$

where $u = f(x)$ and $v = g(x)$, then

$$\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

That is, the derivative of the sum (or a difference) of two or more functions equals the sum (or difference) of derivatives of the individual functions.

Example Find $\frac{dy}{dx}$ when:

$$(i) \quad y = (x - 1)(2x - 1) \quad (ii) \quad y = x(1 + x)(1 + 2x)$$

Solution: (i) $y = (x - 1)(2x - 1) = 2x^2 - x - 2x + 1 = 2x^2 - 3x + 1$

Diff. w.r.t x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(2x)^2 - \frac{d}{dx}(3x) + \frac{d}{dx}(1)$$

$$\frac{dy}{dx} = 4x - 3$$

$$(ii) \quad y = x(1 + x)(1 + 2x) = (x + x^2)(1 + 2x) = x + 2x^2 + x^2 + 2x^3$$

$$y = x + 3x^2 + 2x^3$$

Diff. w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x) - \frac{d}{dx}(3x^2) + \frac{d}{dx}(2x^3)$$

$$\frac{dy}{dx} = 1 - 6x + 6x^2$$

Rule 6: Differentiation of product of two functions-Product Rule

Let $y = u \cdot v$

Where $u = f(x)$ and $v = g(x)$ are differentiable function of x , then

$$\frac{dy}{dx} = \frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

That is, the derivative of the product of two functions is equal to the first function into the derivative of the second function plus the second function into the derivative of the first function.

Rule: $\frac{d}{dx}$ (product of two functions)

$$= 1^{\text{st}} \text{ function} \times \frac{d}{dx} (2^{\text{nd}} \text{ function}) + 2^{\text{nd}} \text{ function} \times \frac{d}{dx} (1^{\text{st}} \text{ function})$$

Extension of the above rule

In case the given function is the product of more than two functions, say,

$y = u \cdot v \cdot w$, where u , v and w are function of x , then

$$\frac{dy}{dx} = u \cdot v \cdot \frac{d}{dx}(w) + u \cdot w \cdot \frac{d}{dx}(v) + v \cdot w \cdot \frac{d}{dx}(u)$$

Example Find $\frac{dy}{dx}$ when $y = x^5 (2x^2 + 1)$

Solution: $y = x^5 (2x^2 + 1)$

Diff. w.r.t x, we get

$$\begin{aligned}\frac{dy}{dx} &= x^5 \cdot \frac{d}{dx} (2x^2 + 1) + (2x^2 + 1) \frac{d}{dx} (x^5) \\ &= x^5 \cdot (4x) + (2x^2 + 1) (5x^4) \\ &= 4x^6 + 10x^6 + 5x^4 = 14x^6 + 5x^4\end{aligned}$$

Example If $y = (1 + 3x^2)(1 + 5x)$, find $\frac{dy}{dx}$

Solution: $y = (1 + 3x^2)(1 + 5x)$

Diff. w.r.t x, we get

$$\begin{aligned}\frac{dy}{dx} &= (1 + 3x^2) \frac{d}{dx} (1 + 5x) + (1 + 5x) \cdot \frac{d}{dx} (1 + 3x^2) \\ &= (1 + 3x^2) (5) + (1 + 5x) (6x) \\ &= 5 + 15x^2 + 6x + 30x^2 = 45x^2 + 6x + 5\end{aligned}$$

Rule 7: Differentiation of quotient of two functions-Quotient Rule

Let $y = \frac{u}{v}$, when u and v are functions of x and $v \neq 0$ then

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

That is, the derivative of the quotient of two functions equal:

(The denominator) $\times$ (Derivative of the numerator) – (The numerator) $\times$ (Derivative of the denominator), then this difference is divided by the square of the denominator.

Rule: $\frac{d}{dx} (\text{Quotient}) = \frac{\text{Denom.} \times \text{Deri. of Num.} - \text{Num.} \times \text{Deri. of Denom.}}{(\text{Denominator})^2}$

Example Find $\frac{dy}{dx}$ when $y = \frac{8x}{x^3+1}$

Solution: $y = \frac{8x}{x^3+1}$

Diff. w.r.t x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^3+1)\frac{d}{dx}(8x) - (8x)\frac{d}{dx}(x^3+1)}{(x^3+1)^2} \\ &= \frac{(x^3+1)(8) - (8x)(3x^2)}{(x^3+1)^2} \\ &= \frac{8x^3+8-24x^3}{(x^3+1)^2} = \frac{8-16x^3}{(x^3+1)^2}\end{aligned}$$

Example If $y = \frac{x^2+5x+1}{x+2}$, find $\frac{dy}{dx}$.

Solution: $y = \frac{x^2+5x+1}{x+2}$

Diff. w.r.t x, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x+2)\frac{d}{dx}(x^2+5x+1) - (x^2+5x+1)\frac{d}{dx}(x+2)}{(x+2)^2} \\ &= \frac{(x+2)(2x+5) - (x^2+5x+1)(1)}{(x+2)^2} \\ &= \frac{2x^2+5x+4x+10 - x^2-5x-1}{(x+2)^2}\end{aligned}$$

$$= \frac{\quad}{(x+2)^2}$$

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$$= \frac{x^2 + 4x + 9}{(x + 2)^2}$$

Rule 8: Derivative of a function of a function-Chain Rule

Let $y = f(x)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

That is, the derivative of y w.r.t. x is the multiplication of derivative of y w.r.t. u and derivative of u w.r.t. x . This is called **Chain Rule**.

Extension of the above rule

If $y = f(w)$, $w = f(z)$ and $z = h(x)$, then

$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{dz} \cdot \frac{dz}{dx}$$

Example If $y = u^2 + 4u$ and $u = x^2 + 4$, find $\frac{dy}{dx}$.

Solution: Given $y = u^2 + 4u$, $u = x^2 + 4$

$$\frac{dy}{du} = 2u + 4, \quad \frac{du}{dx} = 2x$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (2u + 4) \cdot (2x)$$

$$= [2(x^2 + 4) + 4] (2x)$$

$$= (2x^2 + 8 + 4) (2x)$$

$$= (2x^2 + 12) (2x)$$

$$= 4x^3 + 24x$$

Rule 9: Derivative of the nth power of a function

Let $y = u^n$ where $u = f(x)$ is a differentiable function of x and n are any number, then

$$\frac{dy}{dx} = \frac{d}{dx}(u^n) = nu^{n-1} \cdot \frac{du}{dx}$$

$$\text{Rule: } \frac{d}{dx}(\text{inside})^n = n(\text{inside})^{n-1} \cdot \frac{d}{dx}(\text{inside})$$

Example Find $\frac{dy}{dx}$ when $\sqrt{5x^2 + 6x + 9}$.

Solution: Given $y = \sqrt{5x^2 + 6x + 9} = (5x^2 + 6x + 9)^{1/2}$

Diff. w.r.t. x , we get

$$\frac{dy}{dx} = \frac{1}{2} \cdot (5x^2 + 6x + 9)^{-1/2} \cdot \frac{d}{dx}(5x^2 + 6x + 9)$$

$$= \frac{1}{2\sqrt{5x^2 + 6x + 9}} \times 10x + 6 = \frac{5x + 3}{\sqrt{5x^2 + 6x + 9}}$$

7.8 Derivative of Implicit Function

If a function is given by the form $f(x, y) = a$, where a is a constant, then it is called an implicit function. Examples of implicit function are: $x^2 + y^2 = a^2$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, etc.

To find $\frac{dy}{dx}$ for the implicit function, every term on both sides of the function is to be differentiated w.r.t. x .

Important Results: The following results will be quite useful for differentiation of the implicit functions:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(y^n) = ny^{n-1} \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}(x^7) = 7x^6$$

$$\frac{d}{dx}(y^7) = 7y^6 \cdot \frac{dy}{dx}$$

Example Find $\frac{dy}{dx}$ when $ax^2 + 2hxy + by^2 = 1$.

Solution: $ax^2 + 2hxy + by^2 = 1$

Diff. both side w.r.t. x, we get

$$\frac{d}{dx}(ax^2) + \frac{d}{dx}(2hxy) = \frac{d}{dx}(by^2) = \frac{d}{dx}(1)$$

$$2ax + 2h \cdot \left[x \frac{dy}{dx} + y \cdot (1) \right] + 2by \cdot \frac{dy}{dx} = 0$$

$$2ax + 2hx \frac{dy}{dx} + 2hy + 2by \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2hx + 2by) = -(2ax + 2hy)$$

$$\therefore \frac{dy}{dx} = \frac{-2(ax + hy)}{2(hx + by)} = -\frac{ax + hy}{hx + by}$$

7.9 Derivative of Parametric Equations

If $x = f(t)$, and $y = \phi(t)$, i.e., both x and y variable are functions of another variable t, then these equations are known as parametric equations. The variable 't' is called parameter. We can find the derivative of y w.r.t. x by using the following formula:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\phi'(t)}{f'(t)} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Example If $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$, show that $\frac{dy}{dx} + \frac{x}{y} = 0$.

Solution: Here $x = \frac{1-t^2}{1+t^2}$

$$\therefore \frac{dx}{dt} = \frac{(1+t^2)(-2t) - (1-t^2).2t}{(1+t^2)^2}$$

$$= \frac{-2t[1+t^2+1-t^2]}{(1+t^2)^2} = \frac{-4t}{(1+t^2)^2}$$

and $y = \frac{2t}{1+t^2}$

$$\therefore \frac{dy}{dt} = \frac{(1+t^2).2 - 2t.2t}{(1+t^2)^2} = \frac{2-2t^2}{(1+t^2)^2} = \frac{2(1-t^2)}{(1+t^2)^2}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4t} = \frac{1-t^2}{-2t}$$

$$\text{Also, } \frac{x}{y} = \frac{1-t^2}{1+t^2} \times \frac{1+t^2}{2t} = \frac{1-t^2}{2t}$$

$$\therefore \text{LHS} = \frac{dy}{dx} + \frac{x}{y} = \frac{1-t^2}{-2t} + \frac{1-t^2}{2t} = 0 = \text{RHS}$$

7.10 Derivative of a function w.r.t. another function

The technique of differentiating a function defined parametrically is also applicable in finding the derivative of one function w.r.t. another function. Suppose, we wish to find

the derivative of $f(x)$ w.r.t. $f(x)$. Then by letting

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$$y = f(x), z = \phi(x)$$

and treating x as the parameter, we have

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx}$$

Example Differentiate $5x^2 - 4x^2 + 7x - 3$ w.r.t. $5x - 6$.

Solution: Let $y = 5x^2 - 4x^2 + 7x - 3$

$$\frac{dy}{dx} = 15x^2 - 8x + 7$$

$$\text{Let } z = 5x - 6$$

$$\frac{dz}{dx} = 5$$

$$\text{Now, } \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{15x^2 - 8x + 7}{5}$$

7.11 Derivative of Logarithmic Function

Case I: Let $y = \log_a x$, where a is the base of $\log$, then

$$\frac{dy}{dx} = \frac{1}{x} \log_a e$$

Case II: Let $y = \log_e x$ or $\log x$, then

$$\frac{dy}{dx} = \frac{1}{x} \log_e e = \frac{1}{x}$$

Case III: Let $y = \log_a u$, where u is function of x , then

$$\frac{dy}{dx} = \frac{1}{u} \log_a e \cdot \frac{du}{dx}$$

$$\text{Rule: } \frac{d}{dx} [\log_a (\text{function})] = \frac{1}{\text{function}} \cdot \log_e e \cdot \frac{d}{dx} (\text{function})$$

Case IV: Let $y = \log u$, where u is a function of x , then

$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$$

$$\text{Rule: } \frac{d}{dx} [\log (\text{function})] = \frac{1}{\text{function}} \cdot \frac{d}{dx} (\text{function})$$

Example Find $\frac{dy}{dx}$ when $y = \log (x^3 + 3x^2)$.

Solution: $y = \log (x^3 + 3x^2)$

Diff. w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x^3 + 3x^2} \cdot \frac{d}{dx} (x^3 + 3x^2) & \left[\therefore \frac{d}{dx} (\log u) = \frac{1}{u} \cdot \frac{du}{dx} \right] \\ &= \frac{1}{x^3 + 3x^2} (3x^2 + 6x) = \frac{3x^2 + 6x}{x^3 + 3x^2} \end{aligned}$$

Example Find $\frac{dy}{dx}$ when $y = \log_a (4x + 8)$.

Solution: $y = \log_a (4x + 8)$

Diff. w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{4x + 8} \cdot \log_a e \cdot \frac{d}{dx} (4x + 8) \\ &= \frac{\frac{dy}{dx}}{(4x + 8)} \cdot \log_a e \end{aligned}$$

7.12 Derivative of Exponential Functions

A function whose base is constant and the exponent (or power) is variable, i.e., (constant)^{variable} is called an exponential function. In this function, constant base is either a or e.

Examples of exponential functions are a_x , a^x , a^{x^2+1} , e^{x^2} etc.

The derivatives of the exponential functions are obtained as below.

Case I: Let $y = a^x$, where a is some constant, then

$$\frac{dy}{dx} = a^x \cdot \log a$$

Case II: Let $y = e^x$ where e is a fixed constant, then

$$\frac{dy}{dx} = e^x \cdot \log e = e^x \quad [\because \log e = 1]$$

Case III: Let $y = a^u$, where u is a function of x, then

$$\frac{d}{dx} = a^u \cdot \log a \cdot \frac{du}{dx}$$

Rule. $\frac{d}{dx}(a^{\text{function}}) = a^{\text{function}} \log a \cdot \frac{d}{dx}(\text{function})$

Case IV: Let $y = e^u$, where u is a function of x, then

$$\frac{dy}{dx} = e^u \cdot \log e \cdot \frac{du}{dx} = e^u \cdot \frac{du}{dx}$$

Rule: $\frac{d}{dx}e^{(\text{function})} = e^{\text{function}} \cdot \frac{d}{dx}(\text{function})$

Example Find $\frac{dy}{dx}$ when $y = a^{x^2}$.

Solution. $y = a^{x^2}$

Diff. w.r.t. x , we get

$$\frac{dy}{dx} = a^{x^2} \log a \cdot \frac{d}{dx}(x^2)$$

$$= a^{x^2} \cdot \log a(2x)$$

$$= 2x \cdot a^{x^2} \log a$$

Example Find $\frac{dy}{dx}$ when $y = e^{6x^2+7x}$.

Solution. $y = e^{6x^2+7x}$.

Diff. w.r.t. x , we get

$$\frac{dy}{dx} = e^{6x^2+7x} \cdot \frac{d}{dx}(6x^2+7x) \quad \left[\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx} \right]$$

$$= e^{6x^2+7x} \cdot (12x+7)$$

Example Find $\frac{dy}{dx}$ of the following functions:

(i) $y = \frac{e^x}{1+x}$

(ii) $y = \frac{3}{4+6e^{3x}}$

Solution. (i) $y = \frac{e^x}{1+x}$

$$\frac{dy}{dx} = \frac{(1+x) \cdot \frac{d}{dx}(e^x) - e^x \cdot \frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{(1+x)e^x - e^x \cdot (1)}{(1+x)^2} = \frac{e^x + xe^x - e^x}{(1+x)^2}$$

$$= \frac{xe^x}{(1+x)^2}$$

$$(ii) \quad y = \frac{3}{4 + 6e^{3x}}$$

$$\frac{dy}{dx} = \frac{\left[(4 + 6e^{3x}) \frac{dy}{dx} (3) - 3 \cdot \frac{d}{dx} (4 + 6e^{3x}) \right]}{(4 + 6e^{3x})^2}$$

$$\frac{dy}{dx} = \frac{(4 + 6e^{3x}) (0) - 3 \cdot (6 \cdot e^{3x} \cdot 3)}{(4 + 6e^{3x})^2}$$

7.13 Logarithmic Differentiation

When the function is of the type (variable)^{Variable}, i.e., x^x , then the derivative can be only found by using the process of logarithmic differentiation. We put the function equal to y , then take log on both sides and then differentiate. Logarithmic differentiation is also useful to differentiate then function which consists of product or quotient of a number of functions involving variables.

Example Find $\frac{dy}{dx}$ when $y = x^x$

Solution $y = x^x$

Taking log on both sides, we get

$$\log y = x \log x \quad [\because \log m^n = n \log m]$$

Diff. w.r.t. x , we get

$$\frac{d}{dx} (\log y) = \frac{d}{dx} (x \log x)$$

$$\frac{1}{y} \frac{dy}{dx} = x \cdot \frac{d}{dx} (\log x) + \log x \cdot \frac{d}{dx} (x)$$

$$\frac{1}{y} \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\frac{1}{y} \frac{dy}{dx} = 1 + \log x$$

$$\frac{dy}{dx} = y[1 + \log x]$$

$$\therefore \frac{dy}{dx} = x^x [1 + \log x]$$

Example If $x^y = e^{x-y}$, show that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$

Solution $x^y = e^{x-y}$

Taking log on both sides, we get

$$y \log x = (x - y) \log e$$

$$y \log x = x - y$$

$$\text{or } y(1 + \log x) = x \quad [\square \log e = 1]$$

$$\text{or } y = \frac{x}{1 + \log x}$$

Diff. w.r.t. x , we get

$$\frac{dy}{dx} = \frac{(1 + \log x) \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(1 + \log x)}{(1 + \log x)^2}$$

$$= \frac{(1 + \log x)(1) - x \cdot \frac{1}{x}}{(1 + \log x)^2}$$

$$= \frac{1 + \log x - 1}{(1 + \log x)^2} = \frac{\log x}{(1 + \log x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$$

Check your Progress- I

Based on your understanding of the previous section, solve these questions:

Q1. Find $\frac{dy}{dx}$ when $x^2 + y^2 = 1$.

Q2. Find $\frac{dy}{dx}$ when $x = 5t^2$ and $y = 3t^2$.

Q3. Find $\frac{dy}{dx}$ when $y = a\sqrt{x^2 + 1}$

7.14 Second Order or Higher Order Derivatives (or Successive Derivatives)

If $y = f(x)$ be a given function, then $\frac{dy}{dx} = f'(x)$ is known as first derivative of y w.r.t. x . Generally, $f'(x)$ is again a function of x and can be differentiated once again.

The next derivative is called the second derivative of y w.r.t. x and is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$. Similarly, $f''(x)$ again can be differentiated once more and its derivatives is called

the third derivative of y w.r.t. x and is denote by $\frac{d^3y}{dx^3}$ or $f'''(x)$ and so on. The process of differentiating a function more that once is called successive differentiation.

The following example illustrate the technique of obtaining higher order derivatives:

Example If $y = 2x^3 + 5x^2 - 7x + 4$, find $\frac{d^2y}{dx^2}$

Solution. Given $y = 2x^3 + 5x^2 - 7x + 4$

Diff. w.r.t. x , we get

$$\therefore \frac{dy}{dx} = 6x^2 + 10x - 7$$

Differentiating it again, we have

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} (6x^2 + 10x - 7)$$

$$= 12x + 10$$

$$\therefore \frac{d^2y}{dx^2} = 12x + 10$$

Example If $y = ae^{mx} + be^{-mx}$, prove that $\frac{d^2y}{dx^2} - m^2y = 0$

Solution. Here $y = ae^{mx} + be^{-mx}$

Diff. w.r.t. x , we get

$$\begin{aligned}\frac{dy}{dx} &= a \frac{d}{dx}(e^{mx}) + b \frac{d}{dx}(e^{-mx}) \\ &= ae^{mx} \cdot \frac{d}{dx}(mx) + be^{-mx} \frac{d}{dx}(-mx) \\ &= ae^{mx} \cdot m + be^{-mx} (-m) \\ &= ame^{mx} - bme^{-mx}\end{aligned}$$

Differentiating again w.r.t. x , we get

$$\begin{aligned}\frac{d^2y}{dx^2} &= am \cdot \frac{d}{dx}(e^{mx}) - bm \frac{d}{dx}(e^{-mx}) \\ &= am \cdot e^{mx} \cdot (m) - bm \cdot e^{-mx} (-m) = m^2 (ae^{mx} + be^{-mx}) = m^2 y \\ \therefore \quad \frac{d^2y}{dx^2} - m^2 y &= 0\end{aligned}$$

Hence proved

7.15 Economic Applications of Simple Differentiation

Simple derivatives have wide range of applications in the following topics in Business and Economics.

1) To find price elasticity of demand when the demand function is given

Price elasticity of demand is defined as the ratio of proportionate change in demand to a proportionate change in price.

If the demand function is $q = f(p)$ where q is the quantity demanded and p the price, then price elasticity of demand is defined as:

$$e_d = - \frac{dq/q}{dp/p} = - \frac{p}{q} \cdot \frac{dq}{dp}$$

Other notations usually used to denote elasticity of demand or e_p or η . The sign of e_d is negative because in general the slope of the demand curve is negative.

Example A demand function is given as: $q = 50 - 5p$. Compute the price elasticity of demand at $p = 5$.

Solution. Given: $q = 50 - 5p$
Diff. it w.r.t. p , we get

$$\frac{dq}{dp} = -5$$

$$\text{At } p = 5, q = 50 - 5 \times 5 = 25$$

Substituting these values in the elasticity formula which is given:

$$\begin{aligned} e_d &= - \frac{p}{q} \cdot \frac{dq}{dp} \\ &= \left(\frac{5}{25} \right) (-5) = 1 \end{aligned}$$

$$\therefore e_d = 1$$

Example If e_1 and e_2 be the price elasticities of demand laws $p = e^x$ and

$$p = \frac{e^x}{x} \text{ show that } e_1 \cdot e_2 = e_1 - e_2.$$

Solution. $p = e^x, \frac{dp}{dx} = e^x$

$$e_1 = \left| \frac{dx}{dp} \cdot \frac{p}{x} \right| = \left| \frac{1}{e^x} \times \frac{e^x}{x} \right| = \frac{1}{x}$$

$$p = \frac{e^x}{x}, \frac{dp}{dx} = \frac{x(e^x) + e^x \cdot (1)}{x^2} = \frac{e^x(1+x)}{x^2}$$

$$e_2 = \left| \frac{dx}{dp} \cdot \frac{p}{x} \right| = \left| \frac{x^2}{e^x(1+x)} \times \frac{e^x}{x \cdot x} \right| = \frac{1}{x+1}$$

$$e_1 e_2 = \frac{1}{x} \cdot \frac{1}{x+1} = \frac{1}{x(x+1)} \quad \dots(i)$$

$$e_1 e_2 = \frac{1}{x} - \frac{1}{x+1} = \frac{x+1-x}{x(x+1)} = \frac{1}{x(x+1)} \quad \dots(ii)$$

From (i) and (ii), we get

$$e_1 e_2 = e_1 - e_2$$

Hence, the result is verified.

2) To find price elasticity of supply when the supply function is given

Price elasticity of supply is defined as the ratio of proportional change in supply to a proportional change in price.

If the supply function is $q = f(p)$ where q is the quantity supplied and p the price, then price elasticity of supply is defined as:

$$e_s = \frac{dq/q}{dp/p} = \frac{p}{q} \cdot \frac{dq}{dp}$$

The sign of e_s is always positive because the slope of the supply curve is positive.

Example If η_1 and η_2 be the price elasticities of supply for the supply laws: $p = e^x$ and $p = \frac{e^x}{x}$ respectively, show that $\eta_1 \eta_2 = \eta_1 - \eta_2$.

Solution. Elasticity of supply is given by $\eta_s = \frac{p}{x} \frac{dx}{dp}$

For the supply law $p = e^x$,

$$\frac{dp}{dx} = e^x \Rightarrow \frac{dx}{dp} = \frac{1}{e^x}$$

$$\therefore \eta_1 = \frac{e^x}{x} \left(\frac{1}{e^x} \right) \Rightarrow \eta_1 = \frac{1}{x} \quad \dots(i)$$

For the supply law $p = \frac{e^x}{x}$,

$$\frac{dp}{dx} = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2} \Rightarrow \frac{dx}{dp} = \frac{x^2}{e^x(x-1)}$$

$$\therefore \eta_2 = \frac{e^x/x}{\left(\frac{e^x(x-1)}{x^2} \right)} = \frac{e^x}{x^2} \times \frac{x^2}{e^x(x-1)} = \frac{1}{x-1} \dots(ii)$$

$$\text{Now, } \eta_1 \eta_2 = \frac{1}{x} \times \frac{1}{x-1} = \frac{1}{x(x-1)} \quad \dots(ii)$$

$$\eta_2 - \eta_1 = \frac{1}{x-1} - \frac{1}{x} = \frac{x - (x-1)}{x(x-1)} = \frac{1}{x(x-1)} \quad \dots(iv)$$

Thus, using (iii) and (iv), we get $\eta_1 \eta_2 = \eta_2 - \eta_1$

Mixed Example on E_d and E_s

Example Find the elasticity of demand and supply at the equilibrium price for the demand function $p = \sqrt{36-x^2}$ and supply function $x = p^2 - 6$, where p and x are price and quantity respectively.

Solution. Demand function, $p = \sqrt{36-x^2}$

Supply function $x = p^2 - 6 \Rightarrow p^2 = x + 6 \Rightarrow p = \sqrt{x+6} \dots(i)$

At equilibrium price, Demand = Supply

$$\therefore \sqrt{36-x^2} = \sqrt{x+6}$$

$$\Rightarrow 36 - x^2 = x + 6 \quad (\text{taking square on both sides})$$

$$\Rightarrow x^2 + x - 30 = 0$$

$$\Rightarrow (x + 6)(x - 5) = 0$$

$$\therefore x = -6 \text{ or } x = 5$$

Since $x > 0$, $x = 5$

$$\text{At } x = 5, p = \sqrt{36-(5)^2} \Rightarrow p = \sqrt{11}$$

(i) To find η_d

We consider the demand function $p = \sqrt{36-x^2}$

$$\frac{dp}{dx} = \frac{1}{2\sqrt{36-x^2}} \times (-2x) \Rightarrow \frac{dx}{dp} = \frac{-\sqrt{36-x^2}}{x}$$

Elasticity of demand is given by $\eta_d = \frac{-p}{x} \cdot \frac{dx}{dp}$

$$\therefore \eta_d = \frac{-\sqrt{36-x^2}}{x} \cdot \frac{-\sqrt{36-x^2}}{x} = \frac{36-x^2}{x^2}$$

$$\text{At } x = 5, \quad \eta_d = \frac{36-(5)^2}{(5)^2} = \frac{11}{25}$$

(ii) To find η_s

We consider the supply function $p = \sqrt{x+6}$

$$\therefore \frac{dp}{dx} = \frac{1}{2\sqrt{x+6}} \Rightarrow \frac{dx}{dp} = 2\sqrt{x+6}$$

Elasticity of supply is given by $\eta_s = \frac{p}{x} \frac{dx}{dp}$

$$\therefore \eta_s = \frac{\sqrt{x+6}}{x} (2\sqrt{x+6}) \Rightarrow \eta_s = \frac{2(x+6)}{x}$$

$$\text{At } x = 5, \quad \eta_s = \frac{2(5+6)}{5} = \frac{22}{5}$$

$\therefore$ The elasticity of demand is $\frac{4}{25}$ and the elasticity of supply is $\frac{22}{5}$

3) To Find Income Elasticity of Demand

The income elasticity of demand is defined as the ratio of proportional change in quantity demanded to a proportional change in income.

If the income demand function is: $x = f(y)$, where x is quantity demand and y is the income level, then income elasticity of demand is defined is

$$e_y = \frac{dx/x}{dy/y} = \frac{y}{x} \cdot \frac{dx}{dy}$$

If $e_y > 1$, the goods are luxury

If $0 < e_y < 1$, the goods are necessities

If $e_y < 0$, the goods are inferior

Example The demand of x as a function of y is given by: $x = 40 + 0.8y$, find the income elasticity of demand when $y = 100$

Solution. Given: $x = 40 + 0.8y$

Diff. w.r.t. y , we get

$$\frac{dx}{dy} = 0.8$$

$$\text{At } y = 100, x = 40 + 0.8 \times 100 = 120$$

Substituting values in the elasticity formula, we get

$$e_y = \frac{y}{x} \cdot \frac{dx}{dy} = \frac{100}{120} \times 0.80 = 0.66 < 1$$

4) To Find Average Revenue and Marginal Revenue

There are three concepts of revenue:

- a) **Total Revenue TR:** Total revenue refers to the total sale receipts of a firm. It is obtained by multiplying the price per unit (p) with quantity of the good sold (q). Mathematically,

$$R = p \cdot q$$

Where R is the total revenue, p is the price of q the quantity sold.

- b) **Average Revenue AR:** Average revenue is the revenue per unit of the quantity sold. It is obtained by dividing total revenue (R) with the quantity sold. Mathematically,

$$AR = \frac{R}{q} = \frac{pq}{q} = p$$

Thus, AR is always equal to price i.e., $AR = p$

- c) **Marginal Revenue MR:** Marginal revenue is defined as change in total revenue (R) when quantity sold changes by a very small amount. Mathematically,

$$MR = \lim_{\Delta q \rightarrow 0} \frac{\Delta R}{\Delta q} = \frac{dR}{dq}$$

Alternatively, MR is expressed as:

$$R = p \times q$$

$$\therefore MR = \frac{dR}{dq} = \frac{d}{dq}(p \times q)$$

$$MR = p \cdot \frac{dq}{dq} + q \cdot \frac{dp}{dq} \quad [\text{Using product rule}]$$

$$MR = p + q \cdot \frac{dp}{dq}$$

Note: This expression $MR = p + q \cdot \frac{dp}{dq}$ is a general result which is true for all demand functions.

Example The demand for a certain product is represented by the equation

$$p = 100 - 5q$$

- i) Find the marginal revenue for any output.
- ii) What is MR when $q = 0$ and $q = 4$?

Solution. Given the demand function, the marginal revenue function is found by first finding the total revenue function and then taking its derivative, as follows:

$$R = Rq = (100 - 5q) q = 100q - 5q^2$$

$$\therefore MR = \frac{dR}{dq} = \frac{d}{dq} (100q - 5q^2)$$

$$\therefore MR = 100 - 10q$$

$$(i) \text{ When } q = 0, MR = 100 - 0 = 100$$

$$\text{When } q = 4, MR = 100 - 40 = 60$$

Example The demand curve for a monopolist is $2p = 100 - q$. Find revenue and marginal revenue. At what value of q will the marginal revenue is zero? Also find the price when marginal revenue zero.

Solution. Given: $2p = 100 - q \Rightarrow p = 50 - \frac{1}{2}q$

$$R = p \times q = \left(50 - \frac{1}{2}q \right) q = 50q - \frac{1}{2}q^2$$

Diff. w.r.t. x, we get

$$MR = \frac{dR}{dq} = 50 - q$$

(ii) As per condition given, $MR = 50 - q = 0$

$$\Rightarrow 50 - q = 0 \quad \therefore q = 50$$

Thus, MR is zero at $q = 50$

(ii) $p = 50 - \frac{1}{2}q$

$$\text{At } q = 50 \text{ (When } MR = 0), p = 50 - \frac{1}{2} \times 50 = 25$$

Thus, price is 25 when MR is zero.

5) Relationship between AR, MR and Elasticity of Demand

An important relationship between the marginal revenue (MR), average revenue (AR) and elasticity of demand (E) is given by the following equation:

$$E = \frac{AR}{MR}$$

The relationship among AR, MR and E can also be expressed as:

$$MR = AR \left(\frac{1}{E} \right)$$

$$\text{Or } MR = p \left(1 - \frac{1}{E} \right)$$

The technique of derivative is quite useful in deriving the above relationship.

Example Verify that the elasticity of demand (E) = $\frac{AR}{AR - MR}$ for the linear demand function $p = a - bx$.

Solution. Given: $p = a - bx$

Diff. w.r.t. x, we get

$$\frac{dp}{dx} = -b \Rightarrow \frac{dx}{dp} = -\frac{1}{b}$$

$$\text{We know that } E = - \frac{dx}{dp} \cdot \frac{p}{x} = - \left(-\frac{1}{b} \right) \frac{p}{x}$$

$$\text{Or } E = \frac{p}{bx}$$

$$E = \frac{a - bx}{bx} \quad [\square p = a - bx]$$

$$\text{LHS} = E = \frac{a - bx}{bx}$$

$$\text{Now } TR = px = x(a - bx) = ax - bx^2$$

Diff. w.r.t. x, we get

$$MR = \frac{dTR}{dx} = a - 2bx$$

$$\text{RHS} = \frac{AR}{AR - MR}$$

$$\square \quad AR = p = a - bx, \quad MR = a - 2bx$$

$$RHS = \frac{a - bx}{(a - bx) - (a - 2bx)} = \frac{a - bx}{bx}$$

From (i) and (ii), we have

$$E = \frac{AR}{AR - MR}$$

Hence, the result is verified.

6) To Find Average Cost and Marginal Cost

Suppose that a firm produces q units of a certain commodity and incurs a total cost of C . The total cost function is given by:

$$C = f(q)$$

Notations usually used to denote cost are: π or C or TC .

$$C = AC \times q$$

- i) **Average Cost:** Average Cost is defined as the cost of production per unit of output. It is given by

$$AC = \frac{C}{q}$$

- ii) **Marginal Cost:** Marginal Cost is defined on the change in total cost (C) when output (q) is changed by a very small amount. It is given by

$$MC = \frac{dC}{dq}$$

Thus, MC is the first derivative of total (C) w.r.t. output q .

- iii) **Average Variable Cost:** AVC is defined and given by

$$AVC = \frac{TVC}{q}$$

iv) Slope of AVC: It is defined and given by

$$\text{Slope of AVC} = \frac{d}{dq}(\text{AVC})$$

v) Slope of AVC: It is defined and given by

$$\text{Slope of AVC} = \frac{d}{dq}(\text{AVC})$$

Note: For minimum AC or AVC, slope of AC or AVC, is zero i.e.,

$$\frac{d}{dq}(\text{AC}) \text{ or } \frac{d}{dq}(\text{AVC}) = 0$$

Example Given the total cost function:

$$C = 60 - 12q + 2q^2$$

Find (i) AC function (ii) MC function

Solution. Given $C = 60 - 12q + 2q^2$

$$\text{i) } AC = \frac{C}{q} = \frac{60 - 12q + 2q^2}{q} = \frac{60}{q} - 12 + 2q$$

$$\text{i) } MC = \frac{dC}{dq} = \frac{d}{dq} (60 - 12q + 2q^2)$$

$$MC = -12 + 4q$$

Example The total cost C is given as

$$C = .005x^3 - .02x^2 - 30x + 3000$$

Find (i) Total cost when output (x) = 4

(ii) AC when output = 10

(iii) MC when output = 3

Solution: (i) Given

$$C = .005x^3 - .02x^2 - 30x + 3000$$

For $x = 4$

$$C = .005 (4)^3 - (0.02) (4)^2 - 30(4) + 3000$$

$$\therefore C = ₹ 2,880$$

$$(ii) \quad AC = \frac{C}{x} = \frac{.005x^3 - 0.02x^2 - 30x + 3000}{x}$$

$$\therefore AC = .005x^2 - .02x - 30 + \frac{3000}{x}$$

For $x = 10$

$$AC = .005 (100) - (0.02) (10) - 30 + \frac{3000}{10}$$

$$AC = 0.5 - 0.2 - 30 + 300$$

$$\therefore AC = ₹ 270.3$$

$$(ii) \quad MC = \frac{dC}{dx} = \frac{d}{dx} (.005x^3 - 0.02x^2 - 30x + 3000)$$

$$MC = 0.015x^2 - 0.04x - 30$$

For $x = 3$

$$MC = 0.135 - 0.12 - 30$$

$$\therefore MC = ₹ 29.985$$

Relation between AC and MC

The relation between AC and MC is given by the following equation:

$$\text{Slope of AC} = \frac{1}{q}(MC - AC)$$

Derivation: The relationship is derived as follows:

$$\text{Slope of AC} = \frac{d(AC)}{dq} = \frac{d\left(\frac{C}{q}\right)}{dq} \text{ where } AC = \frac{C}{q}, C = \text{total cost}$$

$$\begin{aligned} \text{Slope of AC} &= \frac{q \cdot \frac{dc}{dq} - c}{q^2} = \frac{q \cdot \frac{d}{dq}(q) - \frac{dc - \pi}{dq}}{q^2} \quad [\text{using quotient formula (u/v)}] \\ &= \frac{q \cdot \frac{dc}{dq} - \frac{c}{q}}{q^2} = \frac{1}{q} \cdot \left[\frac{dc}{dq} - \frac{c}{q} \right] = \frac{1}{q} (MC - AC) \end{aligned}$$

$$\text{Thus, slope of AC} = \frac{1}{q} (MC - AC)$$

Note: The expression slope of AC = $\frac{1}{q} (MC - AC)$ represents a standard result which is true for all cost functions.

Example Prove that the slope of average cost curve is $\frac{1}{x} (MC - AC)$ for the total cost function $C = 60 - 12q + 2q^2$

Solution: Given: Total Cost = $C = 60 - 12q + 2q^2$

$$\frac{C}{q} = \frac{60 - 12q + 2q^2}{q}$$

$$\therefore AC = \frac{60}{q} - 12 + 2q$$

$$(ii) MC = \frac{dC}{dq} = \frac{d}{dq}(60 - 12q + 2q^2)$$

$$\therefore MC = -12 + 4q$$

$$\begin{aligned}\text{LHS} &= \text{Slope of AC} = \frac{d}{dq}(AC) \\ &= \frac{d}{dq} \left(\frac{60}{q} - 12 + 2q \right) \\ &= \frac{-60}{q^2} + 2 \quad \dots (i)\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{1}{q}(MC - AC) \\ &= \frac{1}{q} \left[-12 + 4q - \frac{60}{q} + 12 - 2q \right] \\ &= \frac{1}{q} \cdot \left[2q - \frac{60}{q} \right] \\ &= -\frac{60}{q^2} + 2 \quad \dots (ii)\end{aligned}$$

From (i) and (ii), we get

$$\text{Slope of AC} = \frac{1}{q}(MC - AC)$$

Check your Progress-II

Based on your understanding of the previous section, solve these questions

Q1. If $y = ae^{2x} + be^{-2x}$, prove that $\frac{d^2y}{dx^2} - 4y = 0$

Q2. If $y = \log[x + \sqrt{1+x^2}]$, show that

$$(1+x^2) \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = 0$$

Q3. If the demand law is $p = (4 - 5x)^2$ for what value of x the elasticity of demand is unity?

Q4. Prove that $E = \frac{AR}{AR - MR}$ or $MR = AR \left(1 - \frac{1}{E}\right)$, where E is the elasticity of demand, AR is average revenue and MR is Marginal Revenue.

7.16 Let us sum up

In this lesson, we studied the fundamental principles of differentiation, including

techniques for differentiating implicit functions, parametric equations, and logarithmic functions. Additionally, we explored the economic applications of basic differentiation, such as the concepts of elasticity of demand and supply, as well as cost and revenue functions.

7.17 Keywords

$$1. \quad y = x^n = \frac{dy}{dx} = \frac{d}{dx} (x^n) = nx^{n-1} \text{ for any positive integer.}$$

$$2. \quad y = (ax + b)^n = \frac{dy}{dx} = \frac{d}{dx} (ax + b)^n = n (ax + b)^{n-1} \cdot a$$

$$3. \quad y = c \text{ where } c \text{ is a constant, } \frac{dy}{dx} = \frac{d(c)}{dx} = 0$$

$$4. \quad y = kx^n, \frac{dy}{dx} = \frac{d}{dx} (kx^n) = k \cdot \frac{d}{dx} (x^n) = k n x^{n-1}$$

$$5. \quad y = u \pm v \text{ where } u = f(x) \text{ and } v = g(x), \text{ then } \frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$6. \quad y = u \cdot v, \text{ then } \frac{dy}{dx} = \frac{d}{dx} (u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$7. \quad y = \frac{u}{v}, \text{ then } \frac{dy}{dx} = \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

$$8. \quad y = f(x) \text{ and } u = g(x), \text{ then } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$9. \quad y = u^n, \frac{dy}{dx} = \frac{d}{dx} (u^n) = nu^{n-1} \cdot \frac{du}{dx}$$

$$10. \quad \text{If } x = f(t), \text{ and } y = \phi(t), \text{ then } \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{\phi'(t)}{f'(t)} = \frac{dy}{dx} \cdot \frac{dt}{dt}$$

$$\frac{dx}{dt} = f(t) \frac{dx}{dt}$$

11. $y = \log_a x$, then $\frac{dy}{dx} = \frac{1}{x} \log_e a$
12. $y = \log_e x$ or $\log x$, then $\frac{dy}{dx} = \frac{1}{x} \log_e e = \frac{1}{x}$
13. $y = \log_a u$, where u is function of x , then $\frac{dy}{dx} = \frac{1}{u} \log_a e \cdot \frac{du}{dx}$
14. $y = \log u$, $\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$
15. $y = a^x$, where a is some constant, then $\frac{dy}{dx} = a^x \cdot \log a$
16. $y = e^x$ where e is a fixed constant, then $\frac{dy}{dx} = e^x \cdot \log e = e^x$
17. $y = a^u$, where u is a function of x , then $\frac{dy}{dx} = a^u \cdot \log a \cdot \frac{du}{dx}$
18. $y = e^u$, where u is a function of x , then $\frac{dy}{dx} = e^u \cdot \log e \cdot \frac{du}{dx} = e^u \cdot \frac{du}{dx}$

7.18 Examination Oriented Questions

- Q1. The demand curve for a monopolist is $2p = 100 - q$. Find revenue and marginal revenue. At what value of q will the marginal revenue is zero? Also find the price when marginal revenue zero.
- Q2. If the demand curve is $x = \frac{9}{\sqrt{p}}$, show that the price elasticity of demand is constant and equal to $-\frac{1}{2}$.
- Q3. From the total variable cost function: $C = 2x^3 - 300x^2 + 60x$, find the slope of AVC and the output level (x) at which AVC is minimum.

Q4. Given the demand curve $p = 400 - 2q - 3q^2$ demonstrate the relationship between MR and elasticity of demand i.e. $MR = p(1 - \frac{1}{E})$

Q5. If $y = (\log x)^2$, prove that $x^2 \cdot \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} = 2$.

Q6. Differentiate $(x^2 + 3x)$ w.r.t $(x^2 + 1)$.

Q6. Solve the following:

1. $y = (x^2 + 1)^2(2x - 5)^{1/2}$
2. $y = (1 + \sqrt{x})/(1 - \sqrt{x})$
3. $y = u^2 + 4u$ and $u = x^2 + 4$
4. $x^2 = 1 + y^{1/2}$
5. $x = a \log t$ and $y = bt^2$
6. $y = \int \log(3x^2 + 7x)$
7. $y = a^x + e^x + a^a + x^a$
8. $y = (\log x)^{\log x}$

7.19 Hints to Check your Progress (CYP)

CYP- I

- Q1. Refer Rule 5 of Section 7.7
- Q2. Refer Section 7.9
- Q3. Refer Section 7.12

CYP – II

- Q1. Refer Section 7.14

Q2. Refer Section 7.14

Q3. Refer Part (1) of Section 7.15

Q4. Refer Part (5) of Section 7.15

7.20 Suggested Readings

- Aggarwal, C.S & R.C. Joshi: Mathematics for Students of Economics (New Academic Publishing Co).
- Allen, R.G. D. ; Mathematical Analysis for Economists (Macmillan).
- Anthony Martin & Norman Biggs; Mathematics for Economics and Finance-Methods and Modeling.
- Black, J & J.F. Bradley : Essential Mathematics for Economists (John Wiley & Sons).
- Dowling, Edward T : Introduction to Mathematical Economics (Tata McGraw).
- Henderson, James M & Richard E Quandt : Microeconomic Theory- A Mathematical Approach (McGraw-Hill International Book Company).
- Kandoi B: Mathematics for Business and Economics with Applications (Himalaya Publishing House).
- Yamane Taro: Mathematics for Economics- A Elementary Survey (Prentice Hall of India Pvt. Ltd).

M.A. Economics	Lesson No. 8
C.No. 103	Semester - 1st
	Unit II

BASIC CONCEPTS OF PARTIAL AND TOTAL DIFFERENTIATION

STRUCTURE

- 8.0 Objectives/Learning Outcomes
- 8.1 Introduction
- 8.2 Function of two variables
- 8.3 The Partial Derivatives
 - 8.3.1 Second order Partial Derivatives
 - 8.3.2 Signs of Partial Derivatives
- 8.4 Total Differential and Total Derivative
 - 8.4.1 Total Derivative
 - 8.4.2 Examples of Total Differential
 - 8.4.3 Examples of Total Derivative
- 8.5 Let us sum up
- 8.6 Examination Oriented Questions
- 8.7 Hints to Check Your Progress
- 8.8 Suggested Readings

8.0 Objectives/Learning Outcomes

After reading this lesson, you would be able to :

- Define Function of several variables.
- Discuss the partial derivatives with examples.
- Explain Total differential, total derivative and examples.
- Describe their Application.

8.1 Introduction

So far we have considered functions of only one independent variables, viz., $y = f(x)$. But in most of the problems in economics, we are frequently confronted with functions of more than one (independent) variable. For example,

- i) Quantity demanded (x) of a commodity depends not only on the price (p) of the commodity but also on the prices ($p_1, p_2, \dots$) of other commodities, income (y) of the consumer etc. i.e. x is a function of more than one (independent) variable e.g., $x = x^d = f(p_1, p_2, \dots, y)$.
- ii) Quantity supplied (x) of a commodity is a function of not only price (P) of the commodity but also a function of the prices ($p^1, p^2, \dots$) of the technology (T) etc. i.e., $x = x^s = f(p_1, p_2, \dots, p_1, p_2, \dots, T)$.
- iii) The utility (u) derived by the consumer depends on the quantity consumed of various commodities ($x_1, x_2, \dots$) i.e. $U = f(x_1, x_2, \dots)$.
- iv) The level of production (or output) Q of a commodity depends on the inputs ($L \& K$) etc. i.e. $Q = f(L, K)$ etc.

8.2 Function of two variables

Let U be a symbol which has one definite value for every permissible pair of values of the independent variables x and y , then u is called a function of two variables x & y , we write it as :-

$$U = f(x, y) \text{ or } u(x, y)$$

Similarly, a function of 'n' variables may be written as:-

$$U = U(x_1, x_2, \dots, x_n) \text{ or } U = f(x_1, x_2, \dots, x_n)$$

Where $x_1, x_2, \dots, x_n$ are the n independent variables and corresponding to each set of values of the n independent variables ($x_1, x_2, \dots, x_n$), we get a definite value of u .

Value of a function of two and more variables

If $U = f(x, y)$ is a function of two variables, then $f(x_1, y_1)$ is known as the value of function at the point (x_1, y_1) i.e. when $x = x_1$ and $y = y_1$

For example :-

$$\text{If } U = x^2 - y^2 + 2xy - 10 \text{ then}$$

$$\text{i) } f(0, 0) = \text{value of the function at } x = 0, y = 0 = 0 + 0 + 2(0)(0) - 10 = -10.$$

$$\text{ii) } f(-5, 2) = (-5)^2 - (2)^2 + 2(-5)(2) - 10 = 25 - 4 - 20 - 10 = -9$$

$$\text{iii) } f(2, -1) = (2)^2 - (-1)^2 + 2(2)(-1) - 10 = 4 - 1 - 4 - 10 = -11.$$

If $U = f(x_1, x_2, x_3)$ is a function of three variables, then $f(x_{11}, x_{21}, x_{31})$ is called value of the function at the point (x_{11}, x_{21}, x_{31}) .

For Examples if $U = x_1^2 + x_2^2 + x_3^3$ then, f

$$(0, 0, 0) = (0)^2 + (0)^2 + (0) = 0$$

$$f(1, 2, 3) = (1)^2 + (2)^2 + (3)^3 = 1 + 4 + 27 = 32$$

$$f(0, 2, 0) = (0)^2 + (2)^2 + (0)^3 = 4.$$

Functions of two and more variables i.e. multi-variable functions in economic theory

Demand Function :-

(i) If there are two consumer goods X_1 and X_2 , with prices $p_1 \times p_2$, and quantity demanded as x_1 and x_2 , also y is the income of the consumer, then

$$x_1^d = f_1(p_1, p_2; y)$$

$$x_2^d = f_2(p_1, p_2; y)$$

b) If there are 'n' consumer good $x_1, x_2, \dots, x_n$ with prices $p_1, p_2, \dots, p_n$ and quantity demanded of the n goods as $x_1, x_2, \dots, x_n$ respectively and y is income of the consumers, then

$$x_1^d = f_1(p_1, p_2, \dots, p_n; y)$$

$$x_2^d = f_2(p_1, p_2, \dots, p_n; y)$$

ii) **The supply function:-**

$$x_1^s = f_1(p_1, p_2; p_1, p_2; T)$$

If there are two consumer goods X_1, X_2 , with quantity demanded and prices of the two goods as x_1, x_2 & p_1, p_2 respectively. p_1, p_2 being prices of 2 inputs and T being the level of technology, then.

$$X_1^s = f_1(p_1, p_2; p_1, p_2; T)$$

$$X_2^s = f_2(p_1, p_2; p_1, p_2; T)$$

iii) **The utility function:-** If there are 2 commodities X_1 and X_2 , with x_1, x_2 as the quantities consumed respectively of the 2 commodities, then the level of satisfaction derived by a consumer i.e. the utility function of the consumer can be written as:-

$$U = f(x_1, x_2)$$

In case of 'n' commodities

$$U = f(x_1, x_2, \dots, X_n) \text{ for 'n' goods}$$

iv) **The production function:-** If x_1, x_2 are the quantities of two inputs and Q is the level of output, then the production function in terms of inputs can be written as:-

$$Q = f(x_1, x_2)$$

$$Q = f(x_1, x_2, \dots, x_n) \text{ for 'n' inputs}$$

v) **The Cost Function :-** If a firm produces 2 commodities X_1 & X_2 with x_1, x_2 as the levels of output of the 2 commodities, then the cost function C can be written as function of level of output $x_1 + x_2$.

$$C = f(x_1, x_2)$$

$$C = f(x_1, x_2, \dots, x_n) \text{ for 'n' commodities}$$

Check Your Progress-I

1. What do you mean by Supply function and the demand function.

2. Explain the meaning of utility function and the production function.

8.3 The Partial Derivatives

Definition:- The partial derivatives of

$U = f(x, y)$ with respect to x at (x, y) is defined as :

$$\lim_{\delta \rightarrow 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

If it exists finitely, then while finding partial derivatives of $U = f(x, y)$ with respect to x at (x, y) , we assume that 'y' is fixed and any change in the function is due to change in x , i.e. x changes from x to $x + \delta x$

The partial derivative of $U = f(x, y)$ with respect to x at (x, y) is denoted by

$$\frac{\partial U}{\partial x} \text{ or } f_x$$

Similarly, the partial derivative of $U = f(x, y)$ with respect to 'y' at (x, y) is defined as:-

$$\lim_{\delta y \rightarrow 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

If it exists finitely, then partial derivative of $U = f(x, y)$ with respect to 'y' at (x, y) is denoted by $\frac{\partial U}{\partial y}$ or f_y

Remarks :- (i) The concept of partial derivatives is applicable to functions of more than one variable, i.e. function of two or more variables.

8.3.1 Second order partial derivatives

The partial derivatives of the first order partial derivatives are known as the second order partial derivatives e. g.

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{\partial U}{\partial x} \right] \text{ or } \frac{\partial^2 f}{\partial x^2} \text{ or } U_{xx} \text{ or } f_{xx} \text{ or } f_{11}$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y} \left[\frac{\partial U}{\partial y} \right] \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } U_{yy} \text{ or } f_{yy} \text{ or } f_{22}$$

$$\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial}{\partial y} \left[\frac{\partial U}{\partial x} \right] \text{ or } \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy} \text{ or } f_{12} \text{ or } U_{12}$$

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial U}{\partial y} \right] \text{ or } \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx} \text{ or } f_{21} \text{ or } U_{21}$$

Examples :- Find the first order and second order partial derivatives, $3x^2y^2 + x^5 + 3y^2$.

Solution :-

Let $U = 3x^2y^2 + x^5 + 3y^2$

$$f(x) = \frac{\partial U}{\partial x} = 6xy^2 + 5x^4$$

$$f(y) = \frac{\partial U}{\partial x} = 6x^2y + 6y$$

$$\begin{aligned} f_{xx} &= \frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{\partial U}{\partial x} \right] \\ &= \frac{\partial}{\partial x} [6xy^2 + 6x^2] \\ &= 6y^2 + 12x \end{aligned}$$

$$\begin{aligned} f_{yy} &= \frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y} \left[\frac{\partial U}{\partial y} \right] \\ &= \frac{\partial}{\partial y} [6x^2y + 6y] \\ &= 6x^2 + 6 \end{aligned}$$

$$\begin{aligned} f_{xy} &= \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial}{\partial y} \left[\frac{\partial U}{\partial x} \right] \\ &= \frac{\partial}{\partial y} [6xy^2 + 6x^2] \\ &= 12xy \end{aligned}$$

$$\begin{aligned} f_{yx} &= \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial U}{\partial y} \right] \\ &= \frac{\partial}{\partial x} [6x^2y + 6y] \\ &= 12xy \end{aligned}$$

From above we find

$$f_{xy} = f_{yx}$$

Example : Find the first order partial derivative

$$U = \frac{2x^2}{x + y + 8}$$

Sol.

$$\frac{\partial U}{\partial x} = \frac{(x + y + 8) \frac{\partial}{\partial x}(2x^2) - 2x^2 \frac{\partial}{\partial x}(x + y + 8)}{(x + y + 8)^2}$$

$$\frac{\partial U}{\partial x} = \frac{(x + y + 8)(4x) - 2x^2}{(x + y + 8)^2} \quad (1)$$

$$= \frac{4x^2 + 4xy + 32x - 2x^2}{(x + y + 8)^2}$$

$$= \frac{2x(x + 2y + 16)}{(x + y + 8)^2}$$

$$\frac{\partial U}{\partial y} = 2x^2 \frac{\partial}{\partial y} (x + y + 8)^{-1}$$

$$= 2x^2 (-1) (x + y + 8)^{-2} \quad (1)$$

$$= - \frac{2x^2}{(x + y + 8)^2}$$

Check Your Progress-II

1. Find the second order partial derivative of $e^{x^2} - y^2$.
2. Show that :

$$\frac{\partial u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4(a^2 + b^2) - 4$$

8.3.2. Signs of Partial Derivatives:-

A function of two variables yields two first order partial derivatives, f_x shows the rate of change of the function $U = f(x, y)$ w. r. t. x , assuming y as a constant and f_y , shows the rate of change of the function, $U = f(x, y)$ w. r. t. y assuming x as a constant f_{xx} shows whether the function is increasing or decreasing or constant rate, when x varies and y remains constant.

- 1) $f_x > 0$ means that the function increases as x increase, y remaining constant ;
 $f_x < 0$ means that the function decreases as x increases y being held constant.
- 2) $f_{xx} > 0$ means that the rate of change of the function increases as x increases y being held constant, $f_{xx} < 0$ means that the function changes at decreasing rate. Similarly we can interpret the signs of f_y & f_{yy} .
- 3) $f_{xy} = f_{yx} < 0$ means both that f_x decreases as y increases & f_y decreases as x increases.
- 4) $f_{xy} = f_{yx} > 0$, means that f_x increases as y increases and f_y increases as x increases.
- 5) $f_{xy} = f_{yx} = 0$ means that there is no interaction between the variables.

Example :- A demand function is given by

$$Q = 5y + 4y^3 - 10p^2 - 80p^5, y > 0, p > 0$$

Where q is the quantity demanded, y is income and p is the price

- a) Find the slope of demand curve.
- b) Is the commodity normal or inferior.
- c) Is the reaction of demand to price independent of the level of income.

Solution :- We have

$$q = 5y + 4y^3 - 10p^2 - 80p^5$$

$$\frac{\partial q}{\partial p} = f_p = -30p - 400p^4$$

∴ Slope of demand curve is

$$f_p = -30p^2 - 400 p^{-6}$$

b) If the rate of change of q w. r. t. 't' increases as y increases, the commodity will be normal, thus.

$$\frac{\partial q}{\partial y} = f_y = 5 + 12 y^2$$

Since y is positive, therefore $f_y > 0$. Hence the commodity is a normal good.

c) $f_{py} = f_{yp} = 0$

When cross partial derivatives are zero, then the reaction demanded to price is independent of the level of income i.e. it neither increases nor decreases as y increases. Also f_y neither increases nor decrease thus, the rate of change of q w. r. t. price is not independent of income level. Derivative of f_p w.r.t. y is zero and derivative of f_y w. r. t. p is also zero. At any particular level of price, the responsiveness of demand to price does not depend upon income level. The level of income will be high or low, the value of f_p will remain the same. In such cases, the value of the function will be the sum of separate effects of the independent variables. Hence our answer to (c) part is yes.

Example:- For the Cobb-Douglas production function $U = A x^\alpha \cdot y^{1-\alpha}$ find the marginal products of labour and capital, where x is labour and y is capital. Also show that

$$x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = U$$

Solution : $U = A x^\alpha y^{1-\alpha}$

MP_L = Marginal product of labour

$$\begin{aligned} \frac{\partial}{\partial x}(U) &= \frac{\partial}{\partial x}(U) = \frac{\partial}{\partial x} [A x^\alpha y^{1-\alpha}] \\ &= A \cdot y^{1-\alpha} \cdot x^{\alpha-1} \end{aligned}$$

$$= A\alpha \cdot \frac{x^\alpha}{x} \cdot y^{1-\alpha}$$

$$= \frac{\alpha}{x} [A \cdot x^\alpha \cdot y^{1-\alpha}]$$

$$= \frac{\alpha}{x} U$$

MP_k = Marginal product of capital

$$\frac{\partial}{\partial y} (U) = \frac{\partial}{\partial y} [A \cdot x^\alpha \cdot y^{1-\alpha}]$$

$$= A \cdot x^\alpha (1-\alpha) y^{-\alpha}$$

$$= \frac{A(1-\alpha)}{y} \cdot x \cdot \frac{y^{1-\alpha}}{y}$$

$$= \frac{1-\alpha}{y} [A \cdot x^\alpha \cdot y^{1-\alpha}]$$

$$= \frac{(1-\alpha) U}{y}$$

$$\text{Hence, } x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = x \left[\frac{\alpha U}{x} \right] + y \left[\frac{(1-\alpha) U}{y} \right]$$

$$= \alpha U + (1-\alpha) U$$

$$= U (\alpha + 1 - \alpha)$$

$$= U = R. H. S.$$

8.4 Total Differentiation and Total Derivative

Definition:- Total differentiation : If $U = f(x, y)$ is a function of two independent variables, x and y , then total differential of U to be denoted by du , is a linear approximation of the change in u or $f(x, y)$ when there is a small change in both x and y and we write it as:-

$$\Delta u = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = f_x dx + f_y dy$$

We shall explain this concept with the help of an example of wheat output (u), land x and labour y. Our problem is to find out as to what will be the change in wheat output (u) when there is a small change in both land (x) and labour (y). We know that $\frac{\partial U}{\partial x}$ denotes the change in (u) due to small unit change in x, keeping y constant. Thus, if land x changes by Δx , keeping y constant, change in u due to a small unit. Change in x y⁰ denoted by

$$\left(\frac{\partial U}{\partial x} \right) \Delta x \quad (1)$$

Similarly, if there is a change of Δy in land y, when the change in u due to a change of Δy in y, keeping x constant.

$$\begin{aligned} &= (\text{Change in U due to small unit change in y}) \times (\text{change in y}) \\ &= \left(\frac{\partial U}{\partial y} \right) (\Delta y) \end{aligned} \quad (2)$$

∴ as a first approximation, we can think that the change in U due to a small change of Δx in x and a small change of Δy in y will be

$$\begin{aligned} \Delta u &= \left[\frac{\partial U}{\partial x} \Delta x \right] + \left[\frac{\partial U}{\partial y} \Delta y \right] \\ &= f_x \Delta x + f_y \Delta y. \end{aligned}$$

i.e. Total change in U [due to small change in x & y] = [Change in U due to (total) change in x (keeping y constant) & change in U due to change in y (Keeping x constant)] i.e. ∂U = [Change in U due to a unit of change in x) (total change in x) + (change in U due to a unit change in y) (total change in y).

$$du = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = f_x dx + f_y dy$$

8.4.1 Total Derivative

a) Let $U = f(x, y)$ be function of two independent variables in x and y , then total change in u w. r. t. one of the variables (say x) when there is a change in both x and y , is called the total derivative of u w. r. t. x and is denoted by $\frac{d u}{d x}$, similarly, the total change u w. r. t. y when there is change in both x and y is called the total derivative of u w. r. t. y and is written as $\frac{d u}{d y}$

$$\begin{aligned} \text{Thus, } \frac{d U}{d x} &= \left(\frac{\partial U}{\partial x} \right) \left(\frac{d x}{d x} \right) + \left(\frac{\partial U}{\partial y} \right) \left(\frac{d y}{d x} \right) = \frac{\partial U}{\partial x} + \left(\frac{\partial U}{\partial y} \right) \left(\frac{d y}{d x} \right) \\ &= f_x + f_y \frac{d y}{d x} \\ \frac{\partial u}{\partial y} &= \left(\frac{\partial u}{\partial y} \right) \left(\frac{d x}{d y} \right) + \left(\frac{\partial u}{\partial y} \right) \frac{d y}{d y} = \left(\frac{\partial u}{\partial x} \right) \frac{d x}{d y} + \left(\frac{\partial u}{\partial y} \right) \\ &= f_x \left(\frac{d x}{d y} \right) + f_y \end{aligned}$$

8.4.2 Illustrative examples of total differential

i) $U = x^3 + 3x^2 y + 6xy^2 + 2y^3$

$$\text{Total differential } \partial U = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = f_x dx + f_y dy$$

$$\frac{\partial U}{\partial x} = 3x^2 + 6xy + 6y^2 = 3x^2 + 6xy + 6y^2$$

$$\frac{\partial U}{\partial y} = 0 + 3x^2 + 12xy + 6y^2 = 3x^2 + 12xy + 6y^2$$

$$\therefore d u = (3x^2 + 6xy + 6y^2) dx + (3x^2 + 12xy + 6y^2) dy$$

$$\text{ii)} \quad U = xy^3 - yx^3$$

$$\frac{\partial U}{\partial x} = y^3 - 3yx^2$$

$$\frac{\partial U}{\partial y} = 3xy^2 - x^3$$

$$dU = (y^3 - 3x^2y) dx + (3y^2x - x^3) dy$$

$$\text{iii)} \quad U = \frac{x^2 + y^2}{x^2 - y^2}$$

$$\frac{\partial U}{\partial x} = \frac{(x^2 + y^2)(2x) - (x^2 - y^2)(2x)}{(x^2 + y^2)^2}$$

$$= \frac{2x}{(x^2 + y^2)^2} [x^2 + y^2 - x^2 + y^2]$$

$$= \frac{2x}{(x^2 + y^2)^2} \cdot 2y^2 = \frac{4xy^2}{(x^2 + y^2)^2}$$

$$\frac{\partial U}{\partial y} = \frac{(x^2 + y^2)(-2y) - (x^2 - y^2)(2y)}{(x^2 + y^2)^2}$$

$$= \frac{-2y}{(x^2 + y^2)^2} [x^2 + y^2 + x^2 - y^2]$$

$$= \frac{-2y(2x^2)}{(x^2 + y^2)^2} = \frac{-4yx^2}{(x^2 + y^2)^2}$$

$$\therefore dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

$$= \frac{4xy^2}{(x^2 + y^2)^2} dx + \frac{(-4yx^2)}{(x^2 + y^2)^2} dy$$

$$= \frac{4xy}{(x^2 + y^2)^2} [y dx - x dy]$$

iv) Find d if $q = f(L, K) = A \cdot L^{3/4} K^{1/4}$

Sol.

$$q = A L^{3/4} K^{1/4}$$

$$\frac{\partial q}{\partial L} = \frac{3}{4} A L^{-1/4} K^{1/4} = \frac{3}{4} A K^{1/4} L^{-1/4}$$

$$\frac{\partial q}{\partial K} = \frac{1}{4} A L^{3/4} K^{-3/4} = \frac{1}{4} A L^{3/4} K^{-3/4}$$

$$dq = \frac{\partial q}{\partial L} dL + \frac{\partial q}{\partial K} dK$$

$$= \left(\frac{3}{4} A K^{1/4} L^{-1/4} \right) dL + \left(\frac{1}{4} A L^{3/4} K^{-3/4} \right) dK$$

$$= \frac{3}{4} A L^{3/4} K^{1/4} \left[\frac{1}{L} dL + \frac{1}{K} dK \right]$$

$$= \frac{3}{4} q \left[\frac{1}{L} dL + \frac{1}{K} dK \right]$$

$$= \frac{3}{4} \left[\frac{q}{L} dL + \frac{q}{K} dK \right]$$

$$= \frac{3}{4} [AP_L dL + AP_K dK]$$

AP_L = Average product of labour, AP_K = Average Product of Capital

v) A production function is given by $q = 3L^{2/3} K^{1/3}$ where $60 - 2L - K = 0$, q = output, L = Labour, K = Capital. Find the least cost combination of labour and capital.

Sol. Given production function

$$q = 3 L^{2/3} K^{1/3} \quad (1)$$

$$\text{Where } 60 - 2L - K = 0 \quad (2)$$

From (1) & (2) we have

$$Z = 3 L^{2/3} K^{1/3} + \lambda (60 - 2L - K) \quad (3)$$

$$\frac{\partial Z}{\partial L} = 3 \cdot \frac{2}{3} L^{-1/3} K^{1/3} - 2\lambda = 2 L^{-1/3} K^{1/3} - 2\lambda = 0 \quad (4)$$

$$\frac{\partial Z}{\partial K} = 3 \cdot L^{2/3} \cdot \frac{1}{3} K^{-2/3} - \lambda = L^{2/3} K^{-2/3} - \lambda = 0 \quad (5)$$

$$\frac{\partial Z}{\partial \lambda} = 60 - 2L - K = 0 \quad (6)$$

From (4) and (5), $2L^{-1/3} K^{1/3} = 2\lambda$, $L^{2/3} K^{-2/3} = \lambda$

Dividing, we get

$$\frac{L^{-1/3} K^{1/3}}{L^{2/3} K^{-2/3}} = \frac{\lambda}{\lambda}$$

$$K = L$$

Substitute $K = L$, in (6), $60 - 2L - L = 0$, $3L = 60$ $L = 20$ $\therefore K = 20$

When $L = 20$, $K = 20$, Then the first condition of tangency is satisfied

$$\text{For Second condition } \frac{d^2 K}{dL^2} > 0$$

$$\frac{f_L}{f_K} = - \frac{2 L^{-1/3} K^{1/3}}{L^{2/3} K^{-2/3}} = -2 L^{-1} K$$

$$\frac{d}{dL} (-2 L^{-1} K) = 2 L^{-2} K + (-2 L^{-1}) \frac{dK}{dL}$$

$$= 2 L^{-2} K - 2 L^{-1} (-2 L^{-1} K)$$

$$= 6 L^{-3} K > 0$$

∴ output is maximum, when $L = 20$, $K = 20$

8.4.3 Examples of total derivative

i) If $U = f(x, y)$ and $y = f(x) = 4x - 7$, find the total derivative $\frac{\partial U}{\partial x}$ of the function.

$$U = x^3 + 3x^2 y + 6xy^2 + 2y^3$$

Sol : If $U = f(x, y)$ and $y = f(x)$, then the total derivative $\frac{\partial U}{\partial x}$ is given by

$$\frac{dU}{dx} = \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dx}$$

$$y = f(x) = 4x - 7$$

$$\therefore \frac{dy}{dx} = 4.$$

$$U = x^3 + 3x^2 y + 6xy^2 + 2y^3$$

$$\frac{\partial U}{\partial x} = 3x^2 + 6xy + 6y^2$$

$$\frac{\partial U}{\partial y} = 3x^2 + 12xy + 6y^2$$

$$\therefore \frac{dU}{dx} = (3x^2 + 6xy + 6y^2) + (3x^2 + 12xy + 6y^2) 4$$

$$= 3x^2 + 6xy + 6y^2 + 12x^2 + 48xy + 24y^2$$

$$= 15x^2 + 54xy + 30y^2$$

Example :- $U = (x^2 + y^2) \sqrt{x + y}$ and $y = 2 - 3x$. find the total derivative (du/dx)

Sol. $y = 2 - 3x$

$$\frac{dy}{dx} = -3$$

$$U = (x^2 + y^2) \sqrt{x + y}$$

$$\frac{\partial U}{\partial x} = (x^2 + y^2)^{\frac{1}{2}} \left[\frac{1}{2} (x + y)^{-\frac{1}{2}} \right] + (x + y)^{\frac{1}{2}} 2x$$

$$= \frac{x^2 + y^2}{2\sqrt{x+y}} + \sqrt{x+y} 2x$$

$$= \frac{x^2 + y^2 + 4(x+y)x}{2\sqrt{x+y}}$$

$$= \frac{x^2 + y^2 + 4x^2 + 4xy}{2\sqrt{x+y}}$$

$$= \frac{5x^2 + 4xy + y^2}{2\sqrt{x+y}}$$

$$\frac{\partial U}{\partial y} = (x^2 + y^2)^{\frac{1}{2}} \left[\frac{1}{2} (x + y)^{-\frac{1}{2}} \right] + (x^2 + y^2)^{\frac{1}{2}} 2y$$

$$= \frac{x^2 + y^2}{2\sqrt{x+y}} + 2y\sqrt{x+y}$$

$$= \frac{x^2 + 4xy + 5y^2}{2\sqrt{x+y}}$$

$$\frac{dy}{dx} = \frac{5x^2 + 4xy + y^2}{2\sqrt{x+y}} + \frac{x^2 + 4xy + 5y^2}{\sqrt{x+y}} \quad (-3)$$

$$= \frac{1}{2\sqrt{x+y}} [5x^2 + 4xy + y^2 - 3x^2 - 12xy - 15y^2]$$

$$= \frac{1}{2\sqrt{x+y}} [2x^2 - 8xy - 14y^2]$$

$$= \frac{1}{\sqrt{x+y}} \left[x^2 - 4xy - 7y^2 \right]$$

Example:- Find the total derivative $\left(\frac{dU}{dt}\right)$ if $U = f(x, y, t)$ where

$$x = a + bt \quad y = c + dt$$

Sol. $U = f(x, y, t; x = a + bt, y = c + dt)$

$$\therefore \frac{dU}{dt} = \frac{\partial U}{\partial x} \frac{dx}{dt} + \frac{\partial U}{\partial y} \frac{dy}{dt}$$

$$\frac{dx}{dt} = b, \frac{dy}{dt} = d$$

$$\text{Also } \frac{\partial U}{\partial x} = f_x(x, y, t), \frac{\partial U}{\partial y} = f_y(x, y, t) : 1$$

$$\therefore \frac{\partial U}{\partial t} = f_x(x, y, t)b + f_y(x, y, t)d$$

$$= f_x b + f_y d$$

ii) If $U = x^2 - 8xy - y^3$, $x = 3t$, $y = 1 - t$

$$\text{We know } \frac{dU}{dt} = \frac{\partial U}{\partial x} \frac{dx}{dt} + \frac{\partial U}{\partial y} \frac{dy}{dt}$$

$$\frac{\partial x}{\partial t} = 3, \quad \frac{\partial y}{\partial t} = -1$$

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial x} (x^2 - 8xy - y^3) = 2x - 8y$$

$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y} (x^2 - 8xy - y^3) = -8x - 3y^2$$

$$\text{Hence } \frac{dU}{dt} = (2x - 8y)(3) + [-8x - 3y^2](-1)$$

$$\begin{aligned}
 & -3(2x - 8y) + 8x + 3y^2 \\
 & = 6x - 24y + 8x + 3y^2 \\
 & = 14x - 24y + 3y^2
 \end{aligned}$$

Check Your Progress-III

1. If $z = \frac{x-y}{x+1}$, find the total derivative.

2. If $z = \frac{9x-7y}{2x+5y}$ and $y = 3x - 4$, find the total derivative.

8.5 Let Us Sum Up

Dear learner, we conclude this lesson by summarizing what we have covered in it :

- i) Partial derivative:- Finding the first order partial derivative of a function $U = f(x, y)$ w. r. t. x , we assume y remains fixed and the change in the function is due

to change in x and denote it by $\frac{\partial U}{\partial x}$ or $f(x)$, similarly if x remains fixed to the

change in the function is due to change in y , then $\frac{\partial U}{\partial y}$ or $f(y)$

- ii) Second order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} \text{ or } U_{xx} \text{ or } f_{xx} \text{ or } f_{11}$$

$$\frac{\partial^2 f}{\partial y^2} \text{ or } U_{yy} \text{ or } f_{yy} \text{ or } f_{22}$$

iii) Second order cross partial derivatives

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 u}{\partial y \partial x} \text{ or } f_{xy} \text{ or } f_{12} \text{ or } U_{12}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial y} \text{ or } f_{yx} \text{ or } f_{21} \text{ or } U_{21}$$

iv) Total derivatives

$$\frac{\partial u}{\partial x} = f_x + f_y \frac{dy}{dx}$$

$$\frac{\partial u}{\partial y} = f_x \frac{dx}{dy} + f_y$$

v) Total differentials

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = f_x dx + f_y dy$$

vi) Economic examples / application of partial derivatives, total derivatives and total differential.

8.6 Examination Oriented Questions

Q.1. Find the total differential of the following functions

i) $\frac{x^2 + y^2}{x - y}$ ii) $\frac{1}{\sqrt{x^2 + y^2}}$

Q.2. Find du, when

(i) $u = e^{2x^2 + 3y^2}$

(ii) $u = \text{Log} (3x^2 - y^2)$

Q.3. Find dq , if the production function be $q = a L^{3/4}, C^{3/4}$

Q.4. Find the total differential of

$$U = x^3 + 3x^2y + 6xy^2 + 2y^3$$

8.7 Hints to Check Your Progress

CYP-1

Q.1. Refer to Demand and Supply function under section 8.2.

Q.2. Refer to utility and production function under section

8.2

CYP-2

Q.1. Solve using the formulae given in section 8.3.1.

Q.2. Solve using the formulae given in section 8.3.1.

8.8 Suggested Readings

Aggarwal, C.S & R.C. Joshi: Mathematics for Students of Economics (New Academic Publishing Co.).

Allen, R.G.D. ; Mathematical Analysis for Economists (Macmillan).

Anthony Martin & Norman Biggs; Mathematics for Economics and Finance-Methods and Modelling.

Black, J & J.F. Bradley : Essential Mathematics for Economists (John Willey & Sons).

Dowling, Edward T : Introduction to Mathematical Economics (Tata Macgraw).

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India Pvt. Ltd).

M.A. Economics	Lesson No. 9
C.No. : Eco-103	Semester - 1st
	Unit II

EXPANSION OF PARTIAL AND TOTAL DIFFERENTIATION BY TAYLOR SERIES AND APPLICATIONS

STRUCTURE

- 9.0 Objectives/Learning Outcomes
- 9.1 Introduction
- 9.2 Taylor Series for Functions of a Single Variable
 - 9.2.1 Total Differential Concept
- 9.3 Taylor Series in Multivariable Functions
 - 9.3.1 Discuss Taylor Series Expansion for Two Variables
 - 9.3.2 Explain Total Differential in Multivariable Context
- 9.4 Discuss Application of Taylor Series in Economics
- 9.5 Let us Sum Up
- 9.6 Examination Oriented Questions
- 9.7 Hints to check your Progress
- 9.8 Suggested Readings

9.0 Objective/Learning Outcomes

After going through this lesson, you shall be able to :

- ☐ Know what is Taylor series.

- Describe Taylor series in single, two and multivariable Context.
- What are the applications of Taylor series in Economics.

9.1 Introduction

The Taylor series is a powerful mathematical tool that allows us to approximate a function as an infinite sum of terms calculated from the values of its derivatives at a single point. For a function $f(x)$ that is infinitely differentiable at a point 'a', its Taylor series is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

When we truncate this series after a few terms, we obtain an approximation that is especially useful when x is “close” to a . In many applications, including economics, a first- or second-order approximation can be sufficient for decision-making or sensitivity analysis.

9.2 Taylor Series for Functions of a Single Variable

9.2.1 Total Differentiation Concept

For a function $f(x)$, a small change Δx induces a change in the function that can be approximated using the Taylor expansion:

- **First-order (linear) approximation:**

$$\Delta f = f'(a)\Delta x$$

- **Second-order approximation:**

$$\Delta f = f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2$$

This expansion is particularly useful for estimating how small changes in the independent variable affect the value of the function.

9.3 Taylor Series in Multivariable Functions

When dealing with functions of several variables, such as $f(x, y)$, we use **partial derivatives** to capture the sensitivity of f with respect to each variable independently.

931 Taylor Series Expansion for Two Variables

For a function $f(x, y)$ that is twice continuously differentiable, its Taylor series expansion around a point (a, b) is given by:

$$\begin{aligned} f(x, y) = & f(a, b) \\ & + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ & + \frac{1}{2} \left[f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2 \right] \\ & + \text{higher order terms} \end{aligned}$$

Where :

- f_x and f_y denote the first-order partial derivatives
- f_{xx} , f_{yy} , and f_{xy} denote the second order partial derivatives

932 Total Differential in Multivariable Context

The **total differential** of $f(x, y)$ provides a linear approximation of the change in the function due to small changes in x and y :

$$d_f = f_x(x, y) dx + f_y(x, y) dy$$

This expression can be viewed as the first-order Taylor expansion (ignoring higher-order terms).

□ Applications in Economics

Taylor series approximations, partial differentiation, and total differentiation are widely applied in economic analysis. Some key applications include:

Marginal Analysis

□ Marginal Utility and Marginal Cost:

When evaluating how small changes in consumption or output affect utility or cost, economists use the first derivative (marginal utility or marginal cost) to approximate the change. For instance, if $U(x)$ represents

a utility function, then a small change Δx leads to an approximate change in utility:

$$\Delta U = U'(x)\Delta x$$

Sensitivity Analysis in Multivariable Models

□ Production Functions:

Consider a production function $Q = f(L, K)$ where L is labor and K is capital. The Taylor expansion provides:

$$\Delta Q = f_L(L, K) \Delta L + f_K(L, K) \Delta K$$

Where f_L and f_K are the marginal products of labor and capital respectively. This linear approximation helps in analyzing how changes in inputs affect total output.

Elasticities and Approximate Changes

□ Elasticity of Demand:

Elasticity measures the responsiveness of quantity demanded to changes in price. A first-order Taylor expansion helps to approximate the percentage change in demand given a small percentage change in price:

$$\text{Elasticity} = \frac{dQ}{dP} \cdot \frac{P}{Q}$$

Examples

Example 1: Single Variable Function

Suppose $f(x) = e^x$ and we want to approximate $f(x)$ near $x = 0$

□ First-Order Approximation:

$$f(x) = f(0) + f'(0)x = 1 + x$$

□ Second-Order Approximation:

$$f(x) = 1 + x + \frac{1}{2}x^2$$

This shows that for small x , e^x can be very accurately estimated by its Taylor series expansion.

Example 2: Function of Two Variables

Consider $f(x, y) = \log(xy)$ and approximate near the point $(1, 1)$

1. Calculate $f(1, 1)$:

$$f(1, 1) = \log(1 \cdot 1) = \log(1) = 0$$

2. Find the Partial Derivatives at $(1, 1)$:

$$\bullet \quad f_x(x, y) = \frac{1}{x} \text{ so } f_x(1, 1) = 1$$

$$\square \quad f_y(x, y) = \frac{1}{y} \text{ so } f_y(1, 1) = 1$$

3. Construct the First-Order Approximation:

$$f(x, y) \approx 0 + 1(x - 1) + 1(y - 1) = (x - 1) + (y - 1)$$

Thus, for small changes near $(1, 1)$, the natural logarithm of the product can be approximated linearly.

Example 3: Economic Application – Production Function

Assume a simple production function:

$$Q = AL^\alpha K^\beta, \quad \alpha = \text{alpha}$$

$$\beta = \text{Beta}$$

Where A , α & β are constants. To see how a small change in labour (L) and capital (K) affects output, we compute the total differentials :

1. Partial Derivatives:

$$\bullet \quad \frac{\partial Q}{\partial L} = \alpha A L^{\alpha-1} K^\beta$$

$$\bullet \quad \frac{\partial Q}{\partial L} = \beta A L^{\alpha} K^{\beta-1}$$

2. Total Differential:

$$dQ = \alpha A L^{\alpha-1} K^{\beta} dL + \beta A L^{\alpha} K^{\beta-1} dK.$$

This expression helps managers and economists to evaluate the impact on output from small changes in inputs L and K.

Check Your Progress

Q.1. Consider a function $f(x, y) = \log(x^2y)$ and approximate near the point (1, 1)

☐ Calculate $f(1, 1)$

☐ Find the partial derivative at (1, 1)

9.5 Let us Sum Up

We conclude their lesson by summarising what we covered in it :

- (i) Taylor series for functions of a single variable
- (ii) Total differential concept
- (ii) Taylor series in multivariable functions
- (iv) Total differential in multivariable context
- (v) Application of Taylor series in Economics

9.6 Examination Oriented Questions

Use the following questions to test your understanding of the concepts discussed:

1. Basic Taylor Series:

- Derive the Taylor series expansion for $\cos(x)$ about $x = 0$ up to the fourth-order term.

2. Single Variable Differentiation:

- For the function $f(x) = f_n(1 + x)$ obtain the first-order and second-order Taylor approximations near $x = 0$. How good is the approximation for $x = 0.1$?

3. Multivariable Expansion:

- For a function $f(x, y) = x^2y + y^3$, derive the Taylor series expansion up to the first-order terms around (1, 2) What is the approximate change in f if x increases by 0.05 and y decreases by 0.1 ?

4. Application in Economics – Total Differential:

- Given a production function $Q = L^{0.5}K^{0.5}$ calculate the marginal products of L and K at $L = 100$ and $K = 400$. Use the total differential to estimate the change in Q if L increases by 5 units while K decreases by 10 units.

5. Elasticity Concept:

- Explain how you would use a first-order Taylor approximation to derive the price elasticity of demand if the demand function is given $Q = 50 - 2P + 0.01 P^2$ at $P = 10$ Compute the elasticity using your approximation.

9.7 Hints to check your Progress

Refer to section 9.3.2 and solve using the technique as given in example-2.

9.8 Suggested Readings

Aggarwal, C.S & R.C. Joshi: Mathematics for Students of Economics (New Academic Publishing Co.).

Allen, R.G.D. ; Mathematical Analysis for Economists (Macmillan).

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M.A. Economics	Lesson No. 10
C.No. : Eco-103	Semester - 1st
	Unit II

CONCEPT OF MAXIMA AND MINIMA, CONSTRAINED AND UNCONSTRAINED, (ECONOMIC APPLICATIONS)

STRUCTURE

- 10.0 Objectives/Learning Outcomes
- 10.1 Introduction
- 10.2 Increasing and Decreasing Functions
 - 10.2.1 Convexity of Curves
 - 10.2.2 Maxima and Minima of Functions of one variable
 - 10.2.3 Maxima and Minima - An Alternative Approach
- 10.3 Constrained Optimization with Lagrange Multiplier
- 10.4 Application of Minima and Maxima in Economic Theory.
- 10.5 Let us sum up
- 10.6 Examination Oriented Questions
- 10.7 Hints to Check Your Progress
- 10.8 Suggested Readings

10.0 Objectives/Learning Outcomes

After reading this lesson, you should be able to :

- Discuss Increasing and decreasing functions

- Describe Convexity of curves
- Define Maxima and minima of functions without constraints.
- Define Maxima and minima of functions with constraints.
- Discuss their economic applications.

10.1 Introduction

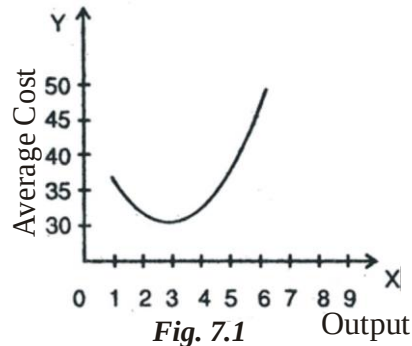
When we first introduced the term equilibrium in the first unit, we made a broad distinction between goal and non goal equilibrium. In the latter type, exemplified by our study of market and national-income models, the interplay of certain opposing forces in the model-e.g., the forces of demand and supply in the market models and the forces of leakages and injections in the income models- dictates an equilibrium state, if any, in which these opposing forces are just balanced against each other, thus obviating any further tendency to change. The attainment of this type of equilibrium is the outcome of the impersonal balancing of these forces and does not require conscious effort on the part of anyone to accomplish a specified goal. True, the consuming households behind the forces of demand and the firms behind the forces of supply are each striving for an optimal position under the given circumstances, but as far as the market itself is concerned, no one is aiming at any particular equilibrium price or equilibrium quantity. Similarly, in national income determination, the impersonal balancing of leakages and injections is what brings about an equilibrium state, and no conscious effort at reaching any particular goal needs to be involved at all.

In the present lesson, however, our attention will be turned to the study of goal equilibrium, in which the equilibrium state is defined as the optimum position for a given economic unit (a household, a business firm, or even an entire economy) and in which the said economic unit will be definitely striving for attainment of that equilibrium. As a result, only in this context, our earlier warning that equilibrium does not imply desirability will become irrelevant and immaterial.

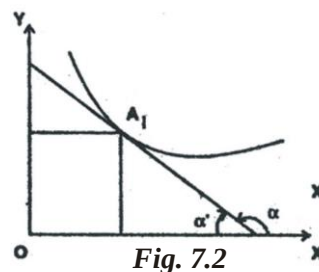
10.2 Increasing and Decreasing Functions

Let us assume x is output and y is average cost. Furthermore, let us assume that the relation between x and y is $y = 40 - 6x + x^2$. This can be thought of as an average cost function. Now, the question is, does y (average cost) increase/decrease or stay

stationary as x (output) increase. Let us first draw a graph (figure 7.1) of this cost function, which shows that as



x (output) increases, y (average cost) decreases, reaches a minimum, and then starts to increase. Graphing the first part of the function, we have figure (7.2). A function (curve) that is downward-sloping like this is called a decreasing function, that is, the value of the function y decreases as x increases. We shall show this mathematically, selecting a point A_1 on the curve and drawing a tangent to the curve. The slope of this tangent is given by $\tan \alpha$.



We also know that the derivative of the function at the point A_1 , is equal to the slope of the curve at that point. Thus

$$\frac{dy}{dx} = \tan \alpha$$

But we see graphically that $\alpha > 90^\circ$. We know from trigonometry that

$$\tan \alpha = \tan (180^\circ - \alpha') = -\tan \alpha'$$

For example, if $\alpha = 135$, then $\alpha' = 45$. Thus,

$$\tan 135^\circ = \tan (180^\circ - 45^\circ) = -\tan 45^\circ = -1 \quad [\square \tan 45^\circ = 1]$$

and the derivative will be equal to

$$\frac{dy}{dx} = \tan \alpha = \tan 135^\circ = -1 < 0$$

In general, when $\alpha > 90^\circ$, then $\frac{dy}{dx} < 0$

As a conclusion, we may say that $y = f(x)$ is a decreasing function at the point x when $f'(x) < 0$. By a similar argument, we may say that $y = f(x)$ is an increasing function at the point x when $f'(x) > 0$

Example : Using our average cost function, let us find whether it is a increasing or decreasing function at the values $x = 4$ and $x = 2$.

a) When $x = 4$, $y = 40 - 6x + x^2$

$$\begin{aligned} \frac{dy}{dx} &= 6 + 2x \\ \left(\frac{dy}{dx} \right)_{x=4} &= -6 + 8 = 2 \end{aligned}$$

b) When $x = 2$

$$\left(\frac{dy}{dx} \right)_{x=2} = -6 + 4 = -2$$

Thus, at $x = 2$, it is decreasing (cost) function. This is interpreted as follow. When x increase by a small amount, y will decrease by 2 units, at the point $x = 2$.

In case of (a), at $x = 4$, it is an increasing function. When $f'(x) = 0$, the slope of the curve is horizontal. That is, $\tan \alpha = \tan 0^\circ = 0$. At this point the curve is stationary.

Example 2 :

$$y = x^2 - 4x$$

$$\therefore \frac{dy}{dx} = 2x - 4$$

When $x = -1$, then $f'(x) = -6 < 0$. Thus, $y = f(x)$ is a decreasing function at the point where $x = -1$

When $x = 2$, then $f'(x) = 0$ and then $y = f(x)$ is a stationary function at $x=2$.

When $x=3$, then $f'(x)=6-4= 2>0$. Thus, $y =f(x)$ is an increasing function at this point.

10.2.1 Convexity of Curves

Consider a car that starts from a stand still position and reaches a certain speed after a certain amount of time. Let y (meters) be the distance the car has travelled and t (seconds) be time. We have the following relationship.

$$y = t^2 \quad \text{..... (1)}$$

Thus, when $t = 1$ second, then $y = 1$ meter; $t = 3$ seconds, then $y = 9$ meters, and so forth.

Let us differentiate with respect to t . Then,

$$\frac{dy}{dx} = 2t \quad \text{.....(2)}$$

dy/dt gives the velocity at time t . For example, when $t = 2$, i.e. 2 seconds after the car has started, $dy/dt = 4$ meters / second, and so forth.

Let us graph equation (1) $dy/dt = 2t$ shows the slope of this curve at various points. For example, at the point (3, 9) we have

$$\frac{dy}{dx} = \tan\alpha = 2 \times 3 = 6$$

$$\alpha = 80^\circ 30' \text{ (approximately)}$$

Now, let us differentiate (2) once more

$$\frac{d^2y}{dt^2} = 2 \quad \dots(3)$$

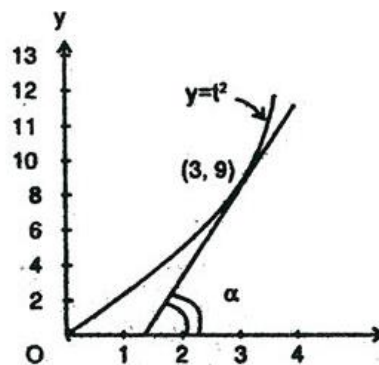


Fig. 7.3

This gives us the change in velocity per second, i.e., acceleration. We have 2 meters per second, as acceleration. This means velocity increases every second by 2 meters per second. Summarizing this in table form, we have

TABLE 1

t(seconds)	0	1	2	3	4
y(meters)	0	1	4	9	16
dy/dt (meters/second)	0	2	4	6	8
d ² y/dt ² (meters/second)	2	2	2	2	2

We are interested in the second derivative $d^2y/dt^2 = f''(t)$. This shows the rate of change. In our present case, $f''(t) = 2$ m/second² and it is constant, that is, the rate of change is the same. Thus, the velocity is increasing at a rate of 2m/ sec every second whether it is one second or four seconds after the start.

Using this discussion, we may define several terms. The function $y = f(x)$ in figure 4(a) is our present case and we shall say y is increasing at an increasing rate. This is shown by

$$\frac{d^2y}{dx^2} = f''(x) > 0$$

The curve $y=f(x)$ lies above the tangent, and we say the curve is concave upward or convex downward.

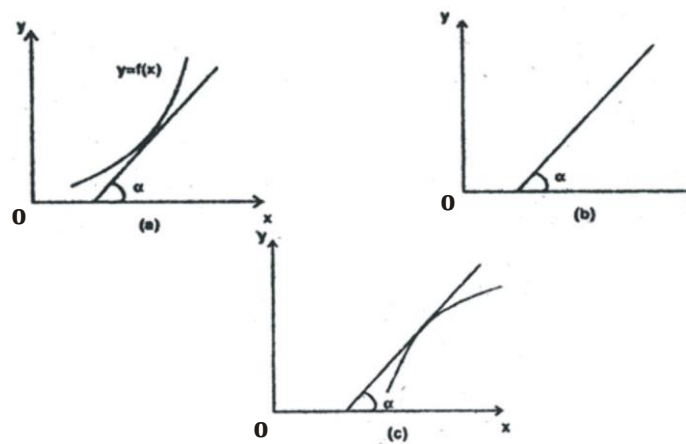


Fig. 7.4

The implication of $f''(x) > 0$ is that $f'(x) = \tan \alpha$ is increasing at the point of tangent. For $f'(x) = \tan \alpha$ to increase, needs to increase towards 90° . This can happen only if the curve gets steeper. Thus, the curve will be convex downwards as in figure 4(a)

Figure 4(b) shows where the rate of increase of y is zero. That is,

$$\frac{d^2y}{dx^2} = f''(x) = 0$$

For example, if

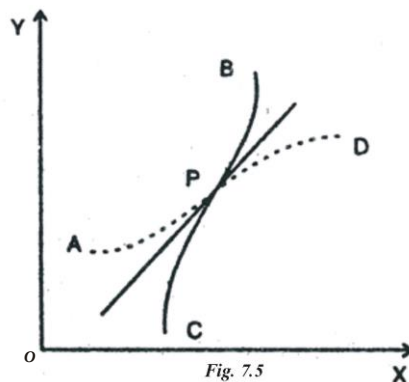
$$y = 2t, \frac{dy}{dt} = 2, \frac{d^2y}{dt^2} = 0$$

There is no curvature of the curve. The curve is a straight line.

Figure 4(c) shows where the rate of increase of y is decreasing. This is written

$$\frac{d^2y}{dt^2} = f''(x) < 0$$

To summarize, the condition for convex downward is $f''(x) > 0$, straight line, $f''(x) = 0$; and convex upwards, $f''(x) < 0$. In the special case where we have two curves tangent to each other, we can draw a tangent going through the point P where they touch. Then the curve APB will be above and the curve CPD will be below the tangent line.



If we look upon CPB as a curve and imagine the point P moving along the curve, then, when P is between C and D of the curve; $f'' < 0$ because CD is convex upward. But, from point P to B, since APB is convex downward, $f'' > 0$. Thus, at point P, f'' changes signs and at that point, $f'' = 0$. This point P is called the **point of inflexion**.

Example-1

$$y = x^2 - 4x, f = 2x - 4, f'' = 2 > 0$$

Thus convex downward

Example-2

$$y = \frac{1}{2}x^4 - 3x^2, f = 2x^3 - 6x, f'' = 6x^2 - 6$$

The point of inflexion is where $f''(x) = 0$.

Thus, let

$$6x^2 - 6 = 0, \quad x^2 - 1 = 0, \quad (x+1)(x-1) = 0$$

$$\therefore x = 1, \quad x = -1$$

Thus, the point of inflexion will be at

$x = 1$ and $x = -1$, For $x = 1$, we have

$$y = \frac{1}{2}x^4 - 3x^2$$

$$= \frac{1}{2} - 3 = -\frac{5}{2} = -2.5$$

and one of the points of inflexion is at (1, -2.5)

10.2.2 Maxima and Minima of functions of one Variable

i) Maxima and Minima of functions

Referring to figure 7.6, we see that points on either side of B_1 in a small neighbourhood are lower than B_1 . Then, the function $y = f(x)$ has a maximum value at the point B_1 . In a similar value, we define B_2 as the point where the function $y = f(x)$ has a minimum value. The maximum and minimum values together are called the extreme values of the function. As can be seen, as the domain of x is enlarged, other maximum and minimum values may occur at different points. To emphasize that B_1 and B_2 may not be the only extreme values, we sometimes say relative maximum or minimum values or relative extreme values.

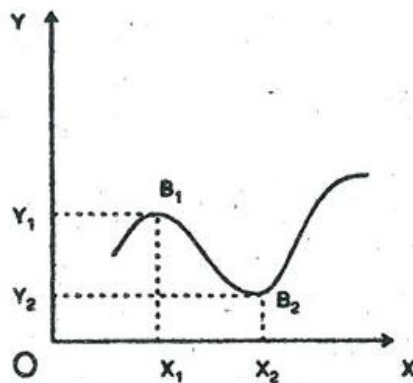


Fig. 7.6

Symbolically, this may be shown as follows:

Let $\epsilon > 0$ be a small number, Then,

$$f(x_1 - \epsilon) < f(x_1) > f(x_1 + \epsilon) \quad \dots \text{Maximum}$$

$$f(x_2 - \epsilon) < f(x_2) > f(x_2 + \epsilon) \quad \dots \text{Minimum}$$

ii) Necessary and sufficient conditions for an extreme value

Assume we have two functions as shown in figures 7.7(a) and 7.7(b). From our previous discussion, we know that when $dy/dx = 0$, the tangent to the curve at that point will be parallel to the x-axis. Let A and

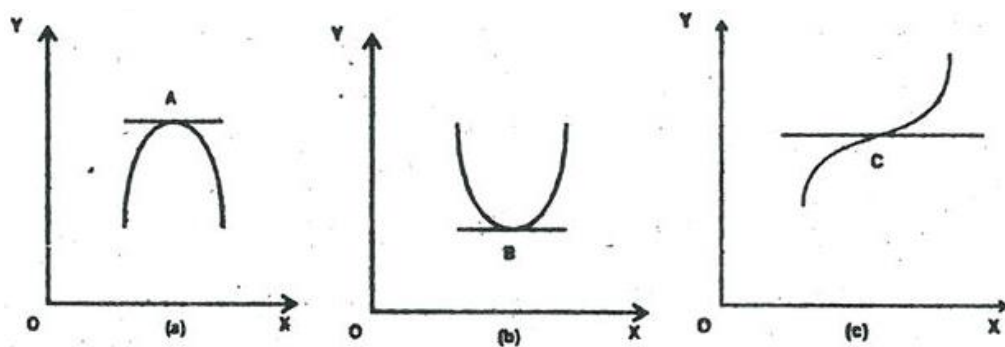
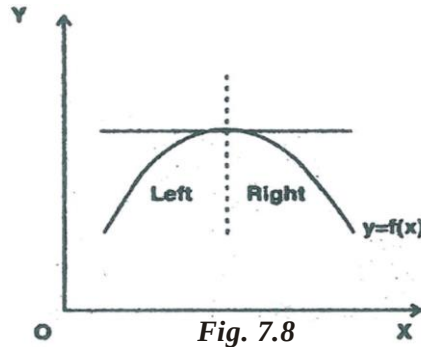


Fig. 7.7

B be two such point, then the tangents are parallel to the x-axis. As is evident from the graph, point A is a maximum and point B a minimum. In other words, the necessary conditions for a certain point such as A or B to be an extreme value is that $dy/dx = 0$ at that point.

From our discussion of the convexity of curves, however, we may have a situation such as in figure 7.7(c) where the tangent going through the point of inflexion is parallel to the x-axis. Thus, $dy/dx = 0$ is a necessary condition for $y = f(x)$ to have an extreme value but it is not a necessary and sufficient condition. Therefore, two questions arise: (1) how may we be certain that we do not have a point of inflexion such as 7.7(c) and (2) how to tell whether it is maximum or minimum? For this we can use our knowledge of the curvature of curves. Looking at figure 7.8 below we see that, for the left side of the curve, we have an increasing curve, that is, $dy/dx > 0$. But we also see that it is increasing at a decreasing rate. That is, $d^2y/dx^2 < 0$, and thus the slope dy/dx must gradually approach zero.



Looking on the right side of the curve, we see it is a decreasing curve. Thus $\frac{dy}{dx} < 0$. But we see it is decreasing at an increasing rate. Since $\frac{dy}{dx} < 0$ this means that $\frac{d^2y}{dx^2} < 0$. Putting the two sides together, the curvature of this curve is shown by $\frac{d^2y}{dx^2} < 0$. So, when we have a point that satisfies.

$$\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} < 0$$

We must have a maximum, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ are the necessary and sufficient conditions for the occurrence of a maximum. Note that $\frac{d^2y}{dx^2} < 0$ alone tells us the curvature of the curve. Only when we have the necessary condition $\frac{dy}{dx} = 0$ does $\frac{d^2y}{dx^2} < 0$ have meaning as a sufficient condition with respect to the occurrence of a maximum.

In a similar manner, we find that

$$\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} > 0$$

are the necessary and sufficient conditions for a minimum.

As was just discussed, the necessary and sufficient conditions need to be considered together. But, in many cases in economics, the $\frac{d^2y}{dx^2} > 0$ needs to be discussed for its implications. We shall call this, and its corresponding part in functions of more than two variables, the sufficient conditions with the understanding of the preceding discussion.

Example: Our cost function was

$$y = 40 - 6x + x^2$$

Then the necessary condition for this to have an extreme value is

$$\frac{dy}{dx} = -6 + 2x$$

$$\frac{dy}{dx} = 0$$

$$\therefore -6 + 2x = 0$$

$$x = 3$$

Now, is the value of y a maximum or minimum at $x = 3$?

For this we determine the second derivative to find the curvature of the curve. This is

$$\frac{d^2y}{dx^2} = 2 > 0$$

Thus, the curve is convex downwards and the function has a minimum value at $x = 3$, that is, at $x = 3$, we have the minimum average cost which is

$$y = 40 - 6x + x^2 = 40 - 6(3) + 3^2$$

$$= 40 - 18 + 9$$

$$= 31$$

10.2.3 Maximum and Minima - An Alternative Approach

An alternative way is to consider the change of signs of $f'(x)$

Maximum : $f'(x)=0$, $f'(x)$ changes from + to -

Minimum : $f'(x)=0$, $f'(x)$ changes from - to +

A heuristic explanation of this is obtained by studying figure 7.6

Example 1. Find the maximum and minimum value of

$$2x^3 - 15x^2 + 36x + 20$$

Sol : Let $y = 2x^3 - 15x^2 + 36x + 20$

$$\begin{aligned}\frac{dy}{dx} &= 6x^2 - 30x + 36 \\ &= 6(x^2 - 5x + 6)\end{aligned}$$

For stationery value, $\frac{dy}{dx} = 0$

$$6(x^2 - 5x + 6) = 0$$

$$6(x - 3)(x - 2) = 0$$

Thus, stationary values occur at $x=2$ and $x=3$

Now, $\frac{d^2y}{dx^2} = 12x - 30$

For $\frac{d^2y}{dx^2} = 24 - 30 = -6 < 0$

At $x = 2$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} < 0$ Hence y is maximum and maximum

value of y is

$$\begin{aligned}y &= 2(2)^3 - 15(2)^2 + 36(2) + 20 \\ &= 2 \times 8 - 15 \times 4 + 72 + 20 \\ &= 16 - 60 + 72 + 20 = 48\end{aligned}$$

At $x = 3$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} < 0$ Hence y is minimum and value is

$$y = 2(3)^3 - 15(3)^2 + 36(3) + 20$$

$$= 54 - 135 + 108 + 20$$

$$= 47$$

Example 2. Show that the maximum value of the function $y = x^3 - 27x + 108$ is 108 more than the minimum value

Sol: Let $y = x^3 - 27x + 108$

$$\frac{dy}{dx} = 3x^2 - 27$$

$$\therefore \frac{dy}{dx} = 0$$

$$3x^2 - 27 = 0$$

$$x^2 = 9$$

$$\therefore x = \pm 3$$

Now, $\frac{d^2y}{dx^2} = 6x$

For $x = \pm 3$

$$\frac{d^2y}{dx^2} = 18 > 0,$$

At $x = 3, \frac{dy}{dx} = 0, \frac{d^2y}{dx^2} > 0$ Hence at $x = 3$, y is minimum and value is

$$\begin{aligned} y &= x^3 - 27x + 108 \\ &= 27 - 27(3) + 108 \\ &= 27 - 81 + 108 \\ &= 54 \end{aligned}$$

At $x = -3, \frac{dy}{dx} = 0, \frac{d^2y}{dx^2} > 0$ Hence at $x = -3$, y is maximum and max. value is

$$\begin{aligned}
 y &= x^2 - 27x + 108 \\
 &= (-3)^2 - 27(-3) + 108 \\
 &= 9 + 81 + 108 \\
 &= 198
 \end{aligned}$$

∴ Max. value (198) is 90 more than min. value (108)

Check Your Progress-I

- Find the maximum and minimum value of $(x - 2)^6 (x - 3)^5$

- Show that $x^3 - 3x^2 + 3x + 7$ has no maximum and no minimum value

10.3 Constrained Optimization with Lagrange Multiplier

Differential calculus is also used to maximize or minimize a function subject to a constraint. Given a function $f(x, y)$, subject to a constraint $I(x, y) = k$ (constant), a new function F can be formed (1) setting the constraint equal to zero (2), multiplying it by λ (the Lagrange multiplier), and (3) adding the product to the original equation

$$F(x, y, \lambda) = f(x, y) + \lambda[k - g(x, y)]$$

Here $F(x, y, \lambda)$ is the Lagrangian function, $f(x, y)$ is the original or objective function, and $g(x, y)$ is the constraint. Since the constraint is always set equal to

zero, the product $\lambda[k-g(x,y)]$ also equals zero and the addition of the term does not change the value of the objective function. Critical values x_0 , y_0 and λ_0 , at which the function is optimized, are found by taking the partial derivatives of F with respect to all three independent variables, setting them equal to zero, and solving simultaneously.

$$F_x(x, y, \lambda) = 0; \quad F_y(x, y, \lambda) = 0; \quad F_\lambda(x, y, \lambda) = 0$$

Second order conditions differ from those of unconstrained optimization and are treated as under. The second order conditions can now be expressed in terms of a bordered Hessian H

$$|\bar{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & F_{xx} & F_{xy} \\ g_y & F_{yx} & F_{yy} \end{vmatrix}$$

which is simply, the plain Hessian $\begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix}$ bordered by the first derivative

of the constraint with zero on the principle diagonal. The order of a bordered principle minor is determined by the order of the principle minor being bordered.

Hence $|\bar{H}|$ above represents a second bordered principal minor H_2 because the principal minor being bordered is 2×2 .

For a function in n variables $f(x_1, x_2, \dots, x_n)$ subject to $g(x_1, x_2, \dots, x_n)$

$$|\bar{H}| = \begin{vmatrix} F_{11} & F_{12} & \dots & F_{1n} & g_1 \\ F_{21} & F_{22} & \dots & F_{2n} & g_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{n1} & F_{n2} & \dots & F_{nn} & g_n \\ g_1 & g_2 & \dots & g_n & 0 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 0 & g_1 & g_2 & \dots & g_n \\ g_1 & F_{11} & F_{12} & \dots & F_{1n} \\ g_2 & F_{21} & F_{22} & \dots & F_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & F_{n1} & F_{n2} & \dots & F_{nn} \end{vmatrix}$$

where $|\bar{H}| = |\bar{H}_n|$ because of the $n \times n$ principle minor being bordered.

If all the principle minors are negative i.e. if $H_2 < 0$, $H_3 < 0$, $H_4 < 0$, the bordered definite Hessian always satisfies the sufficient condition for a relative minimum.

If the principle minors alternate consistently in sign from positive to negative i.e., if $H_2 > 0$, $H_3 < 0$, $H_4 > 0$, etc. the bordered Hessian is negative definite and a negative definite Hessian always meets the sufficient condition for a relative maximum.

Example : Optimize the function, $Z = 4x^2 + 3xy + 6y^2$ subject to the constraint $x + y = 56$

Solution: Setting the constraint equal to zero

$$56 - x - y = 0 \quad \text{---- (1)}$$

Multiply this difference by λ and add the product of the two to the objective functions in order to form the Lagrangian function Z

$$Z = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y) \quad \text{---- (2)}$$

$$Z_x = 8x + 3y - \lambda = 0 \quad \text{---- (3)}$$

$$Z_y = 3x + 12y - \lambda = 0 \quad \text{---- (4)}$$

$$Z_\lambda = 56 - x - y = 0 \quad \text{---- (5)}$$

Subtracting (4) from (3), to eliminate λ , gives

$$5x - 9y = 0$$

$$5x = 9y$$

$$x = \frac{9}{5}y$$

Substitute in (5) $\frac{9}{5}y$

$$56 - x - y = 0$$

$$56 - 1.8y - y = 0$$

$$2.8y = 56$$

$$y = 20$$

we can also find $x = 36$

Substitute the critical values in (2)

$$\begin{aligned} Z &= 4(36)^2 + 3(36)(20) + 6(20)^2 + 348(56-36-20) \\ &= 4(1296) + 3(720) + 6(400) + 348(0) \\ &= 9744 \end{aligned}$$

Also, we can find

$$Z_{xx} = 8, \quad Z_{yy} = 12, \quad Z_{xy} = Z_{yx} = 3$$

From the constraint, $x + y = 56$, we get $g_x = 1$ and $g_y = 1$. Thus,

$$|H| = \begin{vmatrix} 8 & 3 & 1 \\ 3 & 12 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$|\bar{H}_2| = |\bar{H}| = 8(-1) + 3(1) + 1(3 - 12)$$

$$= -8 + 3 - 9$$

$$= -14$$

with $|\bar{H}_2| < 0$ $|\bar{H}|$ is positive definite, which means that Z is at a minimum

Example : The generalized Cobb-Douglas production function $q = K^{0.4}L^{0.5}$, given a budget constraint of Rs. 108 when $P_K = 3$, $P_L = 4$

Solution : $Q = (K^{0.4})(L^{0.5})$

$$Q_K = 0.4(K)^{-0.6}, L^{0.5};$$

$$Q_L = 0.5, K^{0.4}, L^{-0.5}$$

$$Q_{KK} = (0.4)(-0.6) K^{-1.6}, L^{0.5};$$

$$Q_{LL} = (0.5)(-0.5) K^{0.4}, L^{-1.5}$$

$$Q_{KK} = -0.24 K^{-1.6}, L^{0.5};$$

$$Q_{LL} = -0.25, K^{0.4}, L^{-1.5}$$

$$\begin{aligned} \text{Also } Q_{KL} &= Q_{LK} = (0.4)(0.5) K^{-0.6}, L^{-0.5} \\ &= 0.20 K^{-0.6}, L^{-0.5} \end{aligned}$$

and from the constraint $3k + 4L = 108$

$$g_K = 3, g_L = 4$$

$$H = \begin{vmatrix} -0.24K^{1.6} L^{0.5} & 0.20K^{-0.6} L^{-0.5} & 3 \\ 0.2K^{-0.6} L^{-0.5} & -25K^{0.4} L^{-1.5} & 4 \\ 3 & 4 & 0 \end{vmatrix}$$

Starting with $|H_2|$ and expanding along the third row

$$\begin{aligned} H &= 3(0.8K^{-0.6} L^{-0.5} + 0.75K^{-0.4} L^{-1.5}) \\ &\quad - 4(-0.96K^{-1.6} L^{0.5} - 0.6K^{-0.6} L^{-0.5}) \\ &= 2.25K^{0.4} L^{-1.5} + 4.8K^{-0.6} L^{-0.5} + 3.84K^{-1.6} L^{0.5} \\ &= \frac{2.25K^{0.4}}{L^{1.5}} + \frac{4.8}{K^{0.6} L^{0.5}} + \frac{3.84L^{0.5}}{K^{1.6}} > 0 \end{aligned}$$

with $|H_2| > 0$; $|H|$ is negative definite and Q is maximized

Examples of constrained maxima and minima

Example : Find the extremum (if any) of the function by Lagrange's method

$U = f(x, y) = 12xy - x^2 - 3y^2$ subject to the constraint $x + y = 16$

Solution : The given objective function is

$$U = f(x, y) = 12xy - x^2 - 3y^2 \quad \text{--- (1)}$$

$$\text{and the constraint is } \phi(x, y) = x + y - 16 \quad \text{--- (2)}$$

$\therefore$ The Lagrange's function is

$$F = F(x, y; \lambda) = f(x, y) - \lambda \phi(x, y) \quad \text{--- (3)}$$

$$= 12xy - x^2 - 3y^2 - \lambda(x + y - 16) \quad \text{--- (4)}$$

a) First order conditions

$$\left. \begin{aligned} F_x = 12y - 2x - \lambda &= 0 \\ F_y = 12x - 6y - \lambda &= 0 \\ F_\lambda = -(x + y - 16) &= 0 \end{aligned} \right\} \text{ or } \begin{aligned} 12y - 2x &= \lambda & \text{--- (5)} \\ 12x - 6y &= \lambda & \text{--- (6)} \\ x + y &= 16 & \text{--- (7)} \end{aligned}$$

from (5) and (6)

$$12y - 2x = 12x - 6y$$

$$\text{or } 14x = 18y$$

$$\text{or } 7x = 9y \text{ or } y = 7/9x \quad \text{--- (8)}$$

Putting this value of y in (7), we get

$$\begin{aligned} x + \frac{7}{9}x &= 16 \\ \frac{16}{9}x &= 16 \quad \text{or } x = 9 \end{aligned}$$

from (8), $y = 7$

From (5), $12(7) - 2(9) = \lambda$; $\lambda = 66$

Hence, there is one critical point $(x^*, y^*) = (9, 7)$ and $\lambda = 66$

b) Second order conditions (SOC) $d^2F < 0$ for max/min
since $\phi(x, y) = x + y - 16$, $\phi_x = 1$, $\phi_y = 1$

$$F_{xx} = -2, F_{yy} = -6, f_{xy} = 12 = f_{yx}$$

Applying bordered Hessian Determinant (BHD) method

At the C.P. (9, 7), $\phi_1 = 1, \phi_2 = 1, F_{11} = -2, F_{12} = 12 = F_{21}, F_{22} = -6$

$$\therefore \Delta \left[\text{or } H \right] = \begin{vmatrix} 0 & \phi_1 & \phi_2 \\ \phi_1 & F_{11} & F_{12} \\ \phi_2 & F_{21} & F_{22} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 12 \\ 1 & 12 & -6 \end{vmatrix}$$

$$\begin{aligned}
 &= 0 () -1 [(-6-12) + 1(12+2)] \\
 &= -1 (-18) + 1(14) \\
 &= 18 + 14 \\
 &= 32 > 0
 \end{aligned}$$

which $\Rightarrow$ that the function $U = f(x, y) = 12xy - x^2 - 3y^2$ subject to the constraint $\phi(x, y) = x + y - 16 = 0$ is maximum at the C.P (9, 7) and the constrained maximum value = $\text{Max } f = f(9, 7) = 12(9)(7) - (9)^2 - 3(7)^2$

$$= 756 - 81 - 147$$

$$= 756 - 228 = 528$$

10.4 Application of Minima and Maxima in Economic Theory

Maxima and Minima is a potent device with the students of economics confronting problems in economic theory. It is a common experience that a rational consumer thinks always in terms of maximum utility, whereas a prudent producer strives at maximizing profits and choosing the least cost combination. As explained in the previous sub-unit, the technique of determining maximum and minimum value of a function of one variable. Even this limited knowledge will suffice to illustrate the use of this technique in solving many economic problems.

1.a) Maximum Revenue

When the demand function takes the form $P = K - aQ$, then

$$TR = KQ - aQ^2 \quad [\because TR = \text{Price} \times \text{Quantity}]$$

This is a relation between TR and Q and TR will be at a maximum when the first and second-order conditions hold, i.e. when

$$d(TR)/dQ = 0$$

$$\text{and} \quad d^2(TR)/dQ^2 < 0$$

$$\text{when } \frac{d(TR)}{dQ} = 0,$$

$$K - 2aQ = 0 \text{ or } Q = K/2a$$

$$d^2(TR)/dQ^2 = -2a < 0$$

Thus, TR is maximum when $Q = K/2a$

1.(b) Maximum Revenue and Elasticity

The first order condition for a maximum is $d(TR)/dQ = 0$ or $MR = 0$, TR remains unchanged as Q changes and $E_D = -1$

In the above example, TR is at a maximum when $Q = K/2a$

$$E_D \frac{P}{Q} \frac{dQ}{dP} = \frac{K - aQ}{Q} \left(-\frac{1}{a} \right)$$

Since $P = K - aQ$ and $dQ/dP = -1/a$

When $Q = K/2a$

$$\begin{aligned} E_D &= \frac{K - a \cdot \frac{K}{2a}}{\frac{K}{2a}} \left(-\frac{1}{a} \right) \\ &= \frac{K - K/2}{\frac{K}{2a}} \left(-\frac{1}{a} \right) \\ &= \frac{K/2}{\frac{K}{2a}} \left(-\frac{1}{a} \right) \\ &= a(-1/a) = -1 \end{aligned}$$

2 Minimum Average Costs

If a firm's TC function is

$TC = K + aQ + bQ^2$, where K , a and b are positive constants, then

$$AC = TC/Q$$

$$= \frac{K}{Q} + a + bQ$$

Average costs are at a minimum when $d(AC)/dQ = 0$ and $d^2(AC)/dQ^2 > 0$

$$d \frac{(AC)}{dQ} = -\frac{K}{Q^2} + b = 0$$

$$\text{i.e. } Q^2 = \frac{K}{b}$$

$$Q = \sqrt{K/b}$$

Second order condition

$$d^2 \frac{(AC)}{dQ} = \frac{2K}{Q^3} > 0 \text{ for all } Q > 0$$

Thus AC is at a minimum when $Q = \sqrt{K/b}$

3. Minimum Marginal Costs $\sqrt{K/b}$

If a firm's TC function is

$TC = K + aQ - bQ^2 + cQ^3$, where K , a , b , and c are positive constant, then

$$d(TC)/dQ = a - 2bQ + 3cQ^2 = MC$$

MC is at a minimum when

$$d(MC)/dQ = 0$$

$$\text{and } d^2(MC)/dQ^2 > 0$$

when $d(MC)/dQ = 0$

$$-2b + 6cQ = 0$$

$$Q = \frac{2b}{6c} = \frac{b}{3c}$$

$$d(MC)^2/dQ^2 = 6c > 0 \text{ } c \text{ is positive}$$

Thus, MC is at a minimum when $Q = b/3c$

4.a) Profit Maximization

The conditions for a maximum profit derived can be used

$$\text{Profit } S = TR - TC$$

If $TC = K + aQ + bQ^2$ and the demand function takes the form $P = L - nQ$, where k, a, b, L and n are positive constants

$$\begin{aligned} TC &= P \times Q \\ &= LQ - nQ^2 \end{aligned}$$

Then

$$\begin{aligned} S &= LQ - nQ^2 - K - aQ - bQ^2 \\ &= -(b+n)Q^2 + (L-a)Q - K \end{aligned}$$

S is now a function of Q and takes a maximum when

$$dS/dQ = 0 \text{ and } d^2S/dQ^2 < 0 \text{ when } \frac{dS}{dQ} = 0$$

$$-2(b+n)Q + L - a = 0$$

$$Q = \frac{L - a}{2(b + n)}$$

$$d^2S/dQ^2 = -2(b+n) < 0 \text{ since } b \text{ and } n \text{ are positive}$$

Therefore profits are maximized when

$$Q = (L-a)/2(b+n)$$

4.b) The Profit Maximization Conditions

$$S = TR - TC$$

$$\text{and } dS/dQ = \frac{d(TR)}{dQ} - \frac{d(TC)}{dQ}$$

Since S, TR and TC are all functions of Q.

When $dS/dQ = 0$,

$$\frac{d(TR)}{dQ} - \frac{d(TC)}{dQ} = 0$$

$$\text{or } MR - MC = 0$$

Consequently, the first-order condition for a maximum is $MR = MC$

We know that revenue is maximized when $MR = 0$, or $E_D = -1$. A profit maximizing firm will be in equilibrium when $MR = MC$ and since MC must be positive MR will take a positive value in equilibrium. Consequently, a profit maximizing firm will be in equilibrium when $-E_D > 1$. When

$$= \frac{d^2(TR)}{dQ^2} - \frac{d^2(TC)}{dQ^2} \text{ or } \frac{d(MR)}{dQ} - \frac{d(MC)}{dQ} \text{ must be } < 0$$

Hence profits are maximized when $MR = MC$ and $d(MR)/dQ - d(MC)/dQ$ is negative. Figure (7.9) shows that profits are maximized at M. At M, $MC = MR$,

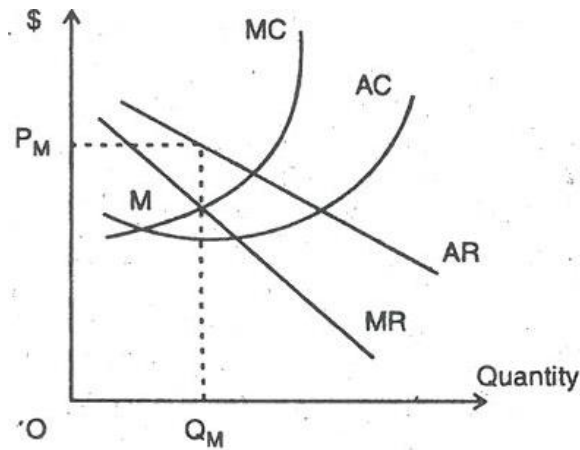


Fig. 7.9

MR is downwards sloping, i.e. the slope of the tangent to the MR curve is negative at every point, thus

$$d(MR)/dQ < 0 \text{ at } M$$

MC is upward sloping at M, thus

$$d(MC)/dQ > 0$$

and
$$\frac{d(MR)}{dQ} - \frac{d(MC)}{dQ} < 0$$

In general, however, we require only that $d(MR)/dQ - d(MC)/dQ < 0$, so that $d(MC)/dQ$ need not be positive.

5. Effect of Taxation on the output of a profit maximizing firm

Earlier in this unit, we saw that profits are maximized when

$$Q = \frac{L - a}{2(b + n)}$$

given that $TC = K + aQ + bQ^2$ and $P = L - nQ$. The levying of a lumpsum tax, i.e. a tax which does not depend on output, on this firm, will have no effect on the equilibrium output. This tax will increase the value of the constant K in the TC

function and consequently decrease profits by the amount of tax. However, the equilibrium output will remain unchanged since the constant term disappears with differentiation.

A tax which varies with output will affect the equilibrium output and profits. If the government imposes a tax of 't' per unit of quantity produced, then TC will increase by tQ or S will decrease by tQ, i.e.

$$\begin{aligned} S &= -(b+n) Q^2 + (L-a) Q - K - tQ \\ &= -(b+n) Q^2 + (L-a-t) Q - K \end{aligned}$$

when $dS/dQ = 0$

$$\frac{dS}{dQ} = -2(b+n)Q + (L-a-t) = 0$$

Thus $Q = \frac{L-a-t}{2(b+n)}$

$$\frac{d^2S}{dQ^2} = -2(b+n) < 0, \text{ since } b+n \text{ are } +ve$$

Thus profits are maximized when

$Q = \frac{L-a-t}{2(b+n)}$. The optimum output falls as a result of the per unit tax: t is subtracted

from the numerator and has the effect of decreasing Q. A per unit subsidy would have the opposite effect.

6. Maximization of Tax Revenue

A per unit tax on quantity produced causes a profit maximizing firm to cut back production. Since total tax revenue T depends upon the tax rate t and the output level Q, i.e. $T = tQ$, it is possible to find the tax rate which maximizes total tax revenue from the point of view of the exchequer.

In the previous example, equilibrium quantity depended on the tax rate t,

i.e., $Q = (L-a-t)/2(b+n)$

$$T = tQ = t \cdot \left(\frac{L - a - t}{2(b + n)} \right)$$

$$= \frac{L - at - t^2}{2(b + n)}$$

We now have T as a function of t, alternatively T could be expressed as a function of Q.

Tax revenue T is maximized when $dT/dt = 0$ and $d^2T/dt^2 < 0$. When $\frac{dT}{dt} = 0$

$$\frac{dT}{dt} = \frac{t - a - 2t}{2(b + n)} = 0$$

i.e. $L - a - 2t = 0$

$$L - a = 2t$$

$$t = \frac{L - a}{2}$$

$$\frac{d^2T}{dt^2} = -\frac{2}{2(b + n)} = -\frac{1}{b + n} < 0$$

Thus T is maximized when $t = (L - a)/2$

When $t = (L - a)/2$

$$Q = \frac{L - a - (L - a)/2}{2(b + n)}$$

$$= \frac{2L - 2a - L - a}{4(b + n)}$$

$$= \frac{L - a}{4(b + n)}$$

T = tQ, thus

$$T = \left(\frac{L-a}{2} \right) \left(\frac{L-a}{4(b+n)} \right)$$

$$= \frac{(L-a)^2}{8(b+n)}$$

7. The supply function and a per unit tax

Consider an industry with the following demand and supply functions

$$P = K - \alpha Q_D$$

$$\text{and } P = -K_1 + \beta Q_S$$

where K , K_1 , α and β are positive constants. If the government imposes a tax of 't' per unit of quantity produced and P is the market price to consumers, then the effective price for producers is $(P-t)$. Therefore, the supply function which includes the tax is $P-t = -K_1 + \beta Q_S$

$$\text{or } P = -K_1 + \beta Q_S + t$$

At equilibrium, $Q_S = Q_D$, so

$$K - \alpha Q = K_1 + \beta Q_S + t$$

$$Q = \frac{K + K_1 - t}{\alpha + \beta}$$

Equilibrium Q is reduced as a result of the tax. To find the tax rate which maximizes total tax revenue T , one can express T as a function of t . Alternatively T can be expressed as a function of Q . At equilibrium

$$t = K + K_1 - (\alpha + \beta)Q$$

$$T = t Q$$

Thus

$$T/Q = K + K_1 - (\alpha + \beta)Q$$

$$T = KQ + K_1 Q - (\alpha + \beta)Q^2$$

T is at a maximum when $DT/dQ = 0$ and $d^2T/dQ^2 < 0$.

When $dT/dQ = 0$

$$\frac{dT}{dQ} = K + K_1 - 2(\alpha + \beta)Q$$

$$K + K_1 - 2(\alpha + \beta)Q = 0$$

i.e. $Q = \frac{K + K_1}{2(\alpha + \beta)}$

$$\frac{d^2T}{dQ^2} = -2(\alpha + \beta) < 0 \quad \alpha + \beta \text{ are +ve}$$

Thus, T is maximized when

$$Q = (K + K_1) / 2 (\alpha + \beta)$$

Once Q is known, t and T can be found by substitution

The conditions for a maximum or minimum discussed above apply only to continuous and smooth, i.e. without sharp bends. Most of the functions dealt with in this unit are quadratics so that the relative maximum or minimum of any of these functions is also the absolute maximum or minimum. With functions of degree 3 the relative maximum or minimum will not be the absolute maximum or minimum.

Cubic Functions

$$Y = ax + bx^2 + cx^3$$

where $c > 0$; thus

$$dy/dx = a + 2bx + 3cx^2$$

This function will have a maximum and minimum when

$$a + 2bx + 3cx^2 = 0$$

Assuming real roots, the values of x which satisfy this equation will only provide a relative maximum or minimum. There is no limit to the value of y in the

upward or downward direction, i.e. no absolute maximum or minimum, since larger positive values for x will keep increasing y and larger negative values will keep decreasing y .

Numerical Exercises

1. Example: If $P = \frac{121}{q+4} - 1$. Find the output level at which total revenue is maximum also find maximum revenue.

Sol. $P = \frac{121}{q+4} - 1$

$$R = \frac{121 \times q}{q+4} - q$$

$$\frac{dR}{dq} = \frac{484}{(q+4)^2} - 1$$

$$\frac{d^2R}{dq^2} = \frac{-968}{(q+4)^3}$$

For stationary values, $\frac{dR}{dq} = 0$

$$\therefore \frac{484}{(q+4)^2} - 1 = 0$$

$$484 = (q+4)^2$$

$$22 = q+4$$

$$q = 18$$

when $q = 18$, $\frac{d^2R}{dq^2}$ is -ve

$\therefore R$ is maximum at $q = 18$ and maximum revenue is

$$R = \frac{121 \times 18}{22} - 18$$

$$= \frac{18[121-18]}{22}$$

$$= \frac{18 \times 99}{22} = 81$$

2 Example: For a firm under perfect competition total cost function is given by

$$\pi = \frac{1}{25} q^3 - \frac{9}{10} q^2 + 10q + 12. \text{ If the price is Rs. 4 per unit, will the firm continue}$$

production?

$$\text{Sol. } MC = \frac{d\pi}{dq} = \frac{3}{25} q^2 - \frac{9}{5} q + 10$$

The first order condition for a firm under perfect competition for profit maximization is $MC = \text{Price}$

$$\text{i.e. } \frac{3q^2}{25} - \frac{9}{5} q + 10 = 4$$

$$\frac{3q^2}{25} - \frac{9}{5} q + 6 = 0$$

$$\frac{q^2}{25} - \frac{3}{5} q + 2 = 0$$

$$q^2 - 15q + 50 = 0$$

$$\begin{aligned} \therefore q &= \frac{15 \pm \sqrt{(15)^2 - 4(1)(50)}}{2} \\ &= \frac{15 \pm \sqrt{225 - 200}}{2} = \frac{15 \pm \sqrt{25}}{2} \\ &= \frac{15 \pm 5}{2} = 5, 10 \end{aligned}$$

The second order condition for profit maximization is

$$\frac{d^2\pi}{dq^2} \text{ must be + ve}$$

Now

$$\text{For } q = 5, d^2\pi/dq^2 = \frac{6}{25} \times 5 - \frac{9}{5} = -\frac{3}{5} \text{ which is - ve}$$

$$q = 10, d^2\pi/dq^2 = \frac{6}{25} \times 5 - \frac{9}{5} = -\frac{3}{5} \text{ which is + ve}$$

Profits are maximum when $q = 10$

$$\begin{aligned} \text{At } q = 10, \text{ Profit} &= Pq - \left[\frac{1}{25}q^3 - \frac{9}{10}q^2 + 10q - 12 \right] \\ &= 40 - \left[\frac{1}{25}(10)^3 - \frac{9}{10}(10)^2 + 10(10) - 12 \right] \\ &= 40 - [40 - 90 + 100 - 12] \\ &= 40 - [62] = -22 \end{aligned}$$

Maximum profit is a loss of Rs. 22. The fixed cost in this case is Rs. 12 and loss of more than the fixed cost. If the firm discontinues production when $p=4$, its loss is reduced to Rs. 12. Hence, the firm will produce nothing if $p=4$

Check Your Progress-II

1. Determine the minimum value of the cost function

$$C = q^2 - 4q + 100$$

-
-
2. Given $p = 200 - .02q$ and $c = 100q + 30000$. A tax of Rs. 50 per unit of output is levied. Determine the quantity at which profit is maximum. Find the maximum profit and price.
-
-
-
-

10.5 Let Us Sum Up

We conclude this lesson, by summarizing what we covered in it increasing and decreasing function, maxima and minima without constraints and maxima and minimum with constraints and application of minima and maxima in economic theory.

10.6 Examination Oriented Questions

- Q.1 State the conditions for the existence of maxima and minima of a function $y = f(x)$
- Q.2 What do you mean by constraint maximization and also write the necessary and sufficient conditions of constrained optimization.
- Q.3 Show that the demand curve
- $$y = 80 - 5x - x^2$$
- is downward sloping and concave from below.
- Q.4 A fixed plant is used to manufacture radio sets and, if x sets are turned out per week, the total variable cost is

$$RS \left(3x + \frac{1}{25}x^2 \right)$$

Show that the average a variable cost increases steadily with output

- Q.5 Show that the maximum value of $\left(\frac{1}{x} \right)^x$ occurs at $x = 1/2$
- Q.6 Divide 24 into 2 parts, each part being positive real number, such that the product is maximum.
- Q.7 Find the maximum or minimum values of $x^2 \cdot e^{1/x^2}$
- Q.8 Find the minimum value of the function $U = x^2 + 3y^2 + 5z^2$ under the condition $2x + 3y + 5z = 100$
- Q.9 If $U = x^2y$ is a utility function find the equilibrium if budget constraint of $x + 2y = 4$
- Q.10 Given the utility function $U = (x+2)(y+1)$ and the budget constraint $2x + 5y = 51$, find the optimal levels of x and y purchases by the consumer.

10.7 Hints to Check Your Progress

CYP-I

Solve using example 1 and 2 given under section

10.2.3.

CYP-2

Solving using examples given under section 10.4

10.8 Suggested Readings

Aggarwal, C.S & R.C. Joshi: Mathematics for Students of Economics (New Academic Publishing Co.)

Allen, R.G.D. ; Mathematical Analysis for Economists (Macmillan)

D:\Himmat 2025\Less 2025\M.A. Semester-I Economics (ECO-103) 2025\M.A. Semester-I Economics (ECO-103) 2025 (Pages 136-349).p65 (**Set-HS**)

Anthony Martin & Norman Biggs; Mathematics for Economics and Finance-Methods and Modeling

Black, J & J.F.Bradley :Essential Mathematics for Economists (John Willey & sons)

Dowling, Edward T: Introduction to Mathematical Economics (Tata Macgraw)

Henderson, James M & Richard E Quandt:Microeconomic Theory- AMathematical Approach (Mcgraw-Hill International Book Company)

Kandoi B: Mathematics for Business and Economics with Applications (Himalaya Publishing House)

Yamane Taro: Mathematics for Economics-A Elementary Survey (Prentice Hall of India Pvt. Ltd.).

Aggarwal S.C., Rana R.K., Gupta Leena : Mathematics for Economists (VK Global Publications Pvt. Ltd.)

M.A. Economics	Lesson No. 11
C.No. : Eco-103	Semester - 1st
	Unit II

PRINCIPLE OF INTEGRATION INDEFINITE, DEFINITE, ECONOMIC APPLICATION

Structure

- 11.0 Objectives/Learning Outcomes
- 11.1 Introduction
- 11.2 Definition
 - 11.21 Standard Forms of Integrals
 - 11.22 Some more Standard Forms
- 11.3 Definite Integrals
- 11.4 Economic Application of the Integral
- 11.5 Let us sum up
- 11.6 Examination Oriented Questions
- 11.7 Hints to Check Your Progress
- 11.8 Suggested Readings

11.0 Objectives/Learning Outcomes

After going through this lesson, you would be able to :

- Discuss Principles of integration
- Define Definite Integrals

- Explain Indefinite integrals
- Describe Economic application

11.1 Introduction

The first limiting process discussed was differentiation. From a geometrical standpoint, differentiation was a study of the tangent of a curve. The second limiting process we take up is integration. In geometrical terms will be a study of area under a curve. Analytically speaking, however, integration and differentiation do not depend on geometry. A geometrical interpretation is used only to help foster intuition.

11.2 Definition

If $\frac{d}{dx} [f(x)] = F(x)$, then $f(x)$ is the integral (or indefinite integral) of

$F(x)$ w.r.t. x and we write it as :

$$\int F(x) dx = f(x)$$

It is read as integral of $F(x) dx$ is $f(x)$ The symbol $\int$ stands for integral of $F(x)$ and dx . Here $F(x)$ is called the integrand. The process of finding the integral of a function is called integration.

11.2.1 Standard Form of Integrals

Suppose we wish to find the integral of X^2 ; i.e. X^2 is the derivative of a function and we want to discover the function.

$$X^2 = f(X)$$

$$f(X) = \int f(X) dx = ?$$

We know that $\frac{d}{dx} X^2 = 2X$, thus

$$\frac{d}{dx} \left(\frac{X^3}{3} \right) = \frac{3X^2}{3} = X^2$$

Consequently, the function whose derivative is x^2 equals $x^3/3$, $x^3/3$, is not integral of x^2 but an integral. Consider the function.

$$\frac{x^3}{3} + K$$

Where K is constant. The derivative of $(x^3/3)+K$ with respect to X is also X^2 . Hence the function whose derivative equals x^2 is $(x^3/3)+K$, where K is called the constant of integration. Further information is required before a definite value can be assigned to K.

The function whose derivative is X^{10} is given by

$$f(x) = \int X^{10} dx = \frac{X^{11}}{11} + K$$

Where K is the constant of integration

It is possible to check by taking the derivative of $(X^{11}/11+K)$, with respect

$$\text{to } x, \text{ e.g. } \frac{d}{dx} \left[\frac{X^{11}}{11} + K \right] = \frac{11X^{10}}{11} = X^{10}$$

Suppose we wish to find the integral of X^n , i.e.

$$X^n = f(x)$$

then

$$f(x) = \int X^n dx = \frac{X^{n+1}}{n+1} + K$$

Because

$$\frac{d}{dx} \left[\frac{X^{n+1}}{n+1} + K \right] = \frac{(n+1)X^n}{n+1} = X^n$$

When $n=4$, then

$$f(x) = \int x^4 dx = \frac{x^{4+1}}{4+1} = K = \frac{x^5}{5} + K$$

When $n = -2$, then

$$f(x) = \int x^{-2} dx = \frac{x^{-2+1}}{-2+1} + K = -\frac{1}{x} + K$$

This works for all values of n except $n = -1$

1. Integral of a Constant

If the constant 'b' is the derivative of a function, i.e. $b = f'(x)$, then.

$$f(x) = \int b \cdot dx = bx + K$$

$$\text{because } \frac{d}{dx}[bx + K] = b$$

2. Integral of Sum :

The integral of $x^\alpha + x^\beta + x^\gamma + b$, where, α, β, γ and b are constant is

$$\begin{aligned} \int x^\alpha dx + \int x^\beta dx + \int x^\gamma dx + \int b dx \\ = \frac{x^{\alpha+1}}{\alpha+1} + \frac{x^{\beta+1}}{\beta+1} + \frac{x^{\gamma+1}}{\gamma+1} + bx + K \end{aligned}$$

3. Integral of Multiple

The integral of a constant multiple of any integrand is the constant times the integral of that integrand and, e.g.

$$\begin{aligned} \int b(x^\alpha + a) dx &= b \int (x^\alpha + a) dx \\ &= b \left[\frac{x^{\alpha+1}}{\alpha+1} + ax + K \right] \end{aligned}$$

Where k is the constant of integration

1122 Some Standard Forms :

$$1. \int \frac{1}{x} dx = \log x + k \quad \left[\frac{d}{dx} (\log x + k) = \frac{1}{x} \right]$$

$$2 \quad \int a^x dx = \frac{a^x}{\log a} + k \left[\frac{d(a^x + k)}{dx} = a^x \log a \right]$$

$$3 \quad \int e^x dx = e^x + k \left[\frac{d(e^x + k)}{dx} = e^x \right]$$

1. Example : Integrate the following w.r.t. x

$$\frac{a+x}{x}$$

$$= \int \left(\frac{a+x}{x} \right) dx$$

$$\left(\frac{a}{x} + 1 \right) dx = a \frac{1}{x} dx + 1 dx = a \log x + x$$

$$= \int \left(\frac{a}{x} \right) dx + \int 1 dx$$

$$2. \quad \int \frac{1}{\sqrt{x-1} - \sqrt{x+1}} dx$$

$$= \int \left[\frac{1}{\sqrt{x-1}\sqrt{x+1}} \times \frac{\sqrt{x+1} + \sqrt{x+1}}{\sqrt{x-1} + \sqrt{x+1}} \right] dx$$

$$= \int \frac{\sqrt{x-1} + \sqrt{x+1}}{\underline{x-1-x-1}} dx$$

$$= \frac{1}{2} \int (\sqrt{x-1} + \sqrt{x+1}) dx$$

$$= \frac{1}{2} \left[\int (x-1)^{1/2} dx + \int (x+1)^{1/2} dx \right]$$

$$= -\frac{1}{2} \left[\frac{(x-1)^{3/2}}{3/2} + \frac{(x+1)^{3/2}}{3/2} \right]$$

$$= -\frac{1}{2} \left[(x-1)^{3/2} + (x+1)^{3/2} \right]$$

A. Integration By Partial Fraction

$$\int \frac{1}{x^2 - 16} dx$$

$$\text{Let } I = \int \frac{1}{x^2 - 16} dx = \int \frac{1}{(x + 4)(x - 4)} dx$$

Now let

$$\therefore \frac{1}{x^2 - 16} = \frac{A}{x + 4} + \frac{B}{x - 4}$$

$$1 = A(x - 4) + B(x + 4)$$

Put $x = 4$, we get $B = 1/8$

$x = -4$, we get $A = -1/8$

$$\begin{aligned} \therefore \int \frac{1}{x^2 - 16} dx &= \int \left[\frac{1}{8(x + 4)} + \frac{1}{8(x - 4)} \right] dx \\ &= -\frac{1}{8} \int \frac{1}{x + 4} dx + \frac{1}{8} \int \frac{1}{x - 4} dx \\ &= -\frac{1}{8} \log(x + 4) + \frac{1}{8} \log(x - 4) \\ &= \frac{1}{8} [-\log(x + 4) + \log(x - 4)] \\ &= \frac{1}{8} [\log(x - 4) - \log(x + 4)] \\ &= \frac{1}{8} \log \frac{x - 4}{x + 4} \end{aligned}$$

B. Integration By Substitution

Suppose, we have to evaluate, $\int \frac{1}{(a+x)^2} dx$ if we put $x = \phi(z)$ and $dx = \phi'(z) dz$

and evaluate the resulting integral $\int \frac{1}{x} dx$, the process of integration becomes

simple. Illustrating by taking a example.

$$\text{Show that } \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} \quad n \neq -1$$

Let $ax+b = z$ differentiating both sides

$$a dx = dz$$

$$dx = \frac{dz}{a}$$

$$\text{Now, } \int (ax+b)^n dx = \int z^n \frac{dz}{a} = \frac{1}{a} \int z^n dz$$

$$= \frac{1}{a} \frac{z^{n+1}}{n+1} \text{ provided } n \neq -1$$

$$= \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} \quad \square Z = ax + b$$

$$\text{Evaluate } \int x\sqrt{2x+1} dx$$

$$\text{Let } \sqrt{2x+1} = z$$

Squaring both sides

$$2x+1 = z^2$$

$$2dx = 2zdz$$

$$dx = z.dz$$

$$\text{From above, } x = \frac{z^2 - 1}{2}$$

$$\int x\sqrt{2x+1} dx = \int \frac{z^2 - 1}{2} z.z.dz$$

$$= \frac{1}{2} \left[\int (z^2 - 1)z^2 dz \right]$$

$$\begin{aligned}
 &= \frac{1}{2} \left[\frac{z^5}{5} - \frac{z^3}{3} \right] \\
 &= \frac{1}{2} \left[\frac{(2x+1)^{5/2}}{5} - \frac{(2x+1)^{3/2}}{3} \right] \\
 &= \frac{(2x+1)^{3/2}}{2} \left[\frac{2x+1}{5} - \frac{1}{3} \right] \\
 &= \frac{(2x+1)^{3/2}}{2} \left[\frac{6x+3-5}{15} \right] \\
 &= \frac{(2x+1)^{3/2}}{2} \cdot \frac{(6x-2)}{15} \\
 &= \frac{1}{15} (2x+1)^{3/2} (3x-1)
 \end{aligned}$$

C. Three Important Forms:

$$\begin{aligned}
 1. \quad \int [f(x)]^n f'(x) dx &= \frac{[f(x)^{n+1}]}{n+1} \quad (n \neq -1) \\
 2. \quad \int \frac{f'(x)}{f(x)} dx &= \log f(x) \\
 3. \quad \int e^{f(x)} \cdot f'(x) dx &= e^{f(x)}
 \end{aligned}$$

Examples : Integrate w.r.t.x

$$1. \quad \int \frac{x}{\sqrt[3]{x^2+1}}$$

$$x(x^2+1)^{-1/3} dx = \frac{1}{2} \int (x^2+1)^{-1/3} 2x dx$$

$$\int \left\{ \text{Form} \int [f(x)]^n \cdot f'(x) dx \right\}$$

$$= \frac{1(x^2+1)^{-1/3+1}}{2 - 1/3 + 1}$$

$$= \frac{1(x^2+1)^{2/3}}{2 - 1/3 + 1} = \frac{3}{4} (x^2+1)^{2/3}$$

$$2. \quad \int \frac{ax^{n-1}}{bx^n + c} dx = a \int \frac{x^{n-1}}{bx^n + c} dx$$

$$= \frac{a}{bn} \int \frac{bn \cdot x^{n-1}}{bx^n + c} dx$$

$$\left[\text{Form} \int \frac{f'(x)}{f(x)} dx \right] \\ = \frac{a}{bn} \log(bx^n + c)$$

$$3. \quad \int x^2 \cdot e^{2x^3+4} dx$$

$$= \frac{1}{6} \int 6x^2 \cdot e^{2x^3+4} dx$$

$$\left\{ \text{Form} \int f'(x) \cdot e^{f(x)} dx \right\}$$

$$= \frac{1}{6} \cdot e^{2x^3+4}$$

D. Integration by Parts

Integral of the product of two functions

= first function x integral of the second - Integral of (the derivative of first x Integral of second)

$$\int \left[\int f(x) \cdot g(x) dx - \int f(x) dx \cdot \int g(x) dx \right] dx$$

Example : $\int x^2 \cdot e^x dx$

$$= x^2 \int e^x dx - \int 2x \cdot e^x dx$$

Taking x^2 as the first function

$$= x^2 \cdot e^x - 2 \int x \cdot e^x dx$$

Taking x as the first function

$$= x^2 \cdot e^x - 2 \left\{ x \cdot \int e^x dx - \int e^x dx \right\}$$

$$= x^2 \cdot e^x - 2xe^x + 2e^x$$

$$= [x^2 - 2x + 2] \cdot e^x$$

Check Your Progress-I

1. Solve the following:

a) $\int (x^2 - 2x + 3) dx$

b) $\int x \cdot \log x dx$

c) $\int \frac{x}{x^2 - 3x + 2} dx$

11.3 The Definite Integrals

The area under a graph of a continuous function from a to b ($a < b$) can be expressed more clearly as the definite integral of $f(x)$ over the interval a to b. Mathematically,

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} \sum_{i=1}^n f(x_i) \Delta x_i$$

Here the left hand side is read, "the integral from 'a' to 'b' of f of x dx". Here 'a' is called the lower limit of integration and 'b' the upper limit of integration. Unlike the indefinite integral which is a set of functions containing all the antiderivatives of $f(x)$ as explained before, the definite integral is a real number which can be evaluated by using the fundamental theorem of calculus. This theorem states that the numerical value of the definite integral of continuous function $f(x)$ over the interval from a to b is given by the indefinite integral $F(x) + c$, evaluated at the upper limit of integration b, minus the same indefinite integral $F(x) + c$ evaluated at the lower limit of integration a. Since c is common to both, the constant of integration is eliminated in subtraction expressed mathematically

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

Where the symbol $\int_a^b$ indicates that b and a are to be substituted successively for x.

Example: the definite integral $\int_1^4 10x dx$ are evaluated as follows

$$\begin{aligned}\int_1^4 10x \, dx &= 5x^2 \Big|_1^4 = 5(4)^2 - 5(1)^2 \\ &= 5 \times 16 - 5 \\ &= 80 - 5 \\ &= 75\end{aligned}$$

2. The definite integral $\int_1^3 (4x^3 + 6x) \, dx$

Evaluated

$$\begin{aligned}\int_1^3 (4x^3 + 6x) \, dx &= [x^4 + 3x^2]_1^3 \\ &= [3^4 + 3 \cdot 3^2] - [1^4 + 3 \cdot 1^2] \\ &= [81 + 27] - [1 + 3] \\ &= 108 - 4 = 104\end{aligned}$$

Properties of definite integrals

- (i) Reversing the order of the limits changes the sign of the definite integral

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx$$

- (ii) If the upper limit of integration equals the lower limit of integration the value of the definite integral is zero.

$$\int_a^a f(x) \, dx = F(a) - F(a) = 0$$

- (iii) The definite integral can be expressed as the sum of component sub integrals.

$$\begin{aligned}\int_a^c f(x) \, dx &= \int_a^b f(x) \, dx + \int_b^c f(x) \, dx \\ a &\leq b \leq c\end{aligned}$$

- (iv) The sum or difference of two definite integrals with identical limits of integration is equal to the definite integral of the sum or difference of the two functions.

$$\int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx = \int_a^b [f(x) \pm g(x)] \, dx$$

- (v) The definite integral of the constant times a function is equal to the constant times the definite integral of the function.

$$\int_0^n f(x) dx = k \int_0^b f(x) dx$$

Numerical examples based on properties of definite integrals.

Example : Show that

$$\int_{-4}^4 (8x^3 + 9x^2) dx = \int_{-4}^0 (8x^3 + 9x^2) dx + \int_0^4 (8x^3 + 9x^2) dx$$

$$\begin{aligned} \text{L.H.S. } \int_{-4}^4 (8x^3 + 9x^2) dx &= 8 \frac{x^4}{4} + \frac{9x^3}{3} \Big|_{-4}^4 \\ &= 2x^4 + 3x^3 \Big|_{-4}^4 \\ &= 2(4)^4 + 3(4)^3 - [2(-4)^4 + 3(-4)^3] \\ &= 2(256) + (64) = [2(256) - 3(64)] \\ &= 512 + 192 - [512 - 192] \\ &= 704 - 320 \\ &= 384 \end{aligned}$$

$$\begin{aligned} \text{R.H.S. } \int_{-4}^4 (8x^3 + 9x^2) dx &= \int_{-4}^0 (8x^3 + 9x^2) dx + \int_0^4 (8x^3 + 9x^2) dx \\ &= \left[\frac{8x^4}{4} + \frac{9x^3}{3} \right]_{-4}^0 + \left[\frac{8x^4}{4} + \frac{9x^3}{3} \right]_0^4 \\ &= 2x^4 + 3x^3 \Big|_{-4}^0 + 2x^4 + 3x^3 \Big|_0^4 \\ &= -[2(-4)^4 + 3(-4)^3] + 2(4)^4 + 3(4)^3 \\ &= -[512 - 192] + (512 + 192) \\ &= -320 + 704 \\ &= 384 = \text{RHS} \end{aligned}$$

Example : Integrate the following definite integral by means of the substitution method

$$\int_0^3 \frac{6x}{x^2 + 1} dx$$

Solution : Let $U = x^2 + 1$, $\frac{dU}{dx} = 2x$ and
 $dx = dU / 2x$ substitutive

$$\int \frac{6x}{x^2 + 1} dx = \int \frac{6x}{2u} dx = 3 \int x \cdot u^{-1} du$$

Integrating with respect to u

$$3 \int u^{-1} du = 3 \ln|u|$$

Substitutive $U = x^2 + 1$

$$\begin{aligned} \int \frac{6x}{x^2 + 1} dx &= 3 \ln|x^2 + 1| \Big|_0^3 \\ &= 3 \ln|3^2 + 1| - 3 \ln(0^2 + 1) \\ &= 3 \ln 10 - 3 \ln 1, \text{ since } \ln 1 = 0 \\ &= 3 \ln 10 = 6.9078 \end{aligned}$$

Example : Suppose we are asked to find the integral of the MC function.

$$MC = a + bQ$$

Where a and b > 0, over the Q = m to Q = n where m and n are constants, m being the smaller. This is written in the form.

$$\int_{Q=m}^{Q=n} (a + bQ) dQ \text{ or } \int_m^n (a + bQ) dQ$$

Q = m the lower limit and

Q = n the upper limit.

The indefinite form of the integral is first of all calculated and bracketed and the

limits are written outside the bracket. The bracket is then evaluated when $Q = m$, the lower limit, and this value is subtracted from the value of the bracket when $Q = n$, e.g.

$$\begin{aligned} \int_m^n (a + bQ) dQ &= \left[aQ + \frac{bQ^2}{2} \right]_m^n \\ &= \left[aQ + 0.5bQ^2 \right]_m^n \\ &= [an + 0.5bn^2] - [am + 0.5bm^2] \\ &= a(n - m) + 0.5b(n^2 - m^2) \end{aligned}$$

The constant of integration need not be included when dealing with definite integrals.

This result gives us a measure of total variable costs when output increases from m to n . The definite integral is thus an area under a given curve since the area under the marginal cost equals total variable cost.

Check Your Progress-II

1. Evaluate

a) $\int_0^4 e^{3x+2} dx$

b) $\int_0^2 2^x 3^{-x} dx$

11.4 Economic Application of The Integral

1. Marginal and Total Costs :-

$$MC = d(TC)/dQ$$

$$\text{Thus } TC = \int MC \, dQ + k$$

Hence if the MC function of a firm is $MC = a + bQ$

$$\begin{aligned} TC &= \int (a + bQ) \\ &= aQ + 0.5b^2Q^2 \end{aligned}$$

2. The MPC and the Consumption Function

Given that the marginal propensity to consume is 0.8, i.e.,

$$dc/dy = 0.8,$$

the consumption function will take the form

$$C = \int MPC \, dy = \int 0.8 \, dy = 0.8y + k$$

The constant of integration k gives the level of consumption when income is zero.

3. Marginal and Total Revenue

$$MR = d(TR)/dQ$$

$$\text{Thus } TR = \int MR \, dQ$$

If a firm's MR function is

$$MR = a - bQ.$$

Where a and b are positive constants, then

$$TR = \int (a - bQ) \, dQ = aQ - 0.5bQ^2 + k$$

k will equal zero because TR is zero when Q is zero

4. Measuring Changes in Capital Stock

Net investment I is the rate of change in capital stock i.e. $dk/dt = I$. If net investment is a function of time, it is possible to calculate the changes in capital stock over some period of time by finding the definite integral of I with respect to time.

$$\begin{aligned} \text{If } I &= \int_0^{t^*} \alpha t^\beta dt = \left[\frac{\alpha t^{\beta+1}}{\beta+1} \right]_0^{t^*} \\ &= \frac{\alpha (t^*)^{\beta+1}}{\beta+1} - 0 = \frac{\alpha (t^*)^{\beta+1}}{\beta+1} \end{aligned}$$

This is also area under the investment curve from $t = 0$ to $t = t^*$.

5. Consumer's Surplus

A demand relation gives the price at which any given quantity could be sold. However, in a competitive market the price does not reflect

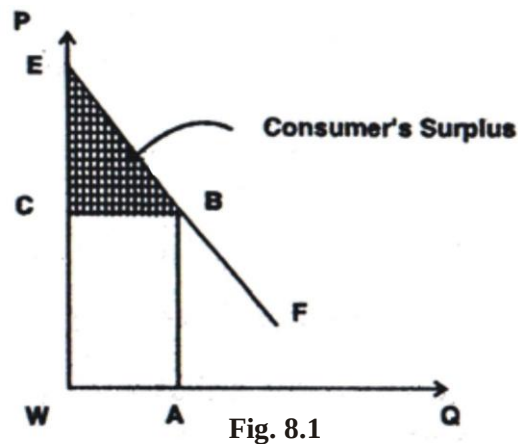


Fig. 8.1

what consumers would be willing to pay for each unit of the good rather than go without. In fact, the price reflects the valuation they place on the last unit they buy.

The demand curve and Consumer's Surplus

The figure 8.1 shows a demand function EF, with price on the vertical and quantity demanded on the horizontal axis. At price WC, quantity demanded is WA. WABC is the amount paid by consumers. However the benefit by consumers is the total area under the demand function over the range WA, i.e.

$$WABE = \text{benefit derived by consumer}$$

Thus

Consumer's Surplus = WABE - WABC = CBE. Assuming the demand function is $P = a - bQ$, where a and b are positive constants, then

$$WABE = \int_0^{q^*} (a - bQ) dQ$$

where $q^* = WA$

$$\begin{aligned} \int_0^{q^*} (a - bQ) dQ &= \left[aQ - 0.5bQ^2 \right]_0^{q^*} \\ &= aq^* - 0.5b(q^*)^2 - a(0) - 0.5b(0) \end{aligned}$$

$$\text{i.e. } WABE = aq^* - 0.5b(q^*)^2$$

$$WABE = P \cdot Q = Pq^*$$

But at B, $P = a - bq^*$, thus

$$WABC = (a - bq^*)q^* = aq^* - b(q^*)^2$$

$$\begin{aligned} \text{Consumer's Surplus} &= aq^* - 0.5b(q^*)^2 - aq^* + bq^2 \\ &= 0.5b(q^*)^2 \end{aligned}$$

6. Producer's Surplus

In figure 8.2, producers will supply quantity WA at WC price. At this price, producer's surplus is the area of the rectangle WABC minus the area under the supply curve DE over the range WA.

$$\text{Producer's Surplus} = WABC - WABD = DBC$$

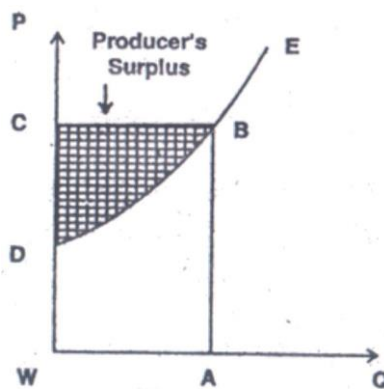


Fig. 8.2

The supply curve and producer's surplus

The supply curve is $P = (a+2Q)^2$ and $WA = q^*$. So that

$$\int (a+2Q)^2 dQ = \frac{(a+2Q)^3}{6} + k$$

Thus

$$\begin{aligned} WABD &= \int_0^{q^*} (a+2Q)^2 dQ = \left[\frac{(a+2Q)^3}{6} \right]_0^{q^*} \\ &= \frac{(a+2q^*)^3}{6} - \frac{a^3}{6} \\ WABC &= P.Q = P.q^* = (a+2q^*)^2 q^* \\ &= (a^2 + 4aq^* + 4q^{*2}) q^* = a^2 q^* + 4a q^{*2} + 4q^{*3} \\ \text{Producer's Surplus} &= q^*(a+2q^*)^2 - \frac{a^3}{6} \\ &= \left(a^2 + 4aq^* + 4q^{*2} \right) q^* - \frac{a^3}{6} \\ &= a^2 q^* + 4a q^{*2} + 4q^{*3} - \frac{a^3}{6} \end{aligned}$$

The integral of $(a-2Q)^2$ with respect to Q is

$$\frac{(a-2Q)^2}{3(-2)} + k = \frac{(a-2Q)^3}{-6} + 6k$$

Applications of Integration in Economics (Examples)

1. The marginal cost function of a product is given by $\frac{d\pi}{dq} = 10 - 0.01q + 0.0009q^2$

Find the total cost function and the average cost function, given that total cost = 105 when $q = 10$.

Sol : Given $d\pi/dq = 10 - 0.01q + 0.0009q^2$

Integrating, both sides w.r.t. q , we have

$$\begin{aligned} \pi &= 10q - 0.01 \frac{q^2}{2} + 0.0009 \frac{q^3}{3} + k \\ \pi &= 10q - .005q^2 + .0003q^3 + k \\ \text{when } q &= 10, \pi = 105 \\ 105 &= 10(10) - .005(10)^2 + .0003(10)^3 + k \\ 105 &= 100 - .5 + .3 + k \\ k &= 5.2 \end{aligned}$$

Hence total cost function

$$\pi = 5.2 + 10q - .005q^2 + .003q^3$$

and average cost function is

$$\frac{\pi}{q} = \frac{5.2}{q} + 10 - .005q + .003q^2$$

2. If the marginal revenue function for output q is given by $R_m = \left(\frac{6}{q+2}\right)^2 - 5$, find the total revenue function by integration. Also deduce the demand function.

Sol. We know that the total revenue function is given by

$$R = \int_0^q RM \, dq$$

The arbitrary constant k in this case is zero as the total revenue is zero at $q = 0$.

$$\begin{aligned} \text{Now } RM &= \frac{6}{(q+2)^3} - 5 \\ \therefore R &= \int_0^q \left(\frac{6}{(q+2)^3} - 5 \right) dq \\ &= \int_0^q \left[\frac{6}{(q+2)^3} \right] dq - \int_0^q 5 dq \\ &= \left[\frac{(q+2)^{-1}}{-1} \right]_0^q - 5q \Big|_0^q \\ &= 6 \left[-\frac{1}{q+2} \right]_0^q - 5q \Big|_0^q \\ &= 6 \left\{ -\frac{1}{q+2} - 5q \right\} \end{aligned}$$

Now, since $R = pxq$

$$p = \frac{R}{q} = \frac{3 - \frac{6}{q+2} - 5q}{q}$$

$$= \frac{3}{q} - \frac{6}{q(q+2)} - 5$$

$$= \frac{3q+6-6}{q(q+2)} - 5$$

$$\therefore p = \frac{3}{q+2} - 5$$

Is the required demand function.

3. If the demand function is $P_D = 10 - Q - Q^2$

and the supply function is $P_s = Q + 2$

Calculate the consumer's surplus at the equilibrium price.

Sol. We have $P_D = 10 - Q - Q^2$ Demand function

and $P_s = Q + 2$ Supply function

At equilibrium

$$P_D = P_s$$

$$10 - Q - Q^2 = Q + 2 \text{ Thus}$$

$$Q^2 + 2Q + 8 = 0$$

$$(Q - 2)(Q + 4) = 0$$

$$Q = 2 \text{ or } -4$$

Q cannot take a negative value, thus $Q = 2$

Consumer's Surplus =

$$\text{Area under demand curve} = \int_0^2 (10 - Q - Q^2) dQ - TE(P, Q), \text{ TE}$$

means Total Expenditure

$$= \left| 10Q - \frac{Q^2}{2} - \frac{Q^3}{3} \right|_0^2 \quad 8$$

$$= 20 - 2 - \frac{8}{3} - 8$$

$$= 15 - \frac{1}{3} - 8 = 7 - \frac{1}{3} \text{ Ans.}$$

11.5 Let Us Sum Up

We conclude this lesson, by summarizing what we have carried in it. Standard form integrals integral of a constant, sum, multiple, integrals by partial fraction, substitution parts definite integrals and the economic application of integrals.

11.6 Examination Oriented Questions

Evaluate

1. $\int x^2 \cdot e^x dx$

2. $\int x\sqrt{2x+1} dx$

3. $\int \frac{1}{x \log x} dx$

Find the value of

a $\int \frac{1}{\sqrt{x-1} - \sqrt{x+1}} dx$

b $\int \frac{\log x}{x^2} dx$

c $\int x \cdot e^{x^2} dx$

d. $\int_1^5 x^2 dx.$

e $\int_0^1 \log(1+x) dx$

f $\int_1^2 \frac{x^2}{\sqrt{x-1}} dx$

4. The following are estimate of MR and MC functions

(1) $MR = 4 - 0.4 Q$ Q is output

(2) $MC = 2 + 0.4 Q$

Find TR when sales are (a) 4 (b) 9 in case of MR function and total variable of MR function and total variable costs when output is (a) 4 (b) 10 from MC function.

5. Given the following demand functions

(i) $Q = 10 - P$

$Q = 64P^{-2}$

Find the consumer surplus when $P = 4$ & $P = 5$

6. The demand function of a monopolist is

$3Q = 60 - 10P$

and his AC function is

$AC = \frac{20}{Q} + 1 + 0.2Q$

If he decides to maximize sales revenue instead of profits, show how this will effect the consumer's surplus.

11.7 Hints to Check Your Progress

CYP-I

Solve using identities under section 11.2.1 (Standard Form of Integrals)

CYP-II

Solve using suitable identities

11.8 Suggested Readings

Aggarwal, C. S & R.C. Joshi : Mathematics for Students of Economics (New Academic Publishing Co.).

Allen, R.G.D. ; Mathematical Analysis for Economists (Macmillan).

Anthony Martin & Norman Biggs; Mathematics for Economics and Finance-Methods and Modeling.

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Dowling, Edward T : Introduction to Mathematical Economics (Tata Mcgraw).

Henderson, James M & Richard E Quandt : Microeconomic Theory- A Mathematical Approach (Mcgraw-Hill International Book Company).

Kandoi B : Mathematics for Business and Economics with Applications (Himalaya Publishing House).

Yamane Taro : Mathematics for Economics-A Elementary Survey (Prentice Hall of India Pvt. Ltd.).

Aggarwal S.C., Rana R.K., Gupta Leena : Mathematics for Economists (VK Global Publications Pvt. Ltd.)

M.A. Economics	Lesson No. 12
C.No. : Eco-103	Semester - 1st
	Unit II

COMPARATIVE STATICS AND ALLIED ECONOMICS

STRUCTURE

- 12.0 Objectives/Learning Outcomes
- 12.1 Introduction
- 12.2 Partial Differentiation and Static Macroeconomic Model
- 12.3 A closed Macroeconomic Model with Government Activity
- 12.4 An open Macro Model with Government Activity
- 12.5 Derivation of Slutsky Equation
- 12.6 IS-LM Analysis
- 12.7 Let us sum up
- 12.8 Examination Oriented Questions
- 12.9 Hints to check your Progress
- 12.10 Suggested Readings

12.0 Objectives/Learning Outcomes

After going through this lesson, we shall be able to understand the application of partial differentiation in :

- ☐ Static macroeconomic model.
- ☐ A closed Macroeconomic model with government activity.

- An open macroeconomic model with government activity.
- IS-LM Model.

We shall also derive the important equations for Slutsky's decomposition of price effect into substitution and income effect.

12.1 Introduction

Economics models have two type of variables-endogenous variables, whose values the model is designed to explain, and exogenous variables, whose values are taken as given from outside the model. The solution values, we obtain for the endogeneous variables with typically depend on the values of the exogeneous variables, and central part of the analysis will often be to show how the solution values of the endogenous variables change with changes in the exogeneous variables. This is the problem of comparative statics equilibrium analysis or comparative statics.

12.2 Partial Differentiation and Static Macroeconomic Models

Taking a closed economic system with no government activity gives

$$Y = C + I$$

Where Y is national Income, C is consumption and I is investment.

Assume I is autonomous at I^* and C is a function of y, i.e $C = C^* + bY$, where C^* is the autonomous consumption and b is the marginal propensity to consume.

$$Y = C + I$$

$$Y = C^* + by + I^*$$

$$(1-b)y = C^* + I^*$$

$$y = \frac{C^*}{1-b} + \frac{I^*}{1-b}$$

$$\partial y / \partial C^* \text{ or } f_c = \frac{1}{1-b}$$

f_c^* is the rate of change in y with respect to changes in C^* ; this is the multiplier

$$\partial y / \partial I^* = f_I = 1/(1-b)$$

Clearly f_c^* and f_I^* take a constant value so long as 'b' remains constant. This is a consequence of the linear relation assumed in the model. In each case, the partial derivative gives the rate of change in national income with respect to a change in any of the autonomous component part of y , assuming all other autonomous part of the consumption function increases by Δc^* then y will increase by $\Delta c^* / 1-c$. Because it analysis only the effects of the changes in autonomous variables on the equilibrium positions for the economy, and does not attempt to deal with the process of transition, this type of analysis is known as comparative statics.

12.3 A Closed Macroeconomic Model with Government Activity

Closed economic system with government activity, the identity now becomes

$$Y = C + I + G$$

Where G is government expenditure, I and G are assumed to be autonomous and consumption C is assumed to be a function of disposable income y_d , i.e

$$C = C^* + b y$$

Where C^* is the autonomous part of consumption and b is the marginal propensity to consume, y_d is national income minus taxes. If

$$T = T^* + t y$$

Where T^* is that part of taxes, which does not depend on income and t is the tax rate, then

$$C = C^* + b (y - T)$$

$$C = C^* + b(y - T^* - t y)$$

We know,

$$y = C^* + b y - b T^* - b t y + I + G$$

$$= C^* + b (1 - t) y - b T^* + I + G$$

$$[1 - b(1 - t)] y = C^* - b T^* + I + G$$

$$y = \frac{C^*}{1-b(1-t)} - \frac{b T^*}{1-b(1-t)} + \frac{1}{1-b(1-t)} + \frac{G}{1-b(1-t)}$$

$f_{T^*} = -\frac{b}{1-b(1-t)}$, i.e. the rate of change in income with respect to changes in T^* is constant. Consequently if T^* increases by ΔT^* , the national income y will decrease by

$$\Delta T^* \frac{b}{1-b(1-t)}$$

f_G gives the rate of change in y with respect to G . If G increases by ΔG then national income Y will increase by

$$\Delta G f_G = \Delta G \frac{b}{1-b(1-t)}$$

If $\Delta G = \Delta T^*$ i.e. an increase in expenditure is financed by an equal increase in the autonomous part of taxation, then the overall change in national income will equal

$$= \Delta T^* \frac{b}{1-b(1-t)} + \frac{1}{1-b(1-t)} \Delta G$$

$$= \Delta G \frac{(1-b)}{1-b(1-t)}$$

$$\Delta y = \frac{\Delta G(1-b)}{1-b+bt}$$

$$\text{but } \frac{(1-b)}{1-b+bt} < 1$$

Since the denominator is greater than the numerator. Consequently the change in income Δy is less than the change in government expenditure ΔG , i.e.

$$\frac{\Delta Y}{\Delta G} < \frac{1-b}{1-b+bt} > 0$$

12.4 An open Macro Model with Government Activity

If the foreign sector is included in the above model, the identity becomes

$Y = C + I + G + X - M$, where X is exports and M imports. Since exports are constant, imports (M) are a function of y_d i.e.

$M = M^* + m y_d$, where M^* is the autonomous part and 'm' is the marginal propensity to import.

Therefore $M = M^* + m (y - T)$ $[\square Y_d = Y - T]$

We know that $T = T^* + ty$

$\therefore$ from eq 1, we have

$$\begin{aligned} M &= M^* + m \{y - (T^* + ty)\} \\ &= M^* + m (y - T^* - ty) \\ &= M^* + my - mT^* - mty \\ &= M^* + m (1 - t)y - mT^* \end{aligned}$$

$$\begin{aligned} \text{Now } y &= C^* + b(1 - t)y - bT^* + I + G + X - M^* \\ &= C^* + b(1 - t)y - bT^* + I + G + X - M^* - m(1 - t)y + mT^* \end{aligned}$$

$$\text{or } Y - b(1 - t)Y + m(1 - t)Y = C^* - bT^* + I + G + X - M^* + mT^*$$

$$\text{or } Y[1 - b(1 - t) + m(1 - t)] = C^* - (b - m)T^* + G + I + X - M^*$$

$$\text{or } Y = \frac{C^* - (b - m)T^* + G + I + X - M^*}{1 - b(1 - t) + m(1 - t)}$$

$$\text{We have } 1 - b(1 - t) + m(1 - t) = 1 - (1 - t)(b - m)$$

$$\text{We have } Y = \frac{C^* - (b - m)T^* + G + I + X - M^*}{1 - (1 - t)(b - m)}$$

Here F_{T^*} is constant, and

$$F_{T^*} = \frac{-(b-m)}{1-(1-t)(b-m)}$$

$$f_{C^*} = F_G = F_t = F_x = -F_{M^*} = \frac{1}{1-(1-t)(b-m)}$$

So that F_{C^*} is also constant.

If the autonomous part of imports M^* increases by ΔM^* , assuming all other components of Y remaining constant, national income will decrease by

$$\Delta M^* \times \frac{1}{1-(1-t)(b-m)}$$

If the autonomous part of the tax function T^* increases by ΔT^* , national income will decrease by

$$\Delta T^* \times \frac{b-m}{1-(1-t)(b-m)}$$

12.5 Derivation of Slutsky Equation

Comparative statics analysis examines the effect of changes on the solution variables (such as price and incomes) on the solution values for the endogenous variables (namely, quantities) changes in prices and income will normally alter the consumers expenditure pattern, but the new quantities (and prices and incomes) will always satisfy the first order conditions.

$$U = f(q_1, q_2) \text{ objective function} \quad (i)$$

$$y = p_1 q_1 + p_2 q_2 \text{ constant equation} \quad (ii)$$

where y is total money income, U is total satisfaction, p_1 = price of commodity one, p_2 is the price of second commodity, and q_1, q_2 are the amount of commodity one and two, $p_1 q_1$ is the amount of expenditure on commodity one and $p_2 q_2$ is the amount of expenditure commodity two. Here p_1, p_2 and y are exogenous.

Variables and q_1, q_2 are endogenous variables from the objective function and the constraints we form a function.

$$V = f(q_1, q_2) + \lambda(y - p_1 q_1 - p_2 q_2)$$

Which satisfy the following conditions for maximizing the total utility function.

$$\begin{aligned} \partial v / \partial q_1 &= f_1 - \lambda p_1 = 0 \\ \partial v / \partial q_2 &= f_2 - \lambda p_2 = 0 \quad \dots\dots\dots (3) \\ \partial v / \partial \lambda &= y - p_1 q_1 - p_2 q_2 = 0 \end{aligned}$$

Now, in order to find the magnitude of the effect of price and income changes on the consumers purchases, we allow all variables to vary simultaneously. This can be done by total differentiation of the equation(3)

$$\begin{aligned} f_{11}dq_1 + f_{12}dq_2 &= p_1 d\lambda = \lambda dp_1 \\ f_{21}dq_1 + f_{22}dq_2 &= p_2 d\lambda = \lambda dp_2 \quad \dots\dots\dots (4) \\ -p_1 dq_1 - p_2 dq_2 &= -dy + q_1 dp_1 + q_2 dp_2 \end{aligned}$$

In order to solve this system of three equations for the three unknown ($dq_1, dq_2, d\lambda$), we take the help of Cramm's rule. In matrix notation, we may write (4) as under

$$\begin{aligned} \begin{bmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \\ d\lambda \end{bmatrix} &= \begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ -dy + q_1 dp_1 + q_2 dp_2 \end{bmatrix} \quad \dots\dots\dots (5) \\ \text{or } \begin{bmatrix} dq_1 \\ dq_2 \\ d\lambda \end{bmatrix} &= \frac{1}{|D|} \begin{bmatrix} D_{11} & D_{21} & D_{31} \\ D_{12} & D_{22} & D_{32} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ -dy + q_1 dp_1 + q_2 dp_2 \end{bmatrix} \quad \dots\dots\dots (6) \\ \text{where } |D| &= \begin{vmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} \end{aligned}$$

and $D_{11}, D_{12}, \dots$ are the co-factors of matrix $[D]$. By inter changing the rows and columns of the co-factor matrix, we get Adj. matrix and if we divide it by the original matrix $[D]$ then we get the inverse of the original matrix.

Therefore,

$$dq_1 = \frac{D_{11}\lambda dp_1 + D_{21}\lambda dp_2 + D_{31}[-dy + q_1 dp_1 + q_2 dp_2]}{[D]} \dots\dots\dots (7)$$

$$dq_2 = \frac{D_{12}\lambda dp_1 + D_{22}\lambda dp_2 + D_{32}[-dy + q_1 dp_1 + q_2 dp_2]}{[D]} \dots\dots\dots (8)$$

Dividing both the sides of (7) by dp_1 and assuming that P_2 and Y do not change [or $dy = dp_2 = 0$], we have

$$\frac{\partial q_1}{\partial p_1} = \frac{D_{11}\lambda}{[D]} + q_1 \frac{D_{31}}{[D]} \dots\dots\dots (9)$$

(Price effect)

The partial derivatives on the left hand side of equation (9) is the rate of change of the consumers purchase of q_1 with respect to change in p_1 , all other things being equal. Ceteris paribus, the rate of change with respect to income is

$$\frac{\partial q_1}{\partial y} = -\frac{D_{31}}{[D]} \dots\dots\dots (10)$$

(Income effect)

Consider a price change that is compensated by an income change that leaves the consumer on his initial indifference curve. An increase in the price of a commodity is, accompanied by a corresponding change in his income, such that $dU = 0$

$$dy = -q_1 dp_1 - q_2 dp_2$$

$$-dy + q_1 dp_1 + q_2 dp_2 = 0$$

Hence from equation 7, we have the substitution effect

$$\left(\frac{\partial q_1}{\partial p_1} \right)_{u=\text{constant}} = \frac{D_{11}\lambda}{[D]}$$

Now, the Slutsky equation can be written in terms of substitution effect and income effect as:

$$\left(\frac{\partial q_1}{\partial p} \right) = \left(\frac{\partial q_1}{\partial p} \right) \Big|_{u=\text{constant}} - q_1 \left(\frac{\partial q_1}{\partial Y} \right) \Big|_{p=\text{constant}}$$

Price effect
substitution effect
income effect

Check Your Progress-I

Q.1. How Slutsky's equation decomposes price effect into substitution and income effect.

□ What is the basic difference between a closed and an open macroeconomic model.

12.6 IS-LM Analysis

IS -LM analysis studies the equilibrium values of national income 'y' and the interest rate 'r', in terms of certain policy, parameters and behavioral parameters. 'IS' stands for 'Investment Savings' and 'LM' for 'Liquidity money'.

First, consider equilibrium in the 'goods sector'. This can be represented by the equation

$$Y = C + I + G \quad \dots\dots\dots (1)$$

Which simply says that national income 'Y' is equal to the national expenditure. Expenditure can be split into consumption C, investment I, and governments spending G, the last being considered, as a policy parameter. Behavioral parameters enter when we consider how consumption is related to the national income, and how investment is affected by the interest rate. It is usual to assume that both relationships are linear, that is.

$$C = C_o + bY \quad \dots\dots\dots (2)$$

$$I = I_o - a r \quad \dots\dots\dots (3)$$

(a, b, co, I_o positive constants)

Thus the equilibrium condition becomes

$$Y = (C_o + bY) + (I_o - a r) + G$$

$$Y - b Y + a r = C_o + I_o + G$$

$$(I-b) Y + a r = C_o + I_o + G \quad \dots\dots\dots (4)$$

Equilibrium in the money sector is represented by the statement that the supply of money is equal to the demand for money. Ms = Md. the money supply Ms is assumed to be given; it is another policy parameter. The money demand Md is assumed to depend on y and r, and again we assume a linear relationship.

$$Md = M_o + fY - g r \quad \dots\dots\dots (5)$$

(f, I, M_o positive constant)

Thus, the equilibrium condition becomes

$$Ms = M_o + fY - g r$$

$$fY - g r = Ms - M_o \quad \dots\dots\dots (6)$$

The two equilibrium conditions can be written as a system of two linear equation in the unknown Y and r

$$f Y - g r = Ms - Mo \quad \dots\dots\dots (6)$$

$$(I-b) Y + a r = C_o + I_o + G \quad \dots\dots\dots (4)$$

In matrix terms this system is

$$\begin{pmatrix} f & -g \\ I-b & a \end{pmatrix} \begin{pmatrix} Y \\ r \end{pmatrix} = \begin{pmatrix} M_s - M_o \\ C_o + I_o + G \end{pmatrix} \quad \dots\dots\dots (7)$$

and the solution is

$$\begin{aligned} \begin{pmatrix} y \\ r \end{pmatrix} &= \begin{pmatrix} f & -g \\ 1-b & a \end{pmatrix}^{-1} \begin{pmatrix} M_s - M_o \\ C_o + g_o + G \end{pmatrix} \\ &= \frac{1}{fa + g(1-b)} \begin{pmatrix} a & -g \\ 1-b & f \end{pmatrix} \begin{pmatrix} M_s - M_o \\ C_o + g_o + G \end{pmatrix} \dots\dots\dots (8) \end{aligned}$$

Thus, we have explicit formula for the equilibrium values y and r

$$y^* = \frac{a(M_s - M_o) + g(c_o + g_o + G)}{fa + g(1-b)}$$

$$r^* = \frac{(1-b)(M_s - M_o) + f(c_o + g_o + G)}{fa + g(1-b)}$$

These expression enable us to answer questions about what happens when the policy parameters or the behavioural parameters change.

Example: Suppose that the government decides to allow an increase in the money supply M_s . What will be the effect on the equilibrium value of the national income?

From the formula for y^* , we find

$$\frac{\partial y^*}{\partial M_s} = \frac{a}{fa + g(1-b)}$$

Thus an increase in money supply will result in an increase in y^* if

$$\frac{a}{fa + g(1-b)}$$

is positive. The behavioural parameters a , b , f & g are assumed to be positive, this condition will certainly be satisfied if $1-b > 0$ that is $b < 1$.

In fact, this condition will certainly hold in any realistic model. We have $y = C + I + G$, so $C < Y$ and $C = C_o + bY$ so $bY < c$, Hence $bY < Y$ or $b < 1$.

Note : Partial Differentiation and static macroeconomic models, IS-LM Analysis

totally dependent on Anthony Martin & Norman Biggs; Mathematics for economics and finance-Methods and Modeling

Black, J & J.F. Bradley : Essential Mathematics for Economists (John Willey & Sons) Slutsky Equation, Heavily Depended on H.S. Aggarwal Book - A Mathematical Approach to Economic Theory]

12.7 Let Us Sum Up

This chapter gives idea about the comparative statics and the allied economics by explaining the sections - 9.3, 9.4, 9.5, 9.6 and 9.7.

12.8 Examination Oriented Questions

Q1. Explain IS-LM Model Mathematically

Q2. Examine Static Macro-Economic Modes with the help of Partial Differentiation.

Q3. Derive the Slutsky Equation.

13.6 Hints to Check Your Progress

CYP-I

Q1. Go through the derivation of Slutsky equation under section 12.5.

CYP-2

Q2. Go through sections 12.3 and 12.4 simultaneously.

12.10 Suggested Readings

Aggarwal, C.S. & R.C.Joshi : Mathematics for Students of Economics (New Academic Publishing Co.).

Allen, R.G. D. : Mathematical Analysis for Economists (Macmillan).

Anthony Martin & Norman Biggs : Mathematics for Economics and Finance-Methods and Modeling.

Black, J & J.F. Bradley : Essential Mathematics for Economists (John Willey & Sons).

Dowling, Edward T : Introduction to Mathematical Economics (Tata McGraw).

Henderson, James M & Richard E Quandt : Microeconomic Theory- A Mathematical Approach (McGraw-Hill International Book Company).

Kandoi B : Mathematics for Business and Economics with Applications (Himalaya Publishing House).

Yamane Taro : Mathematics for Economics - A Elementary Survey (Prentice Hall of India Pvt. Ltd.).

M.A. Economics	Lesson No. 13
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Semester - 1st	

COBB-DOUGLAS AND CES PRODUCTION FUNCTION

STRUCTURE

- 13.0 Objectives/Learning Outcomes
- 13.1 Introduction
- 13.2 The general form of the Cobb-Douglas Production Function
 - 13.2.1 Properties of CDPF
- 13.3 The CES Production Function
 - 13.3.1 Properties of C.F.S.P.F.
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13.0 Objectives/Learning Outcomes

After going through their lesson the shall be able to examine Cobb-Douglas production function alongwith its properties and also examine CES production function alongwith its limitation and properties.

13.1 Introduction

Economists have developed a number of production functions. Their production functions are based on the empirical analysis of industry and agriculture in a given economy or a region. It is difficult to study all such production functions here and nor they are relevant for us. In the following analysis, we discuss only the important production functions, namely (i) the Cobb-Douglas Production Function and (ii) the Constant Elasticity of Substitution Production Function or simply the C.E.S. Production Function.

13.2 The Cobb-Douglas Production Function

The Cobb-Douglas Production Function is based on the empirical study of the American Manufacturing Industry made by Paul H. Douglas and C.W. Cobb. It is a linear homogeneous production function which takes into account only two inputs, labour and capital for the entire output of the manufacturing industry.

The general form of Cobb-Douglas production function is

$$Q = AL^{\alpha}K^{\beta}, \text{ where}$$

$$Q = \text{output}$$

$$L = \text{Labour}$$

$$K = \text{Capital}$$

$$A, \alpha, \beta = \text{Positive constants}$$

The Cobb-Douglas Production function can be estimated by regression analysis by first converting into the following form

$$\log Q = \log A + \alpha \log L + \beta \log K$$

The function in log form is a linear function.

Properties of Cobb-Douglas Production Function

1. The sum of the exponents of factors in Cobb-Douglas production ($Q =$

$AL^\alpha K^\beta$), i.e. $\alpha + \beta$ measures returns to scale

If $\alpha + \beta = 1$, returns to scale are constant

If $\alpha + \beta > 1$, returns to scale are increasing

If $\alpha + \beta < 1$, returns to scale are decreasing

2. In Cobb-Douglas Production function, there is an important relationship between average products of the inputs with the marginal products.

$$\text{Since } Q = AL^\alpha K^\beta$$

$$AP_L = \frac{Q}{L}$$

$$MP_L = \frac{\partial Q}{\partial L} = \alpha AL^{\alpha-1} K^\beta$$

$$= \alpha AL^\alpha L^{-1} K^\beta$$

$$= \frac{\alpha}{L} AL^\alpha K^\beta$$

$$= \frac{\alpha}{L} Q = \alpha \cdot \frac{Q}{L}$$

$$= \alpha \cdot AP_L$$

$$\text{ii } MP_L = \alpha \cdot AP_L$$

$$\text{Similarly, } MP_K = \beta \cdot AP_K$$

Thus, marginal product of labour is α times the average product of labour and the marginal product of capital is β times the average product of capital.

3. In a linear homogeneous Cobb-Douglas production function $Q = AL^\alpha K^\beta$, average and marginal products of a factor depend on ratio of factors used in production and are independent of the absolute quantities of the factors used. In the linear homogeneous Cobb-Douglas production function, $Q = AL^\alpha K^{1-\alpha}$, where

$$\alpha + 1 - \alpha = 1$$

$$\begin{aligned}
 \text{Average product of labour} &= \frac{Q}{L} \\
 &= \frac{AL^\alpha K^{1-\alpha}}{L} \\
 &= \frac{AK^{1-\alpha}}{L^{-\alpha} \cdot L} = \frac{AK^{1-\alpha}}{L^{1-\alpha}} \\
 &= A \left(\frac{K}{L} \right)^{1-\alpha}
 \end{aligned}$$

Since A & α are constants, average product of labour will depend on the ratio of the factors $\left(\frac{K}{L} \right)$ used and will not depend upon their absolute quantities.

Similarly, the average product of capital will depend on the ratio of the factors $\left(\frac{L}{K} \right)$ used and will not depend upon their absolute quantities.

Now, marginal product of a factor, say labour, is first order partial derivative of the production function with respect to labour.

$$\text{i.e. If } Q = AL^\alpha K^{1-\alpha}$$

$$\begin{aligned}
 \text{Marginal product of labour } \left(\frac{\partial Q}{\partial L} \right) &= A\alpha L^{\alpha-1} K^{1-\alpha} \\
 &= A\alpha L^\alpha \cdot L^{-1} K^{1-\alpha} \\
 &= A\alpha \frac{L^\alpha}{L} K^{1-\alpha} \\
 &= A\alpha \frac{L^\alpha}{L} \cdot \frac{L^{-\alpha}}{L^{-\alpha}} K^{1-\alpha}
 \end{aligned}$$

$$\begin{aligned}
 &= A\alpha \frac{L^{\alpha-\alpha}}{L^{1-\alpha}} K^{1-\alpha} \\
 &= A\alpha \frac{K^{1-\alpha}}{L^{1-\alpha}} \\
 &= A\alpha \left(\frac{K}{L} \right)^{1-\alpha}
 \end{aligned}$$

Since A and α are constants, marginal product of labour will depend on capital-labour ratio $\left(\frac{K}{L} \right)$, i.e. capital per worker and is independent of the absolute quantities of the factors employed. Similarly, marginal product of capital will depend on labour-capital $\left(\frac{L}{K} \right)$, i.e. labour per capital.

4. Linear Homogeneous Cobb-Douglas Production Function exhibits diminishing returns to a variable factor.

To prove that the linear homogeneous Cobb-Douglas production function exhibits diminishing marginal product of a variable factor, we have to show that the second derivative of the function with respect to a variable factor (say, labour) is negative

Let $Q = AL^{0.75}K^{0.25}$, where $0.75 + 0.25 = 1$

$$\begin{aligned}
 \text{Marginal product of labour (MP}_L) &= \frac{\partial Q}{\partial L} \\
 &= 0.75 AL^{0.75-1} K^{0.25}
 \end{aligned}$$

$$\text{second derivative, } \frac{\partial^2 Q}{\partial L^2} = -0.25 \times 0.75 AL^{-1.25} \times K^{0.25}$$

Since second derivative of Cobb-Douglas production is negative, it shows diminishing marginal returns to the variable factor i.e. labour.

5. The exponents of labour and capital in Cobb-Douglas production function measure output elasticities of labour and capital respectively.

Output elasticity of a factor shows the percentage change in output resulting from a given percentage change in the quantity of a factor, other factors remaining constant.

Thus, output elasticity of labour

= $\frac{\partial Q}{\partial L} \cdot \frac{L}{Q}$, where $\frac{\partial Q}{\partial L}$ is the marginal productivity of labour.

Since $MP_L = \frac{\partial Q}{\partial L}$ (solved earlier), we substitute this $\frac{\partial Q}{\partial L} \cdot \frac{L}{Q}$

$$= \alpha \cdot \frac{L}{Q} = \alpha \quad \dots(i)$$

Thus, exponent 'α' of labour in the Cobb-Douglas production is equal to the output elasticity of labour.

Similarly, output elasticity of capital = $\frac{\partial Q}{\partial K} \cdot \frac{K}{Q}$

Marginal productivity of capital, $\frac{\partial Q}{\partial K} = \beta \cdot \frac{Q}{K}$

Therefore, output elasticity of capital

$$= \beta \cdot \frac{K}{Q} = \beta \quad \dots(ii)$$

It follows from eqm (i) above that 1% increase in labour in Cobb-Douglas production function, holding capital constant, causes α% increase in output. Similarly, from equ. (ii), it follows that 1% increase in capital, holding labour constant, causes β% increase in output.

6. When the sum of exponents (α + β) in the two factor linear homogeneous Cobb-Douglas production $Q = AL^\alpha K^\beta$ is equal to one, it would show constant returns to scale.

Proof. Since $\alpha + \beta = 1$
 $\Rightarrow \beta = 1 - \alpha$
 Now $Q = AL^\alpha K^\beta$

If the inputs of labour (L) & capital (K) are increased by a constant 'g', then the quantity of output will be increased to

$$A(gL)^\alpha (gK)^{1-\alpha} = Ag^\alpha L^\alpha \cdot g^{1-\alpha} K^{1-\alpha}$$

since $g^\alpha \cdot g^{1-\alpha} = g$

$$\text{Therefore, } A(gL)^\alpha (gK)^{1-\alpha} = gAL^\alpha K^{1-\alpha}$$

$$= gQ.$$

Thus, when the inputs of labour and capital are increased by a constant 'g', the output Q also increases by 'g'.

7. Elasticity of Substitution between two factors, labour and capital, in Cobb-Douglas production function is equal to unity (one)

Proof.

$$Q = AL^\alpha K^\beta \dots (i)$$

$$\text{Elasticity of substitution (e}_s\text{)} = \frac{\Delta \left(\frac{K}{L} \right)}{K/L} \div \frac{\Delta \text{MRTS}_{LK}}{\text{MRTS}_{LK}}$$

$$\text{MRTS}_{LK} = \frac{MP_L}{MP_K} \dots (ii)$$

from equ. (i), $MP_L = \frac{\partial Q}{\partial L} = \alpha AL^{\alpha-1} K^\beta$

$$MP_K = \frac{\partial Q}{\partial K} = \beta AL^\alpha K^{\beta-1}$$

(ii) from eq^h (2), $\text{MRTS}_{LK} = \frac{MP_L}{MP_K} = \frac{LAL^{\alpha-1}K^\beta}{\beta AL^\alpha K^{\beta-1}}$

$$\begin{aligned}
 &= \frac{\alpha}{\beta} L^{\alpha-1} K^{\beta} \cdot K^{-\beta} \cdot K \\
 &= \frac{\alpha}{\beta} L^{-1} K - \frac{\alpha}{\beta} \cdot \frac{K}{L} \\
 \text{Now, } e_s &= \frac{\Delta \left(\frac{K}{L} \right)}{\Delta MRTS} \times \frac{MRTS_{LK}}{\frac{K}{L}} \\
 &= \frac{\Delta \left(\frac{K}{L} \right)}{\Delta \left(\frac{\alpha}{\beta} \cdot \frac{K}{L} \right)} \times \frac{\frac{\alpha}{\beta} \left(\frac{K}{L} \right)}{\left(\frac{K}{L} \right)} \\
 &= \frac{\Delta \left(\frac{K}{L} \right)}{\frac{\alpha}{\beta} \Delta \left(\frac{K}{L} \right)} \times \frac{\alpha}{\beta} \\
 &= \frac{\frac{\alpha}{\beta}}{\frac{\alpha}{\beta}} = \frac{\cancel{\alpha}}{\cancel{\beta}} \times \frac{\beta}{\cancel{\alpha}} = 1 \text{ hence proved}
 \end{aligned}$$

If one input is zero, output will also be zero.

Check Your Progress-I

1. Explain the property - I of CDPF.

-
-
2. Elucidate the property - V of CDPF.
-
-
-
-

13.3 The C.E.S. Production Function

A production function, which assumes that the basic measure of the degree of substitution is called but it not restricted 'a priori' to any value. It is called the 'Constant Elasticity of Substitution (C.E.S.) production function'. Clearly, the Cobb-Douglas and Leontief's production functions are special cases of the CES relation.

When the elasticity of substitution specified as constant, it only assumes that changes in relative factor inputs and prices do not alter the elasticity. The value of the elasticity is determined by the underlying technology; and changes in the underlying technology affect variations on the elasticity for every level of factor inputs and prices. Hence, the constancy of the elasticity refers to its invariance with respect to changes in relative factor supplies and not to transformations of the underlying technology.

The characteristics of an abstract technology are identified by the C.E.S. production function. That is to say that it permits us to measure changes in the efficiency of a technology, changes in the technologically determined returns to scale, changes in the capital intensity of a technology and changes in the substitution of labour for capital etc.

The C.E.S. production function was derived independently by two different groups of economists: one consisting of K.J. Arrow, H.B. Chenery, B.S. Minhas and Solow; and the other group consist of Murraray Brown and De Cani. The two derivations are dissimilar; also the latter permits any degree of returns to scale. Murray Brown and De Cani (1963) used the function in an ambitious attempt to separate the effect

of output change; economics of scale, technical change and changes in the relative factor prices on labour demand; with U.S. economy data over the period 1890-1958.

Further, some of the recent studies use the constant elasticity of substitution (C.E.S.) function with substitution elasticity below unity is claimed to be more suitable for production functions than the Cobb-Douglas form. Under the Cobb-Douglas production function, elasticity of substitution is equal to unity but in C.E.S. production function, as the name suggests, the elasticity of substitution is constant and not necessarily equal to unity. Four economists, Arrow, Chenery, Minhas and Solow have proposed this C.E.S. production function.

The equation of this function is.

$$X = \gamma [KC^{-\alpha} + (1-K)N^{-\alpha}]^{\frac{1}{\alpha}} \quad \dots(1)$$

Where, X = output,

C = capital input,

N = labour input,

γ = Coefficient of technical efficiency or scale effect (or simply the efficiency parameter). This parameter plays the same role in C.E.S. function as the coefficient 'A' in the Cobb-Douglas function. It serves as an indicator of the general state of technology. Hence the greater the value of γ , the larger will be the output, regardless of inputs.

K = Capital intensity factor coefficient. It is also known as the distribution parameter and is similar to β in the Cobb-Douglas production function. This parameter shows the relative contribution of capital input (K) and labour input (L), to the total output (X).

$(1-K)$ = Labour intensity coefficient.

α = It is known as the substitution parameter and is closely related to the constant elasticity of substitution. The value of e^s (elasticity of substitution) depends on the value of this parameter.

ν = It represents the degree of homogeneity of the function or the degree of returns to scale.

Hence, this function consists of three variable (X, C and N) and four parameters (γ , K, α and ν). The variables are measured in index number terms with a common base period. And the four important parameters in the function represent the four different characteristics of an abstract technology. Thus is, in brief, the beauty of this particular production function over the Cobb-Douglas (or other types) production function.

Relation (1) can be written in any one of the following forms.

$$i) \quad X = \gamma [K C^{-\alpha} + (1-K) N^{-\alpha}]^{-\frac{\nu}{\alpha}}; \alpha' = -\left[1 - \frac{1}{\sigma}\right]$$

$$ii) \quad X = \gamma [K C^{\alpha} + (1-K) N^{\alpha}]^{-\frac{\nu}{\alpha}}; \alpha' = \left[1 - \frac{1}{\sigma}\right]$$

$$iii) \quad X = \gamma \left[\frac{N^{\alpha} C^{\alpha}}{K N^{\alpha} + (1-K) C^{\alpha}} \right]^{-\frac{\nu}{\alpha}}; \alpha' = -\left[1 - \frac{1}{\sigma}\right]$$

$$iv)^2 \quad X = \gamma [K_1 C^{-\alpha} + K N^{-\alpha}]^{-\frac{\nu}{\alpha}}; \alpha' = \left[1 - \frac{1}{\sigma}\right]$$

where σ = elasticity of substitution

All the above four forms of the function will give us the same type of result.

13.3.1 Properties of C.E.S. Production Function:

1) The value of the elasticity of substitution depends upon the value of α
We define the elasticity of substitution as

$$\sigma = \frac{\frac{\partial \left(\log \frac{K}{L} \right)}{\partial \log R}}{\frac{\partial \left(\frac{K}{L} \right)}{\partial R}} = \frac{\frac{\partial \left(\frac{K}{L} \right)}{\partial R}}{\frac{\partial \log R}{\partial R}}$$

Where K/L = factor quantity ratio, R = factor price ratio

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Now our function is,

$$X = \gamma [KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{\gamma}{\alpha}} \quad \dots (2)$$

Taking the partial derivatives of X w.r.t. N we get

$$\begin{aligned} \frac{\partial X}{\partial N} &= \gamma \left[-\frac{\gamma}{\alpha} \right] [KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{\gamma}{\alpha}-1} [-\alpha N^{-\alpha-1}(1-K)] \\ &= \gamma \gamma [KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{\gamma}{\alpha}-1} [-\alpha N^{-\alpha-1}(1-K)] \quad \dots (3) \end{aligned}$$

Now we are again using equation n(2) as

$$X = \gamma [KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{\gamma}{\alpha}}$$

OR

$$\frac{X}{\gamma} = [KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{\gamma}{\alpha}}$$

OR

$$\left[\frac{X}{\gamma} \right]^{-\frac{\alpha}{\gamma}} = [KC^{-\alpha} + (1-K)N^{-\alpha}]$$

OR

$$\left[\frac{X}{\gamma} \right]^{\frac{-\alpha(-\frac{\gamma}{\alpha}-1)}{\gamma}} = [KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{\gamma}{\alpha}-1}$$

OR

$$= \left[\frac{X}{\gamma} \right]^{1+\frac{1}{\gamma}} = [KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{\gamma}{\alpha}-1} \quad \dots (4)$$

Putting the value of (4) in equation (3) we get,

$$\frac{\partial X}{\partial N} = \gamma \left(\frac{X}{N} \right)^{1+\alpha} N^{-\alpha-1} (1-K) = \gamma \frac{\alpha}{1+\alpha} \frac{X}{N} - \alpha - 1(1-K) \dots (5)$$

Similarly we can find,

$$\frac{\partial X}{\partial C} = \gamma \frac{\alpha}{1+\alpha} \frac{X}{C} - \alpha - 1(K) \dots (6)$$

Now we can define the marginal rate of substitution as

$$MRS = \frac{\partial X}{\partial C} \bigg| \frac{\partial X}{\partial N} = R \quad [\text{Because MRS is equal to the price ratio}].$$

Putting the values from equations (5) and (6) we have,

$$\therefore R = \frac{\gamma \frac{\alpha}{1+\alpha} \frac{X}{C} - \alpha - 1(K)}{\gamma \frac{\alpha}{1+\alpha} \frac{X}{N} - \alpha - 1(1-K)}$$

$$\therefore \text{Denoting } \frac{K}{1-K} = K' \text{ and } \frac{N}{C} = U \text{ we get, } R = K' U^{1-\alpha}$$

Taking the log on both the sides,

$$\log R = \log K' + (1-\alpha) \log U$$

$$\text{OR } \frac{1}{R} \frac{\partial R}{\partial U} = (1-\alpha) \frac{1}{U} \quad \text{or} \quad \frac{\partial R}{R} = (1-\alpha) \frac{dU}{U}$$

$$\text{OR } \frac{\partial U}{U} \bigg/ \frac{\partial R}{R} = \frac{1}{(1-\alpha)} \quad \text{where } \alpha > 0. \quad \text{or} \quad \sigma = \frac{1}{1-\alpha}$$

$$\text{Also } \alpha = \begin{bmatrix} 1 & -1 \\ \sigma & \end{bmatrix} = \begin{bmatrix} 1 \\ 1 & \sigma \end{bmatrix} \dots (7)$$

If $\alpha = 0$ then $\sigma = 1$ and we shall get the situation of Cobb-Douglas production function. Hence the value of σ depends upon the value of α . It assumes whatever the value of σ may be, (depending upon α) but it is always constant.

2) The marginal products should be positive:

Our function is, $X = \gamma [KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{1}{\sigma}}$

Or Now $MP_L = \frac{\partial X}{\partial N} = \gamma^{-\frac{\alpha}{\sigma}} \gamma^{\frac{1}{\sigma}} N^{-\alpha-1} (1-K)$

[Please see property No.1 for its calculation]

Or $\frac{\partial X}{\partial N} = h_1 X^{\frac{1+\alpha}{\sigma}} N^{-\alpha-1}$ where $h_1 = \gamma^{-\frac{\alpha}{\sigma}} \gamma^{\frac{1}{\sigma}} (1-K)$ (8)

If there are constant returns to scale then $\sigma = 1$ (Here σ refers to returns to scale or the degree of homogeneity of the function).

If $\sigma = 1$ then, $\frac{\partial X}{\partial N} = h_1 X^{1+\alpha} N^{-\alpha-1}$

$= h_1 \left(\frac{X}{N} \right)^{1+\alpha} = h_1 \left(\frac{X}{N} \right)^{\frac{1}{\sigma}}$ where $\sigma = \frac{1}{1+\alpha}$ (9)

Similarly, the Marginal Product of Capital is,

$MP_K = \frac{\partial X}{\partial C} = \gamma^{-\frac{\alpha}{\sigma}} \gamma^{\frac{1}{\sigma}} X^{1+\alpha} C^{-\alpha-1} K$ [See property No.1]

$= h_2 X^{\frac{1+\alpha}{\sigma}} C^{-\alpha-1}$ where $h_2 = K \gamma^{-\frac{\alpha}{\sigma}} \gamma^{\frac{1}{\sigma}}$

Again, if there are constant returns to scale, then $\sigma = 1$ and

$\frac{\partial X}{\partial C} = h_2 X^{1+\alpha} C^{-\alpha-1}$ or $\frac{\partial X}{\partial C} = h_2 \left(\frac{X}{C} \right)^{\frac{1}{\sigma}}$

()

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Where $\sigma = \frac{1}{1-\alpha}$ [See property No.1] ... (10)

It is now obvious that the marginal products are positive.

3) The marginal product curves are sloping downward or

$$\frac{\partial^2 X}{\partial N^2} < 0 \text{ and } \frac{\partial^2 X}{\partial C^2} < 0$$

4) The marginal product of each factor will increase for increase in the other factor inputs.

To satisfy this property, the RTS (or MRS) of labour (N) for capital © can be derived in terms of C.E.S. production function by taking the ratio of MP_L and MP_K as :

$$\frac{\partial X}{\partial C} / \frac{\partial X}{\partial N} = R = K' U^{1-\alpha} \quad \text{where} \quad K' = \frac{K}{1-K} U = \frac{N}{C}$$

[See property No.1]

Or $R = K' U^{1/\alpha}$ where $\sigma = \frac{1}{1+\alpha}$ (11)

This plays an important role in the following analysis. If the production process is highly L-intensive (small capital is used per unit of labour), the MP_L is high compared to MP_K for each N/C ratio ; thus a unit reduction in the labour rate has to be compensated for larger increase in the rate of capital than if the process were to be less L-intensive. In this sense, K is a measure of capital intensity.

If σ is high, then capital is easily substitute for labour and vice-versa. The expression for R tells us that if we reduce the rate capital inputs by one unit, we have to increase the rate of labour input by more when factors are easily substitutable

for each other then when they are ceteris paribus. Perhaps the easiest way to rationalize this is to recall that the more easily substitutable are the factors for each other, the more similar they are forms for each other, the more similar they are for an economic point of view. If σ is low then the factors are dissimilar.

13.3.2 Advantages of C.E.S. Production Function over the Cobb-Douglas Production Function and Limitations :

Advantages of CES Production Function

- 1) C.E.S. function represents a production function where all types of returns may be analyzed. Since σ is not necessarily be equal to one ($\sigma \neq 1$), it represents a more general form of production techniques.
- 2) As we have seen, the C.E.S. production function takes into consideration a number of important parameters. It, therefore, covers a wide range of variety, substitutability, and efficiency.
- 3) Estimation of this C.E.S. function is very easy. Of course, some transformation is needed, if we write output per unit of labour as a function of capital per unit of labour, that is,

$$\frac{X}{L} = f\left(\frac{K}{L}\right)$$

then this production function becomes very easy.

- 4) It removes all the difficulties of Cobb-Douglas production function and free from the unrealistic assumptions of the Cobb-Douglas production function.

Limitations of the C.E.S. Production Function :

- 1) The C.E.S. production function combines in one parameter, σ , two forces that affect it. In the first place, economies of scale can result from an expansion in the scale of operation for a given technology. Alternatively, given the scale

of operation, a technological change can alter the rate of output. In empirical application both forces may affect the homogeneity parameters, ν , and it may not be possible to distinguish between them.

- 2) Prof. H. Uzawa has attempt this function and concludes that it is difficult to generalize it to n-factors of production.
- 3) A limitation of the C.E.S. production function is associated with its principal virtue - the specification of elasticity of substitution which is invariant to changes in factor proportions. Recall that we have allowed the elasticity of substitution (σ) to changes in response to changes in factor proportions. But this is an 'a priori specification': we really do not know whether the elasticity of substitution (σ) should vary when factor proportions change. It is true structure prescribes a variable elasticity due to changes in factor proportions and we claim that elasticity in changing for technological reasons then we are ascribing to technological change more than is due to it. Unless a completely general function is specified - a polynomial of degree n-it seems this difficulty must be accepted. Since it is impossible with the available data and statistical techniques to obtain estimates of completely general production and since they do not necessarily satisfy all neo-classical criteria (properties of C.E.S. function), we are forced for the immediate future to utilize the C.E.S. production function and live with the potential specification error.
- 4) A fourth difficulty to this C.E.S. function is that K , the capital intensity parameter is not dimensionless.

Beside from there theoretical difficulties, there is an empirical problem; the C.E.S. production function is relatively difficult to fit to data.

[Note: Both the Cobb-Douglas and CES production function heavily depended on H.S. Agarwal book - A Mathematical approach to Economic Theory]

Check Your Progress-II

1. Explain the advantages and disadvantages of C.E.S.P.F.

2. Narrate the first property of C.E.S.P.F.

13.4 Let us Sum Up

This chapter deals with the non-Linear Production Functions and narrate their importance to the reader.

13.5 Examination Oriented Questions

- Q1. Explain the Cobb-Douglas production function and its properties.
- Q2. Elasticity of substitution is equal to one in case of Cobb-Douglas production function.
- Q3. Examine C.E.S production function and its properties.

13.6 Hints to Check Your Progress

CYP-I

- Q1. Go through section 13.2 and study the property given at serial no. 1.
- Q2. Go through section 13.2 and study the property given at serial no. 5.

CYP-II

- Q.1. Go through section 13.3.2.
- Q.2. Go through section 13.3.1 under properties of C.E.S. Production Function.

13.7 Suggested Readings

Aggarwal, C.S & R.C. Joshi : Mathematics for Students of Economics (New Academic Publishing Co.).

Allen, R.G.D. ; Mathematical Analysis for Economists (Macmillan).

Anthony Martin & Norman Biggs; Mathematics for Economics and Finance-Methods and Modeling.

Black, J & J.F. Bradley : Essential Mathematics for Economists (John Willey & Sons).

Dowling, Edward T : Introduction to Mathematical Economic (Tata McGraw).

Henderson, James M & Richard E Quandt : Microeconomic Theory- A Mathematical Approach (McGraw-Hill International Book Company).

Kandoi B : Mathematics for Business and Economics with Applications (Himalaya Publishing House).

Yamane Taro : Mathematics for Economics-A Elementary Survey (Prentice Hall of India Pvt. Ltd.).

M.A. Economics	Lesson No. 14
C.No. : Eco-103	Semester - 1st
	Unit III

DIFFERENTIAL EQUATIONS DEFINITIONS AND CONCEPTS

STRUCTURE

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- 14.1 Learning Outcomes
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 - 14.4.1 Order of Differential Equations:
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 - 14.5.1 Initial Conditions in Differential Equations
 - 14.5.2 Boundary Conditions in Differential Equations
- 14.6 Key words
- 14.7 Lets Sum Up
- 14.8 Examination oriented questions
- 14.9 Hints to check your progress

14.10 Suggested Readings

14.0 Objectives

The main objectives of this lesson are to enable you to:

- ☐ State the definition of a differential equation.
- ☐ Differentiate between ordinary and partial differential equations.
- ☐ Recognize the order and degree of differential equations.
- ☐ Solve first-order differential equations using initial conditions.
- ☐ Discuss the behavior of systems using numerical methods for solving complex differential equations that cannot be solved analytically.

14.1 Learning Outcomes

By the end of this lesson, you should be able to:

1. Define differential equations and explain their significance in mathematics and real-world applications.
2. Differentiate between ordinary and partial differential equations.
3. Solve basic first-order differential equations using appropriate methods and apply initial conditions.
4. Apply differential equations to model real-world phenomena such as population growth and Newton's law of cooling.

14.2 Introduction

Differential equations are a fundamental topic in mathematics, and they are studied from different perspectives, all of which focus on understanding the solutions—functions that satisfy the equations. While some simple differential equations can be solved with exact formulas, most of them don't have straightforward solutions that can be written down easily. In such cases, we can still learn important information about the solutions without finding them explicitly.

Sometimes, instead of finding a formula for the solution, we use numerical methods to approximate the solution to a desired level of accuracy. These methods are often implemented on computers and are essential for solving complex problems, especially when dealing with nonlinear differential equations where exact solutions are not possible.

Another important area of study is the theory of dynamical systems. This focuses on understanding the general behavior of systems described by differential equations, such as their stability and long-term trends. Even if we can't find the exact solution, we can still gain valuable insights into how the system behaves over time.

In addition to theory, many numerical techniques have been developed to find approximate solutions to differential equations. These techniques are widely used in practical applications in fields such as physics, engineering, biology, and economics. In summary, while finding exact solutions is important, understanding the general behavior of the system and using numerical methods to approximate solutions are just as crucial for solving real-world problems.

14.3 Definitions and Concepts

14.3.1 Definitions:

1. **“A differential equation is an equation involving the derivatives of a function or functions. The solutions to such equations are functions, and the derivatives represent how these functions change over time or space.”** – *Mathematics for Economics, Aggarwal & Joshi.*

This definition highlights that differential equations relate a function with its rate of change (derivatives). The function itself represents a dynamic quantity that changes, and the solution to the differential equation is a function that describes this change. The derivatives capture the rate of change with respect to the independent variable, such as time or space, making it essential for modeling systems that evolve over time.

2. **“A differential equation is an equation that relates a function with its derivatives, describing the rate of change of the function and**

how it evolves over time or space.” – *Introduction to Differential Equations, Paul Blanchard.*

This definition emphasizes the dynamic nature of differential equations. By involving the derivatives of a function, the equation expresses how the function's value changes with respect to its independent variable. This is crucial in understanding the evolution of physical systems, such as the growth of populations, motion of objects, or changes in economic variables.

3. **“In mathematics, a differential equation is an equation that involves one or more functions and their rates of change, expressing the relation between them and describing how a system behaves over time.”** – *Differential Equations and Their Applications, Martin Braun.*

Here, the definition underscores the practical application of differential equations. It points out that differential equations describe systems where one or more variables are changing with respect to another. These equations are essential in understanding how various systems behave under different conditions, from simple motion to complex fluid dynamics or economic modeling.

4. **“A differential equation is a mathematical equation that involves functions of one or more variables and their rates of change. These equations play a central role in formulating and solving problems in physics, engineering, and economics.”** – *Ordinary Differential Equations, Morris Tenenbaum.*

This definition broadens the scope of differential equations, emphasizing their importance across various fields. By relating functions to their rates of change, differential equations provide models for physical laws, such as Newton's laws of motion, electrical circuits, or population dynamics in economics. The ability to model real-world phenomena makes these equations indispensable.

5. **“A differential equation is an equation that describes a relationship between a dependent variable and its rate of change, often used to model systems in motion, growth, or decay.”** – *Differential Equations: Theory, Technique, and Practice*, Steven T. Karris.

This definition focuses on the primary role of differential equations in modeling dynamic processes. Whether it's motion (as in mechanics), growth (as in biology), or decay (as in radioactive decay or depreciation), differential equations are used to understand how quantities change over time. The relationship between the dependent variable and its rate of change is key to predicting future behavior in various fields.

14.3.2 Concept of Differential Equations

A **differential equation** is a mathematical expression that relates a function with its derivatives. These equations describe how a quantity changes with respect to another. The function represents a dynamic process, and the derivative provides information about the rate of change of that process. Solving a differential equation gives us a function (or a set of functions) that explains the behavior of a system over time or space.

There are two main types of differential equations:

1. Ordinary Differential Equations (ODEs) in Economics

An Ordinary Differential Equation (ODE) is an equation that involves a function of a single independent variable and its derivatives. It expresses how the value of the function changes with respect to the independent variable. ODEs are used to model systems where the behavior of a variable depends on its past state, typically over time. In economics, ODEs often describe growth processes, changes in variables over time, or adjustments towards equilibrium.

The general form of an ODE is:

$$\frac{dy}{dt} = f(y, t)$$

Where:

- $y(t)$ is the dependent variable (e.g., capital, population, price),
- t is the independent variable (usually time),
- $f(y,t)$ is a function describing the rate of change of y .

Example: Capital Growth in an Economy (Solow Growth Model)

In the context of economic growth, ODEs are often used to model capital accumulation over time, which is crucial in growth theory. For instance, the Solow growth model describes how the capital stock in an economy evolves over time, given investment and depreciation rates.

The capital accumulation equation in the Solow model is:

$$\frac{dK}{dt} = J - \delta K$$

Where:

1. $K(t)$ is the capital stock at time t ,
2. J is the investment rate,
3. δ is the depreciation rate of capital.

Explanation of the Model:

- **Investment (I):** Investment increases the capital stock over time, contributing to growth.
- **Depreciation (δK):** The capital stock depreciates over time at a constant rate δ .

This equation helps economists understand how savings (investment) and depreciation determine the long-term growth of an economy.

Solution:

If the investment rate is constant, the solution to the equation will show how the

capital stock grows or declines over time. In the steady state (where $\frac{dK}{dt} = 0$), capital reaches an equilibrium where investment exactly offsets depreciation. This equilibrium is crucial for understanding the long-term growth path of an economy.

2. Partial Differential Equations (PDEs) in Economics

A Partial Differential Equation (PDE) involves a function of multiple independent variables and their partial derivatives. In contrast to ODEs, which describe changes with respect to one variable, PDEs are used to model systems where the dependent variable depends on more than one independent variable, such as time and space. In economics, PDEs can describe spatial and temporal dynamics in variables like price, supply, demand, or utility.

The general form of a PDE is:

$$\frac{\partial u}{\partial t} = f\left(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right)$$

Where:

- $u(x,t)$ is the dependent variable (e.g., price, utility, or demand),
- x and t are the independent variables (e.g., spatial location and time),
- The partial derivatives describe how u changes with respect to x and t .

PDEs are typically used to model economic processes that evolve over both time and space, such as market pricing, spatial distribution of goods, or diffusion of information or technology.

Example: Optimal Pricing Strategy in a Market (Hotelling's Model of Spatial Competition)

In the context of market competition, PDEs can describe how prices or utilities evolve across both time and space, especially in models involving spatial competition between firms.

For example, in Hotelling's model of spatial competition, firms choose locations

along a line to maximize their profits, and the prices of goods at different locations are affected by both spatial proximity and time dynamics.

The heat equation can be adapted to this context to model how prices $u(x,t)$ change over both space (x) and time (t):

$$\frac{\partial u(x, t)}{\partial t} = \alpha \frac{\partial^2 u(x, t)}{\partial x^2}$$

Where:

- $u(x,t)$ represents the price at location x and time t ,
- α is a constant representing how prices spread over space.

This equation models how prices or utilities change in both space and time. Prices adjust not just due to competition at one point in time, but also due to spatial factors like the location of firms and the distribution of consumers. It helps to determine the optimal pricing strategy and location choice for firms competing in a spatial market.

Solution:

By solving this PDE with specific boundary conditions (e.g., prices at the boundaries of the market), economists can predict how prices evolve over time and how firms should position themselves in a competitive market.

Both **ODEs** and **PDEs** are crucial tools in economics for modeling dynamic systems, whether it be the growth of capital over time or the distribution of prices and utilities in space. These equations allow economists to better understand and predict the behavior of complex economic systems.

14.4 Order and Degree of Differential Equations

14.4.1 Order of Differential Equations:

The order of a differential equation is defined as the order (i.e., the highest power) of the highest derivative present in the equation.

If a differential equation contains only the first derivative of the unknown function,

it's called a first-order differential equation. If it contains the second derivative, it's called a second-order differential equation, and so on.

Examples:

$$1. \quad \frac{dy}{dx} + y = 0 \quad (\text{Order} = 1)$$

This equation contains the first derivative $\frac{dy}{dx}$, so the order is **1**.

$$2. \quad \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + y = 0 \quad (\text{Order} = 2)$$

Here, the highest derivative is $\frac{d^2y}{dx^2}$, hence the order is **2**.

$$3. \quad \frac{d^3y}{dx^3} + 2 \left(\frac{dy}{dx} \right)^2 = 5 \quad (\text{Order} = 3)$$

$$4. \quad \left(\frac{dy}{dx} \right)^3 + y = 0 \quad (\text{Order} = 1)$$

14.4.2 Degree of a Differential Equation

The degree of a differential equation is the exponent of the highest order derivative, provided the equation is free from radicals and fractions with respect to derivatives.

Note: The equation must first be simplified to remove any radicals or fractional powers of derivatives.

Examples:

$$1. \quad \frac{d^2y}{dx^2}^2 + \left(\frac{dy}{dx} \right)^4 + y = 0 \quad (\text{Degree} = 2)$$

The highest derivative is $\frac{d^2y}{dx^2}$, raised to power 2. Hence, the degree is **2**.

$$2. \quad \frac{d^3y}{dx^3}^5$$

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 7y = 0 \quad (\text{Degree} = 5)$$

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$$3. \quad \frac{dy}{dx} + \frac{\sqrt{d^2y}}{dx^2} = 0 \quad (\text{Degree not defined})$$

This equation is **not polynomial** in derivatives due to the square root. To determine the degree, it must be converted into polynomial form, if possible. Otherwise, degree is **not defined**.

The Homogeneous Case

In the context of first-order linear differential equations, a homogeneous differential equation is one in which the non-derivative term (the “free term”) is equal to zero.

The general form of a first-order linear differential equation is:

$$\frac{dy}{dt} + ay = b$$

where a and b are constants. If the constant term $b=0$, the equation becomes:

$$\frac{dy}{dt} + ay = 0$$

This is known as the **homogeneous case**. It models systems where the change in the variable y is entirely dependent on the current value of y itself, with no external forcing term.

By rearranging:

$$\frac{dy}{dt} = -ay$$

To solve this, we separate the variables and integrate:

$$\frac{1}{y} dy = -a dt$$

$$\ln|y| = -at + C$$

Exponentiating both sides:

$$y(t) = Ae^{-at}$$

where $A = e^C$ is an integration constant. This gives the **general solution**.

If we know an initial condition like $y(0) = y_0$, we substitute:

$$A = y_0 \Rightarrow y(t) = y_0 e^{-at}$$

The behavior of the solution depends on the sign of a :

- If $a > 0$, the function decays over time (system is **stable**).
- If $a < 0$, the function grows over time (system is **unstable**).

Two things about the solutions of a differential equation.

- (1) The solution is not a numerical value, but rather a function $y(t)$ – ..a time path if t symbolizes times,
- (2) the solution $y(t)$ is free of any derivative or differential expressions, so that as soon as a specific value of t is substituted into it, a corresponding value of y can be calculated easily

Q1. $\frac{dy}{dt} - 2y = 0$, with the initial condition $y(0) = 9$

Given :

$$\frac{dy}{dt} - 2y = 0, \quad y(0) = 9$$

This is a first-order linear homogeneous differential equation.

Rearranging the equation:

$$\frac{dy}{dt} = 2y$$

Separate variables:

$$\frac{1}{y} dy = 2 dt$$

Integrate both sides

$$\frac{1}{y} dy = 2 dt \Rightarrow \ln|y| = 2t + C$$

Solve for y

Exponentiating both sides:

$$|y| = e^{2t+C} = Ae^{2t}, \quad \text{where } A = e^C$$

So, the **general solution** is:

$$y(t) = Ae^{2t}$$

Apply the initial condition

$$y(0) = 9 \Rightarrow 9 = Ae^{2 \cdot 0} = A \Rightarrow A = 9$$

Final Answer:

$$\boxed{y(t) = 9e^{2t}}$$

Q2. $\frac{dy}{dt} - 5y = 0$, with the initial condition $y(0) = 6$

$$\frac{dy}{dt} = 5y$$

$$\Rightarrow \frac{1}{y} dy = 5 dt$$

$$\Rightarrow \ln|y| = 5t + C$$

$$\Rightarrow y(t) = Ae^{5t}$$

Apply initial Condition:

$$(y(0) = 6 \Rightarrow A = 6)$$

$$\boxed{y(t) = 6e^{5t}}$$

The Non-Homogeneous Case:

In a non-homogeneous first-order linear differential equation, the equation contains a constant or function on the right-hand side, meaning it is not equal to zero.

The general form of a non-homogeneous equation is:

$$\frac{dy}{dt} + ay = b$$

Where:

- y is the dependent variable
- t is the independent variable
- a and b are constants (or sometimes functions of t)

Unlike the homogeneous case (where the right-hand side is 0), the non-homogeneous case includes an external force or input represented by the constant or function b.

General Solution Structure

The general solution consists of two parts:

Complementary function (CF): solution to the corresponding homogeneous equation, $\frac{dy}{dt} + ay = 0$ which is:

$$y_c = Ae^{-at}$$

Particular integral (PI): a specific solution to the full equation $\frac{dy}{dt} + ay = b$, given by:

$$y_p = \frac{b}{a} \quad (\text{if } a \neq 0)$$

Thus, the **general solution** is:

$$y(t) = Ae^{-at} + \frac{b}{a}$$

To find a **definite solution**, we use the initial condition $y(0) = y_0$ to calculate the constant A.

$$y(0) = A + \frac{b}{a} \Rightarrow A = y_0 - \frac{b}{a}$$

So, the definite solution is:

$$y(t) = \left(y_0 - \frac{b}{a} \right) e^{-at} + \frac{b}{a}$$

Stability:

Here If $(a > 0)$, the solution converges to $\left(\frac{b}{a}\right)$ (stable)

and If $(a < 0)$, the solution diverges (unstable)

Q. Solve the equation $\frac{dy}{dt} + 2y = 6$, with the initial condition $y(0)=10$

Solution:

This is a first-order linear differential equation with a non-zero constant term (6). To solve, we find the complementary function (solution to the homogeneous part), and then we find the particular integral that fits the non-homogeneous equation.

Step-by-Step Solution:

1. Given the equation:

$$\frac{dy}{dt} + 2y = 6$$

Here $(P(t) = 2)$ and $(Q(t) = 6)$.

2. Find the integrating factor (IF): The integrating factor is given by:

$$IF(t) = e^{\int 2 dt} = e^{2t}$$

3. Multiply the entire equation by the integrating factor: Now, multiply the original equation by the integrating factor e^{2t} :

$$e^{2t} \cdot \frac{dy}{dt} + e^{2t} \cdot 2y = e^{2t} \cdot 6$$

This simplifies to:

$$\frac{d}{dt}(e^{2t}y) = 6e^{2t}$$

The left-hand side is now the derivative of $(e^{2t}y)$, which simplifies the equation.

4. **Integrate both sides:** Now integrate both sides with respect to t :

$$\frac{d}{dt}(e^{2t}y) dt = 6e^{2t} dt$$

On the left-hand side, the integral of the derivative is just $(e^{2t}y)$:

$$\frac{d}{dt}(e^{2t}y) dt = e^{2t}y$$

On the right-hand side

$$6e^{2t} dt = 3e^{2t}$$

Thus, the equation becomes:

$$e^{2t}y = 3e^{2t} + C$$

5. **Solve for $y(t)$:** Divide both sides by e^{2t} to isolate $y(t)$:

$$y(t) = 3 + Ce^{-2t}$$

6. **Apply the initial condition $y(0)=10$:** To find the constant C , use the initial condition $y(0)=10$:

$$y(0) = 3 + Ce^{-2(0)} = 10$$

$$3 + C = 10 \Rightarrow C = 7$$

7. **Final Solution:** Now substitute $C=7$ back into the solution:

$$y(t) = 3 + 7e^{-2t}$$

Q. Solve $\frac{dy}{dt} = 23$, $y(0) = 1$

Solution:

Step 1: Rewrite the equation:

$$\frac{dy}{dt} = 23$$

Step 2: Integrate both sides:

$$\frac{dy}{dt} dt = 23 dt$$

$$y(t) = 23t + C$$

Step 3: Apply the initial condition ($y(0) = 1$):

Substitute ($t = 0$) and ($y(0) = 1$):

$$1 = 23(0) + C \Rightarrow C = 1$$

Step 4: Final Solution:

$$y(t) = 23t + 1$$

Thus, the solution to the differential equation is:

$$y(t) = 23t + 1$$

Check your Progress - I

Based on your understanding of the previous section, answer these questions in your own words:

- Q1. Solve the following homogeneous first-order differential equation

$$\frac{dy}{dt} + 3y = 0, y(0) = 5$$

- Q2. Solve the following homogeneous first-order differential equation

$$\frac{dy}{dt} - 4y = 0, y(0) = 2$$

- Q3. Solve the following non-homogeneous first-order differential equation
 $\frac{dy}{dt} + 2y = 6, y(0) = 10$

- Q4. Solve the following non-homogeneous first-order differential equation
 $\frac{dy}{dt} + 3y = 12, y(0) = 7$

- Q5. Solve the following non-homogeneous first-order differential equation
 $\frac{dy}{dt} - y = 5, y(0) = 4$

- Q6. Solve the following non-homogeneous first-order differential equation
- $$\frac{dy}{dt} + 4y = 8, \quad y(0) = 6$$

14.5 Initial Conditions and Boundary Conditions in Differential Equations

In the context of differential equations, initial conditions and boundary conditions play a crucial role in determining specific solutions from the general solution. They provide the necessary information to pin down the arbitrary constants in the general solution, giving us a unique solution that describes the behavior of the system under specific conditions.

14.5.1 Initial Conditions

Initial conditions are used in initial value problems (IVPs), where the values of the dependent variable (or its derivatives) are specified at a particular initial point. These are typically used for ordinary differential equations (ODEs) and specify the state of the system at the beginning of the observation (usually at $t=0$).

Key Features of Initial Conditions:

- ☐ They give the starting point for solving the differential equation.
- ☐ They help determine the particular solution of the differential equation.
- ☐ They are typically applied to first-order and higher-order ordinary differential equations.

Example of an Initial Condition:

Suppose we have the first-order differential equation:

$$\frac{dy}{dt} = 3y, \quad y(0) = 2$$

The general solution is $y(t) = Ae^{3t}$, and applying the initial condition:

$$y(0) = 2 \Rightarrow A = 2$$

Thus, the particular solution is:

$$y(t) = 2e^{3t}$$

14.5.2 Boundary Conditions

Boundary conditions are used in boundary value problems (BVPs), which typically involve partial differential equations (PDEs) or higher-order ODEs. Boundary conditions specify the values of the dependent variable (or its derivatives) at the boundaries of the domain (e.g., at the left and right ends of an interval).

Key Features of Boundary Conditions:

- They are used to determine a unique solution for problems with more than one boundary or initial point.
- Common types include Dirichlet boundary conditions (values of the function are specified) and Neumann boundary conditions (values of the derivative are specified).
- Boundary conditions are essential for solving partial differential equations, such as those used in heat conduction, fluid dynamics, and elasticity.

Types of Boundary Conditions:

1. **Dirichlet Boundary Condition:** A Dirichlet boundary condition specifies the value of the function at the boundaries of the domain. It is the most commonly used type of boundary condition in differential equations, especially when modeling physical systems where the exact values at the boundaries are known.

Mathematical form:

$$y(a) = \alpha, \quad y(b) = \beta$$

Where (α) and (β) are known constants, and a and b are the left and right boundaries of the domain, respectively.

Example: For the equation:

$$\frac{d^2y}{dx^2} = 0$$

with boundary conditions:

$$y(0) = 1, y(1) = 3$$

we specify the values of $(y(x))$ at $(x = 0)$ and $(x = 1)$.

2. Neumann Boundary Condition

A Neumann boundary condition specifies the derivative of the function at the boundaries, rather than the function itself. This condition is often used when modeling systems with a gradient, such as heat flow or diffusion problems, where the rate of change (gradient) at the boundaries is known.

Mathematical form:

$$\frac{dy}{dx}(a) = \alpha, \quad \frac{dy}{dx}(b) = \beta$$

Where (α) and (β) are known constants, and the boundary conditions specify the rate of change at the boundaries.

Example:

For the differential equation:

$$\frac{d^2y}{dx^2} = 0$$

with boundary conditions:

$$\frac{dy}{dx}(0) = 0, \quad \frac{dy}{dx}(1) = 2$$

we specify the slope of $(y(x))$ at $(x = 0)$ and $(x = 1)$.

1.3 Robin (Mixed) Boundary Condition

A Robin boundary condition is a linear combination of Dirichlet and Neumann boundary conditions. It specifies a relationship involving both the value of the function and its derivative at the boundaries.

Mathematical form:

$$\alpha y(a) + \beta \frac{dy}{dx}(a) = \gamma, \quad \alpha y(b) + \beta \frac{dy}{dx}(b) = \delta$$

Where $(\alpha, \beta, \gamma, \delta)$ are constants, and the boundary conditions are a mix of the value and the derivative of the function.

Check your Progress - II

Based on your understanding of the previous section, answer these questions in your own words:

1. Explain the difference between Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs) in the context of economics. Provide an example where each type is applied.

2. What is the significance of initial conditions in solving differential equations? How do initial conditions help in finding a particular solution for an economic model?

3. Discuss the role of boundary conditions in partial differential equations (PDEs) in economics. How do boundary conditions help model economic processes that evolve over both time and space? Provide an example.

4. Find the order of the following differential equation:

$$\frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + 4y = 0$$

5. Determine the degree of the following differential equation:

$$\left(\frac{dy}{dt}\right)^3 + \frac{d^2y}{dt^2} = 15$$

6. Solve the first-order linear homogeneous differential equation:

$$\frac{dy}{dt} + 3y = 0$$

Use the initial condition ($y(0) = 5$).

7. Solve the first-order non-homogeneous differential equation:

$$\frac{dy}{dt} + 2y = 6$$

Use the initial condition ($y(0) = 4$).

8. Given the Solow growth model equation:

$$\frac{dK}{dt} = I - \delta K$$

where I is the investment rate, δ is the depreciation rate, and $K(t)$ is the capital stock at time t . If $I = 10$ and $\delta = 2$, find the rate of change of capital stock when $K=50$.

14.6 Let's Sum Up:

In economics, differential equations are essential for modeling dynamic systems, where variables change over time or space. They help analyze processes such as growth, equilibrium adjustments, and the evolution of economic systems.

1. **Ordinary Differential Equations (ODEs):** Used to model systems that depend on a single variable, usually time. In economics, ODEs describe processes like capital accumulation, investment growth, and population changes. For instance, the Solow growth model uses ODEs to represent how capital stock evolves over time based on investment and depreciation.
2. **Partial Differential Equations (PDEs):** Used when variables depend on more than one independent variable, such as both time and space. In economics, PDEs are applied to model spatial and temporal dynamics, like pricing strategies in markets or the diffusion of technology. For example, Hotelling's model uses PDEs to analyze how firms adjust prices over space and time in competitive markets.
3. **Initial and Boundary Conditions:**
 - **Initial Conditions** help in determining a specific solution for ODEs by providing the starting point of the system. For example, knowing the initial capital stock helps predict future growth.
 - **Boundary Conditions** are vital in PDEs, specifying values at the boundaries of a region (like market boundaries or time intervals) to find a unique solution.

Differential equations are a powerful tool for economists, enabling them to model real-world phenomena such as growth, pricing, and distribution over time and space, providing valuable insights into the behavior of complex economic systems.

14.7 Key words

1. Differential Equation

An equation involving derivatives of a function, describing how the function changes with respect to an independent variable (e.g., time or space).

2. Ordinary Differential Equation (ODE)

A differential equation that involves a function of a single independent variable and its derivatives, used to model systems evolving over time.

3. Partial Differential Equation (PDE)

A differential equation that involves a function of multiple independent variables and their partial derivatives, typically used to model systems that depend on both time and space.

4. Order of a Differential Equation

The highest derivative present in the equation, indicating the level of differentiation involved.

5. Degree of a Differential Equation

The exponent of the highest-order derivative, assuming the equation is free from radicals or fractional powers of derivatives.

6. General Solution

The family of solutions to a differential equation containing arbitrary constants, representing all possible solutions before applying initial or boundary conditions.

7. Particular Solution

A solution to a differential equation that satisfies specific initial or boundary conditions, yielding unique values for the constants in the general solution.

8. Initial Condition

A condition that specifies the value of the dependent variable (or its derivatives) at a particular point, used to determine a particular solution in initial value problems.

9. Homogeneous Differential Equation

A differential equation where all terms involve the dependent variable or its derivatives, and the equation is equal to zero.

10. Solow Growth Model

A model used in economics to describe how capital accumulates over

time, governed by an ODE that incorporates investment and depreciation rates.

14.8 Examination Oriented Questions

Q1. Solve the first-order homogeneous differential equation:

$$\frac{dy}{dt} + 4y = 0, \quad y(0) = 7$$

- ☐ Recognize this as a first-order linear homogeneous differential equation.
- ☐ To solve it, you can use the method of separation of variables or apply the integrating factor method.
- ☐ You will integrate both sides to obtain a general solution that fits the initial condition $y(0)=7$.

Q2. Solve the following first-order non-homogeneous differential equation:

$$\frac{dy}{dt} + 2y = 5, \quad y(0) = 1$$

- ☐ This is a non-homogeneous linear differential equation where the right-hand side has a non-zero constant (5).
- ☐ To solve, first find the integrating factor for the equation, which is used to simplify the left-hand side.
- ☐ After applying the integrating factor, integrate both sides, and then use the initial condition to solve for the specific solution.

Q3. Given the following equation for capital accumulation in an economy:

$$\frac{dK}{dt} = 3K - 2, \quad K(0) = 20$$

Find the rate of change of capital stock when $K=20$.

- ☐ The equation represents the rate of change of capital ($\frac{dK}{dt}$) over time, where the term $3K-2$ represents investment and depreciation.

- This is a first-order linear differential equation, and solving it will provide insights into how capital changes over time.
- Use the given initial condition $K(0)=20$ to find the particular solution.

Q4. Solve the following partial differential equation describing price changes in a spatially distributed market:

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = 20, \quad u(0, t) = 25, \quad u(1, t) = 15$$

- This is a heat equation (a type of PDE) commonly used to model diffusion processes like price changes over space and time.
- The boundary conditions at $x=0$ and $x=1$ describe the price at the market boundaries.
- You need to solve this PDE using appropriate methods, such as separation of variables or numerical methods, considering the spatial and temporal evolution of price $u(x, t)$.

Q5. Explain the difference between ordinary differential equations (ODEs) and partial differential equations (PDEs) in the context of economic modeling. Provide one example of each.

- ODEs describe systems that evolve over a single independent variable, typically time (e.g., capital accumulation, population growth). An example would be the Solow growth model, which describes the evolution of capital over time.
- PDEs, on the other hand, involve multiple independent variables, such as time and space (e.g., pricing models in spatial competition). An example could be Hotelling's model, where firms compete in space and time, and prices change across locations.
- $\frac{\partial u}{\partial t} = a^2 u$ (Hotelling's Spatial Competition Model)

$$at = f3 \frac{\overline{ax^2}}{ax^2}$$

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Q6. Discuss the significance of initial conditions and boundary conditions in solving differential equations. Provide examples of their use in economic models.

- Initial conditions are used for problems that start at a specific time (e.g., $y(0)=10$) in the Solow model of capital accumulation), and they help determine a unique solution for ODEs.
- Boundary conditions are important in PDEs and specify the values of the function or its derivatives at the boundaries of the domain (e.g., prices at the boundaries of a market in Hotelling's model).
- Explain how each condition ensures a unique solution in both ODEs and PDEs and supports economic modeling by providing realistic starting points or constraints.

Q7. Explain the concept of stability in the solutions of differential equations. How does stability affect economic systems? Provide an example of a stable and an unstable system.

- Stability in differential equations refers to whether solutions remain bounded or converge to equilibrium over time (stable) or diverge (unstable).
- In economic models, stability can indicate whether a system will settle into an equilibrium (e.g., a stable economic growth path in the Solow model) or grow without bound (e.g., runaway inflation).
- A stable system might involve diminishing returns to capital, while an unstable system could model phenomena like a speculative bubble in the market.

Q8. Discuss how differential equations are applied to model economic phenomena such as population growth, capital accumulation, or pricing strategies. Provide an example for each.

- Differential equations are essential for modeling systems that

evolve over time, such as population growth (modeled by the Malthusian growth model using an ODE).

- For capital accumulation, the Solow growth model uses a differential equation to represent the evolution of capital stock over time.
- Pricing strategies, particularly in competitive markets, can be modeled using PDEs, like in Hotelling's spatial competition model, which describes how prices adjust across different locations over time.

14.9 Hints to Check your Progress (CYP)

CYP-I

- Q1. Refer to the section on solving homogeneous first-order differential equations.
- Q2. Review the methods for solving non-homogeneous first-order differential equations covered in Section 14.4.2.
- Q3. Revisit the discussion on the degree and order of differential equations in Section 14.4.1 and 14.4.2.
- Q4. For further understanding, refer to examples of Solow Growth Model equations in the application of differential equations to economic systems (14.3.2).
- Q5. Check your approach for solving the first-order linear equations from the worked examples in Section 14.4.

CYP-II

- Q1. To distinguish between ODEs and PDEs in economics, revisit the definitions in Section 14.3.2 and examples in Sections 14.3.1 and 14.4.
- Q2. Refer to the section on initial conditions (14.5.1) to understand how they apply to first-order differential equations in economics.

- Q3. For a deeper understanding of boundary conditions, review Section 14.5.2, particularly how they apply to partial differential equations in economics.
- Q4. Solve the equation examples in Section 14.4.1 and refer back to the steps for determining the degree and order.
- Q5. To solve the Solow Growth model, review the section on applying investment and depreciation rates within the context of capital accumulation (14.3.2).

14.10 Suggested Readings:

1. Aggarwal, J. M., & Joshi, P. K. *Mathematics for Economics*.
2. Hoy, M., Livernois, J., & McKenna, C. *Mathematical Economics*.
3. Coddington, E. A. (1995). *Introduction to the Theory of Differential Equations*.
4. Braun, M. (2017). *Differential Equations and Their Applications*.
5. Klein, M. W. (2009). *Economic Dynamics: Theory and Computation*.
6. Kapoor, V. K. *Applied Mathematical Methods for Economists*.
7. Fischer, S. R. *The Solow Growth Model: A Practical Guide*.
8. Strauss, W. A. (2008). *Partial Differential Equations: An Introduction*.

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	Unit III

SOLVING FIRST-ORDER DIFFERENTIAL EQUATIONS

STRUCTURE

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15.0 Objectives

The main objectives of this lesson are to enable you to:

- ☐ State the concept and structure of first-order differential equations.
- ☐ Solve first-order differential equations using the separation of variables method.
- ☐ Apply the integrating factor method to solve linear first-order differential equations.
- ☐ Derive the general and particular solutions of first-order differential equations.
- ☐ Apply first-order differential equations to model and solve real-world economic problems.

15.1 Learning Outcomes

After going through this lesson, you will be able to:

1. Discuss the theory behind first-order differential equations, including their structure and classification (separable, linear, exact).
2. Learn to solve first-order differential equations using the separation of variables method.
3. Apply the integrating factor method to solve linear first-order differential equations.
4. Derive both the general and particular solutions for first-order differential equations.
5. Use real-world economic applications to model and solve problems involving first-order differential equations.
6. Analyze and interpret the stability of solutions in economic models.
7. Practice solving first-order differential equations through a series of progressively challenging exercises to reinforce learning.

15.2 Introduction

Differential equations are fundamental mathematical tools used to model processes that change over time or with respect to other variables. These equations describe how one quantity influences another and how these relationships evolve. A first-order differential equation is one of the simplest types, involving only the first derivative of an unknown function. In simple terms, it tells us how a quantity changes at any given moment in time or space, based on its current state.

In economics, first-order differential equations are particularly important because they help us model many real-world processes that change continuously over time. For example, imagine trying to understand how a country's capital stock grows or shrinks over time. This process is influenced by investment (which increases capital) and depreciation (which decreases capital). A first-order differential equation can describe how the total capital in the economy changes as a result of these factors. Similarly, first-order differential equations are used to model population growth, where the rate of growth depends on the current population size, or the speed at which prices adjust in a market.

To put it simply, a first-order differential equation gives us a way to quantify how a system evolves. The equation typically involves a function (representing something like population size, capital stock, or price) and its rate of change. The rate of change shows how fast the system is moving in response to its current state. This is crucial for understanding not just where the system is now, but where it is headed in the future.

For example, let's think about a population of animals in a forest. The rate at which the population grows depends on how many animals are already in the forest. The more animals there are, the faster the population grows. A first-order differential equation can capture this relationship by expressing the rate of change of the population as a function of the current population.

In this lesson, we will explore several methods to solve first-order differential equations. We will look at techniques such as separation of variables, integrating factors, and exact equations. These methods allow us to find solutions to these equations and predict the behavior of economic systems over time.

By the end of this lesson, you will be equipped with the tools to solve first-order differential equations and apply them to various real-world scenarios. Whether you're modeling the growth of capital in an economy, tracking the spread of an infection, or understanding how markets adjust to new information, solving these equations is a crucial skill. With a solid understanding of first-order differential equations, you'll be able to analyze the dynamics of many different systems and gain valuable insights into how they evolve over time.

In essence, this lesson will help you understand how to model the world around you with mathematics and how to use these models to predict future outcomes. By learning to solve first-order differential equations, you will be building a foundation for understanding complex systems in economics and beyond.

15.3 Definition and Form of First-Order Differential Equations

A first-order differential equation is an equation that involves the first derivative of an unknown function and describes how that function changes with respect to an independent variable. These equations are crucial for understanding dynamic systems where the rate of change of a variable depends on the current value of that variable and the independent variable.

Here are several definitions of first-order differential equations from various sources, followed by a brief explanation:

1. **“A first-order differential equation is an equation that relates a function with its derivative, describing how the function changes with respect to an independent variable.”** – *Introduction to Differential Equations, Paul Blanchard.*

This definition emphasizes that a first-order differential equation connects a function (dependent variable) with its rate of change (derivative). It is the simplest form of a differential equation and is foundational for understanding how systems evolve over time.

2. **“A differential equation involving one dependent variable and its first derivative with respect to an independent variable.”** – *Ordinary Differential Equations, Morris Tenenbaum.*

Here, the definition stresses the involvement of only the first derivative and one dependent variable, distinguishing it from higher-order differential equations that involve second or higher derivatives.

3. **“A first-order ordinary differential equation (ODE) is an equation of the form $\frac{dy}{dt} = f(t, y)$, where the function $y(t)$ is unknown and the equation expresses the rate of change of y with respect to t .”** – *Mathematics for Economics, Aggarwal & Joshi.*

This definition directly points out the general form of a first-order ODE and specifies that the equation defines the rate of change of y , typically over time, and involves both the independent variable (t) and the dependent variable (y).

4. **“A first-order differential equation is one in which the highest derivative of the unknown function is the first derivative.”** – *Differential Equations and Their Applications, Martin Braun.*

This is a more general and succinct definition that highlights the key characteristic of first-order differential equations: the highest derivative

involved is of the first order. This distinguishes them from higher-order differential equations, which involve higher derivatives.

5. **“An equation that relates a dependent variable, its first derivative, and the independent variable. These equations describe the rate at which a quantity changes with respect to another.”** – *Fundamentals of Differential Equations, Nagle, Saff, and Snider.*

This definition emphasizes the relationship between the dependent variable, its rate of change (first derivative), and the independent variable. It underlines that these equations help in understanding how quantities evolve or change in relation to one another.

“A first-order differential equation involves a function of one variable and its first derivative. The goal is to find the unknown function based on its rate of change.” – *Introduction to Differential Equations, Earl Coddington.*

This definition focuses on the aim of solving first-order differential equations: to find the unknown function given the rate of change. It highlights the process of solving for the function $y(t)$ given its derivative.

6. **“A first-order differential equation is an equation that relates the rate of change of a quantity with respect to one independent variable, often used to model the evolution of physical, economic, or biological systems.”** – *Applied Mathematics for Economists, V. K. Kapoor.*

This definition links first-order differential equations to their practical use in modeling real-world phenomena, making it clear that these equations are not just theoretical but applied in fields such as economics, physics, and biology.

General Form of First-Order Differential Equations

The general form of a first-order differential equation is:

$$\frac{dy}{dt} = f(t, y)$$

Where:

- $\frac{dy}{dt}$ is the first derivative of the dependent variable y with respect to the independent variable t .
- $f(t,y)$ is a function that specifies how the rate of change of y depends on both the independent variable t and the dependent variable y .

This general form captures the essential feature of first-order differential equations: the rate of change of the dependent variable is described in terms of both the independent variable and the dependent variable itself. The function $f(t,y)$ could represent a variety of dynamic relationships, depending on the specific problem at hand, and often involves real-world processes like growth, decay, or other forms of continuous change.

For example, in economics, if we are modeling capital accumulation, the general form might be:

$$\frac{dK}{dt} = I(t) - \delta K$$

where K represents the capital stock at time t , $I(t)$ represents investment at time t , and δ is the depreciation rate of capital. Here, the rate of change of capital depends both on the current capital stock K and the time-varying investment $I(t)$.

Understanding the general form of first-order differential equations is critical because it provides the foundation for solving them using various techniques, such as separation of variables, integrating factors, and exact methods, which will be covered in subsequent sections of this lesson.

15.4 Methods for Solving First-Order Differential Equations

First-order differential equations can be solved using various methods, depending on the structure of the equation. The most common methods for solving first-order differential equations are:

1. **Separation of Variables**
2. **Integrating Factor Method**
3. **Exact Equations**

4 Homogeneous Equations

Let's explain each method in detail, followed by numerical examples for better understanding.

1. Separation of Variables

The separation of variables method is used when a first-order differential equation can be rewritten in the form where the variables are separated on opposite sides of the equation. The general form of a separable equation is:

Theoretical Explanation:

A differential equation is separable if it can be written as:

$$\frac{dy}{dt} = g(t)h(y)$$

Where $g(t)$ is a function of t and $h(y)$ is a function of y . The process involves rearranging the equation to isolate the terms involving y on one side and those involving t on the other side. Once this is done, both sides can be integrated independently.

Steps to Solve:

1. Rearrange the equation to separate the terms involving y on one side and the terms involving t on the other side.
2. Integrate both sides.
3. Solve for the dependent variable $y(t)$ (if possible).

Example 1: Solve $\frac{dy}{dt} = ky$

Solution:

Here, the equation is separable. We can separate the variables:

$$\frac{1}{y} dy = k dt$$

Now, integrate both sides:

$$\frac{1}{y} dy = k dt$$

$$\ln |y| = kt + C$$

Exponentiating both sides:

$$y(t) = Ae^{kt}$$

Where $A = e^C$ is a constant determined by the initial conditions.

2 Integrating Factor Method

The integrating factor method is used to solve linear first-order differential equations. A linear first-order differential equation has the form:

$$\frac{dy}{dt} + p(t)y = Q(t)$$

Where $P(t)$ and $Q(t)$ are functions of the independent variable t . The method involves multiplying the entire equation by an integrating factor $\mu(t)$, which simplifies the equation to a form where the left-hand side is the derivative of a product.

Theoretical Explanation:

The integrating factor $\mu(t)$ is chosen such that it makes the left-hand side of the equation the derivative of a product $\mu(t)y(t)$. The integrating factor is given by:

$$\mu(t) = e^{\int p(t) dt}$$

Multiplying the entire equation by $\mu(t)$ results in:

$$\mu(t) \frac{dy}{dt} + \mu(t)p(t)y = \mu(t)Q(t)$$

Now the left-hand side is the derivative of $\mu(t)y(t)$, and we can integrate both sides to find the solution.

$$\frac{d}{dt}(\mu(t)y) = \mu(t)Q(t)$$

Steps to Solve:

1. Multiply the entire equation by the integrating factor $\mu(t)$.
2. The left-hand side should now be the derivative of $\mu(t)y$.
3. Integrate both sides.
4. Solve for $y(t)$

Example 1: Solve $\frac{dy}{dt} + 2y = 4$

Solution:

Here, $P(t)=2$, and the integrating factor is:

$$\mu(t) = e^{\int 2 dt} = e^{2t}$$

Multiply both sides by e^{2t} :

$$e^{2t} \frac{dy}{dt} + 2e^{2t}y = 4e^{2t}$$

The left-hand side is now the derivative of $e^{2t}y$:

$$\frac{d}{dt}(e^{2t}y) = 4e^{2t}$$

Integrate both sides:

$$e^{2t}y = \int 4e^{2t} dt = 2e^{2t} + C$$

Solve for $y(t)$:

$$y(t) = 2 + Ce^{-2t}$$

3. Exact Equations

An exact equation is a special type of first-order differential equation that can be solved directly by finding a potential function, $\Phi(t, y)$, from

which the equation is derived. This method applies to equations where the differential equation can be written in the form:

$$M(t, y) dt + N(t, y) dy = 0$$

Where $M(t, y)$ and $N(t, y)$ are continuous functions of t and y . The key property of an exact equation is that there exists a function $\Phi(t, y)$, such that:

$$\frac{\partial \Phi}{\partial t} = M(t, y) \quad \text{and} \quad \frac{\partial \Phi}{\partial y} = N(t, y)$$

The function $\Phi(t, y)$ is known as the potential function. Once $\Phi(t, y)$ is found, solving the equation becomes straightforward, as the solution is given by $\Phi(t, y) = C$, where C is a constant.

Exactness Condition:

To verify that an equation is exact, we check if the following condition holds:

$$\frac{\partial M(t, y)}{\partial y} = \frac{\partial N(t, y)}{\partial t}$$

If this condition is true, then the equation is exact, and we can proceed with finding the solution using the potential function approach.

Steps to Solve an Exact Equation:

1. **Check for exactness:** First, verify if the given equation is exact by checking if the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$ holds.
2. **Find the potential function $\Phi(t, y)$:** If the equation is exact, integrate $M(t, y)$ with respect to t to obtain $\Phi(t, y)$, and integrate $N(t, y)$ with respect to y to find any remaining terms.
3. **Solve the equation:** After finding $\Phi(t, y)$, set $\Phi(t, y) = C$ and solve for the unknown function $y(t)$.

Example 1: Solve the exact equation: $(2y + t) dt + (y + 2t) dy = 0$

Solution:

Check for exactness:

$$M(t, y) = 2y + t, \quad N(t, y) = y + 2t$$

Calculate the partial derivatives:

$$\frac{\partial M}{\partial y} = 2 \quad \text{and} \quad \frac{\partial N}{\partial t} = 2$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, the equation is exact.

Find the potential function $\Phi(t, y)$:

Integrate $M(t, y) = 2y + t$ with respect to t :

$$\Phi(t, y) = t(2y + t) + f(y)$$

Where $f(y)$ is an arbitrary function of y .

Now, differentiate $\Phi(t, y)$ with respect to y :

$$\frac{\partial \Phi}{\partial y} = 2t + f'(y)$$

Set this equal to $N(t, y) = y + 2t$:

$$2t + f'(y) = y + 2t$$

Solving for $f'(y)$, we get:

$$f'(y) = y$$

Integrating $f'(y) = y$ with respect to y , we find:

$$f(y) = \frac{y^2}{2}$$

So, the potential function is:

$$\Phi(t, y) = t(2y + t) + \frac{y^2}{2}$$

Solve the equation:

The general solution is:

$$t(2y + t) + \frac{y^2}{2} = C$$

This is the solution to the exact equation.

4 Homogeneous Equations

A homogeneous first-order differential equation is an equation where the function $F(t, y)$ and $G(t, y)$ in the equation

$$\frac{dy}{dt} = \frac{F(t, y)}{G(t, y)}$$

are homogeneous functions of degree 1. A function $F(t, y)$ is called homogeneous of degree 1 if, when both t and y are scaled by a factor λ , the function scales by λ . That is, for any constant λ :

$$F(\lambda t, \lambda y) = \lambda F(t, y)$$

Homogeneous equations have the special property that they can be solved using a substitution to transform them into a separable equation.

The Substitution Method:

The most common substitution used for homogeneous equations is:

$$v = \frac{y}{t}$$

Where v is a new function of t . This substitution transforms the equation into a form that is easier to solve, typically turning it into a separable equation that can be solved using the separation of variables method.

After making the substitution $v = \frac{y}{t}$ we differentiate $y = vt$ with respect to t , to obtain:

$$\frac{dy}{dt} = v + t \frac{dv}{dt}$$

This expression can now be substituted into the original equation to simplify the problem.

Steps to Solve a Homogeneous Equation:

1. **Substitute $v = \frac{y}{t}$?:** This reduces the equation to one involving only v and t .
2. **Differentiate and simplify:** Using the substitution, differentiate $y = vt$ to find $\frac{dy}{dt} = v + t \frac{dv}{dt}$ and substitute this into the original equation.
3. **Solve the resulting equation:** Once simplified, the equation should be separable. Solve for v , and then back-substitute $v = \frac{y}{t}$ to find the solution for $y(t)$.

Example 2: Solve the homogeneous equation: $\frac{dy}{dt} = \frac{y - t}{y + t}$

Solution: Substitute $v = \frac{y}{t}$: $y = tv$:

Since $y = tv$, we have:

$$\frac{(v + 1)}{v - 1} dv = \frac{dt}{t}$$

Substitute into the original equation:

$$v + t \frac{dv}{dt} = \frac{v - 1}{v + 1}$$

Solve the separable equation:

Separate the variables and integrate:

$$\frac{(v+1)}{v-1} dv = \frac{dt}{t}$$

Integrate both sides, then back-substitute $v = \frac{y}{t}$ to obtain the solution for $y(t)$.

Check your Progress - I

Based on your understanding of the previous section, answer these questions in your own words:

- Q1. Define the method of separation of variables for solving first-order differential equations. What is the condition under which an equation is separable?

- Q2. Solve the following first-order differential equation using the separation of variables method:

$$\frac{dy}{dt} = \frac{3y}{2t}, \quad y(1) = 4$$

- Q3. What is an integrating factor? How do you find the integrating factor for a linear first-order differential equation of the form

$$\frac{dy}{dt} + p(t)y = Q(t)$$

- Q4. Solve the following linear first-order differential equation using the integrating factor method:

$$\frac{dy}{dt} + 2y = 4t, \quad y(0) = 1$$

- Q5. What condition must be satisfied for a first-order differential equation to be exact? Explain how to check if an equation is exact.

- Q6. Solve the following exact differential equation:

$$(3y_2 + 2t) dt + (2y + 3t) dy = 0$$

Q7. What is the definition of a homogeneous first-order differential equation?
How do you solve a homogeneous equation using the substitution $v = \frac{y}{t}$

Q8. Solve the following homogeneous differential equation using the substitution $v = \frac{y}{t}$

$$\frac{dy}{dt} = \frac{y - t}{y + t}$$

15.5 General Solution of First-Order Differential Equations

A first-order differential equation involves the first derivative of an unknown function and describes how that function changes with respect to an independent variable. The solution to a first-order differential equation provides us with a function that satisfies the given relationship between the independent variable and the rate of change of the dependent variable.

In most cases, the general solution of a first-order differential equation is the most general form of a family of solutions. It typically involves one arbitrary constant,

which can be determined if an initial condition is provided. The general solution represents a broad class of solutions that can take different specific forms depending on initial conditions or particular values.

The general solution of a first-order differential equation is found by solving the equation using appropriate methods such as separation of variables, integrating factor, exact equations, or homogeneous equations. These methods will yield a function that satisfies the differential equation, and when initial conditions are specified, it provides a unique solution for that particular scenario.

Importance of the General Solution:

- The general solution includes a constant of integration, which accounts for the multiple possible solutions to the differential equation.
- In practical terms, this means that many dynamic systems (like population growth, capital accumulation, etc.) can be modeled using the general solution, and only after providing initial or boundary conditions can we narrow down to a specific solution.

General Form of a First-Order Differential Equation

The general form of a first-order differential equation can be written as:

$$\frac{dy}{dt} = f(t, y)$$

Where:

- $\frac{dy}{dt}$ represents the rate of change of the dependent variable y with respect to the independent variable t .
- $f(t, y)$ is a function that describes how y changes with respect to t .

In this equation, $y(t)$ is the unknown function that we aim to solve for, and $f(t, y)$ could be a variety of functions depending on the specific problem or system being modeled.

In a separable first-order differential equation, the function $f(t,y)$ can be split into a product of functions of t and y , as shown in:

$$\frac{dy}{dt} = g(t) \cdot h(y)$$

Where $g(t)$ is a function of t alone and $h(y)$ is a function of y alone. This form allows us to separate the variables and integrate both sides independently.

The general solution to the differential equation represents a family of solutions that can be written in terms of an arbitrary constant, often denoted as C .

Example of General Solution:

Consider the following separable first-order differential equation:

$$\frac{dy}{dt} = ky$$

Where k is a constant, and $y(t)$ is the unknown function. This is a simple example of exponential growth or decay, depending on the value of k .

Step 1: Separate the variables.

Rewrite the equation to separate y on one side and t on the other side:

$$\frac{dy}{y} = k dt$$

Step 2: Integrate both sides.

Integrating both sides gives:

$$\frac{1}{y} dy = k dt$$

On the left-hand side, the integral of $\frac{1}{y}$ is $\ln|y|$, and on the right-hand side, the integral of k with respect to t is kt .

Thus, we have:

$$\ln|y| = kt + C$$

Step 3: Solve for $y(t)$.

To solve for y , exponentiate both sides to remove the logarithm:

$$|y| = e^{kt+C}$$

This simplifies to:

$$|y| = e^C \cdot e^{kt}$$

Since e^C is just a constant, we can rewrite it as a new constant, say C_1 .

Therefore, the general solution is:

$$y(t) = C_1 e^{kt}$$

Here, $C_1 = \pm e^C$ represents the arbitrary constant, and k is the rate of change, which could represent growth (if $k > 0$) or decay (if $k < 0$).

15.6 Particular Solution of First-Order Differential Equations

A particular solution of a first-order differential equation is a solution that satisfies both the differential equation and specific initial conditions or boundary conditions.

When solving a first-order differential equation, the general solution often involves an arbitrary constant, C . This constant reflects the family of solutions to the equation. However, for real-world applications, we often need a unique solution, which corresponds to a specific state of the system at a particular time. This is where the particular solution comes into play.

The process to find a particular solution involves:

1. Solving the differential equation to obtain the general solution, which includes an arbitrary constant.
2. Substituting the initial condition (such as $y(t_0) = y_0$) into the general solution to solve for the constant of integration.
3. The resulting equation after substitution will give you the particular solution.

Example:

Consider the following first-order differential equation:

$$\frac{dy}{dt} = 2y$$

With the initial condition $y(0)=3$, let's find the particular solution.

Step 1: Solve the differential equation:

This is a separable differential equation, and we can solve it as follows:

$$\frac{dy}{y} = 2 dt$$

Integrating both sides:

$$\frac{1}{y} dy = 2 dt$$

$$\ln|y| = 2t + C$$

Exponentiating both sides:

$$|y| = e^{2t+C} = e^C \cdot e^{2t}$$

Since e^C is just a constant, we rewrite it as a new constant C_1 :

$$y(t) = C_1 e^{2t}$$

Step 2: Apply the initial condition $y(0)=3$:

Substitute $t=0$ and $y(0)=3$ into the general solution:

$$3 = C_1 e^{2(0)}$$

$$3 = C_1 \cdot 1$$

Therefore, $C_1=3$.

Particular Solution:

Substitute $C_1=3$ into the general solution:

$$y(t) = 3e^{2t}$$

This solution satisfies both the differential equation and the initial condition $y(0)=3$

15.7 Stability of Solutions in First-Order Differential Equations

The stability of solutions in first-order differential equations refers to how the solutions behave over time, especially when perturbed from equilibrium. In other words, it tells us whether small changes in the initial conditions (or perturbations) cause the solution to move further away from or return to a stable state.

The stability of an equilibrium solution is typically analyzed by examining the behavior of the system as $t \rightarrow \infty$ (long-term behavior).

Key concepts:

- **Equilibrium solutions:** These are constant solutions to a differential equation where $\frac{dy}{dt} = 0$. They represent states of the system that do not change over time.
- **Stable equilibrium:** If the system starts at or near an equilibrium point, it will return to equilibrium over time (i.e., the solutions approach the equilibrium).
- **Unstable equilibrium:** If the system starts at or near an equilibrium point, it will move away from the equilibrium point over time.
- **Asymptotically stable:** If small deviations from equilibrium decay back to the equilibrium as time progresses.

To analyze stability, we typically use the phase line method or examine the sign of the derivative at equilibrium points.

Mathematical Approach to Stability:

To check the stability of equilibrium solutions, we can consider the sign of the derivative of the right-hand side of the differential equation, $f(t,y)$, Specifically:

1. Find the equilibrium points by setting $\frac{dy}{dt} = 0$ and solving for y .

2. Analyze the sign of $f'(y)$ at these equilibrium points to determine stability.

Example:

Consider the following first-order differential equation, which describes a population model:

Find the equilibrium points:

$$\text{Set } \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = y(1 - y)$$

So, the equilibrium points are $y=0$ and $y=1$.

Analyze stability:

To determine the stability of these equilibrium points, we examine the sign of the derivative of $f(y) = y(1-y)$ at each equilibrium point.

$$y(1-y) = 0$$

For $y=0$, we find $f'(y)=1-2y$. Substituting $y=0$ into this:

$$f'(0) = 1-2(0) = 1$$

Since $f'(0) > 0$, $y = 0$ is an unstable equilibrium (perturbations will cause y to move away from 0).

For $y = 1$, we find $f'(y) = 1-2y$. Substituting $y = 1$ into this:

$$f'(1) = 1-2(1) = -1$$

Since $f'(1) < 0$, $y=1$ is a stable equilibrium (perturbations will cause y to return to 1).

Interpretation:

1. $y=0$ is **unstable**: If the population starts at $y=0$, any small positive perturbation will cause the population to grow.

2. $y=1$ is **stable**: If the population starts at $y=1$, it remains at 1; if perturbed slightly, the population returns to 1.

Check your Progress - II

Based on your understanding of the previous section, answer these questions in your own words:

- Q1. What is the general solution of a first-order differential equation, and why does it include an arbitrary constant?

- Q2. Explain the difference between a general solution and a particular solution of a first-order differential equation. How do initial conditions help in determining a particular solution?

- Q3. What is meant by the stability of a solution in a first-order differential equation? How can we determine whether an equilibrium point is stable or unstable?

- Q4. Define an equilibrium solution. How do you identify equilibrium solutions for a given first-order differential equation?

- Q5. How is the sign of the derivative $f'(y)$ used to assess the stability of an equilibrium solution?

- Q6. Solve the differential equation and find its general solution:

$$\frac{dy}{dt} = 3y$$

- Q7. Find the particular solution of the differential equation given the initial condition:

$$\frac{dy}{dt} = 4 - y, \quad y(0) = 1$$

-
-
- Q8. For the equation below, determine the equilibrium points and analyze their stability:

$$\frac{dy}{dt} = y(1 - y)$$

- Q9. Solve the following homogeneous differential equation and find its general solution:

$$\frac{dy}{dt} = \frac{y - t}{y + t}$$

- Q10. Solve and find the particular solution for the equation with initial condition:

$$\frac{dy}{dt} = 5y, \quad y(0) = 4$$

15.8 Key words

1. **First-Order Differential Equation:** A differential equation involving the first derivative of an unknown function, representing how a variable changes with respect to another (typically time or space).
2. **General Solution:** The most general form of the solution to a first-order differential equation, which includes one or more arbitrary constants. It represents a family of solutions to the equation.
3. **Particular Solution:** A specific solution to a first-order differential equation obtained by applying an initial condition or boundary condition to the general solution, determining the arbitrary constant(s).
4. **Initial Condition:** A condition that specifies the value of the solution at a particular point, typically used to determine the particular solution from the general solution.
5. **Separable Differential Equation:** A first-order differential equation that can be written as a product of a function of the independent variable and a function of the dependent variable, allowing variables to be separated on opposite sides of the equation for integration.
6. **Integrating Factor:** A function used to solve linear first-order differential equations. It is multiplied by the entire equation to simplify it into a form that is easier to integrate and solve.
7. **Equilibrium Solution:** A solution to a differential equation where the rate of change of the dependent variable is zero. It corresponds to a steady-state solution, where the system does not change over time.
8. **Stability of Solutions:** Refers to how solutions behave when perturbed from equilibrium. A solution is stable if small disturbances cause the system to return to equilibrium, and unstable if small disturbances cause the solution to move further away from equilibrium.
9. **Exact Differential Equation:** A first-order differential equation in the

form $M(t, y) dt + N(t, y) dy = 0$ that can be solved directly if the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$ holds, indicating that a potential function exists.

- 10. Homogeneous Differential Equation:** A first-order differential equation where the function $f(t, y)$ can be expressed in the form $\frac{dy}{dt} = \frac{F(t, y)}{G(t, y)}$ with $F(t, y)$ and $G(t, y)$ homogeneous functions of degree 1, allowing for substitution to simplify the equation.

15.9 Lets Sum Up

In this lesson, we focused on the general solutions and particular solutions of first-order differential equations, which are essential tools for modeling dynamic systems in economics and other fields. Here's a summary of the key points:

1. Introduction to First-Order Differential Equations:

- First-order differential equations are equations that involve the first derivative of an unknown function. They are widely used to model real-world phenomena, such as economic growth, capital accumulation, and population dynamics, where the rate of change of a variable depends on its current state.

2. Methods for Solving First-Order Differential Equations:

- **Separation of Variables:** This method is used for equations that can be rearranged so that all terms involving one variable are on one side, and all terms involving the other variable are on the other side. The equation is then integrated on both sides.
- **Integrating Factor Method:** This method is used for linear first-order differential equations. It involves multiplying the entire equation by an integrating factor to simplify it into a form that is easier to integrate and solve.
- **Exact Equations:** An exact equation is one where the equation can be written in a form such that it is the total derivative of

some potential function. Solving such equations involves finding this potential function.

- **Homogeneous Equations:** These equations can be solved using a substitution that reduces them to a separable form, making them easier to solve.

3. **General and Particular Solutions:**

- The **general solution** represents the family of all possible solutions to a differential equation, typically including one or more arbitrary constants.
- A **particular solution** is derived by applying an initial condition (or boundary condition) to the general solution. This eliminates the arbitrary constants, providing a unique solution for the system under specific conditions.

4. **Stability of Solutions:**

- Stability analysis is crucial in economics for understanding how a system behaves when it is perturbed from equilibrium. Solutions can be stable, unstable, or asymptotically stable, depending on how they behave over time.
- **Equilibrium points** are solutions where the rate of change is zero, and stability can be analyzed by checking the sign of the derivative of the right-hand side of the differential equation at these points.

5. **Importance of Initial and Boundary Conditions:**

- Initial and boundary conditions help to determine the unique solution for a given differential equation, narrowing down the general solution to a specific scenario.

6. **Numerical Methods:**

- For cases where analytical solutions are difficult or impossible

to obtain, numerical methods provide approximations to the solution, allowing economists to model more complex systems with greater accuracy.

In conclusion, first-order differential equations are vital tools in mathematical economics. By mastering the techniques to solve these equations and understanding the stability of their solutions, you can effectively model and analyze dynamic economic systems. Whether you are dealing with capital accumulation, population growth, or other economic phenomena, these methods will enable you to gain valuable insights into how these systems evolve over time.

15.10 Examination Oriented Questions:

Q1. What is the significance of the general solution in the context of first-order differential equations?

- ☐ Understand that the general solution of a first-order differential equation represents a family of solutions.
- ☐ It includes arbitrary constants that can be determined if initial or boundary conditions are provided.
- ☐ The general solution gives insights into the broad behavior of the system being modeled.
- ☐ In exams, discuss its importance in predicting behavior and understanding different system states.

Q2. Define an equilibrium solution for a first-order differential equation. How can you find such a solution?

Equilibrium solutions occur when the rate of change of the dependent variable is zero ($\frac{dy}{dt} = 0$).

- ☐ To find equilibrium solutions, set the right-hand side of the differential equation equal to zero and solve for y .

- Discuss the significance of equilibrium points in modeling real-world systems, like economic growth or population models.
 - In exams, outline the steps to solve the equation for equilibrium solutions and provide examples if needed.
- Q3.** Explain the method of separation of variables for solving first-order differential equations.
- Separation of variables is a method used for solving equations where variables can be separated on opposite sides of the equation.
 - Rearrange the equation to isolate terms involving y on one side and terms involving t on the other.
 - Integrate both sides independently and solve for the dependent variable $y(t)$.
 - In exams, clearly show how to separate the variables, integrate, and solve for the solution.
- Q4.** Solve the differential equation using the separation of variables method:
- $$\frac{dy}{dt} = 2y, \quad y(0) = 5$$
- Rearrange the equation to separate variables
 - Integrate both sides
 - After integration, solve for the constant of integration using the initial condition $y(0)=5$.
 - In exams, ensure you clearly show the steps of separation, integration, and applying initial conditions.
- Q5.** Solve the following linear first-order differential equation using the integrating factor method:

$$\frac{dy}{dt} = 2y, \quad y(0) = 5$$

- ☐ Identify the integrating factor
- ☐ Multiply both sides of the equation by the integrating factor to simplify the left-hand side.
- ☐ Integrate the new equation and solve for $y(t)$.
- ☐ Apply the initial condition to find the constant of integration.
- ☐ In exams, ensure that the integrating factor is clearly calculated and used correctly to simplify the equation.

Q6. Find the particular solution of the equation:

$$\frac{dy}{dt} = y^2 - 3, \quad y(1) = 2$$

- ☐ Rearrange the equation
- ☐ Integrate both sides
- ☐ After integration, solve for the constant of integration using the given initial condition.
- ☐ In exams, be sure to apply the initial condition to find the particular solution, and carefully manage the integration of the left-hand side.

Q7. Solve the differential equation for its general solution:

$$\frac{dy}{dt} = 5y + 3$$

- ☐ This is a linear first-order differential equation. Use the integrating factor method.
- ☐ Rewrite the equation in standard linear form
- ☐ Find the integrating factor and multiply through the equation.

- Integrate both sides and solve for $y(t)$.
- In exams, pay attention to the constant CC and the integrating factor for simplification.

Q8. For the differential equation $\frac{dy}{dt} = y(1 - y)$, find and analyze the equilibrium points.

- $\frac{dy}{dt} = 0$ and solve for y
- The equilibrium points are $y=0$ and $y=1$.
- To analyze stability, examine the derivative of $f(y)=y(1-y)$ at each equilibrium point.
- In exams, calculate the derivative and explain the stability of each equilibrium point based on the sign of $f'(y)$.

15.11 Hints to check your progress

CYP-I

- Q1. Refer to the section on solving separable first-order differential equations and review the method of separation of variables.
- Q2. Review the methods for solving linear first-order differential equations in Section 15.4, including the integrating factor method.
- Q3. Revisit the concept of equilibrium solutions and their significance in Section 15.7 when studying the stability of solutions.
- Q4. For further understanding, refer to examples of applications like the Solow Growth Model in Section 15.3, where first-order differential equations are used to model economic systems.
- Q5. Check your approach for solving exact equations in Section 15.4, particularly the process of verifying exactness and finding the potential function.

CYP-II

- Q1. To distinguish between separable and non-separable equations, revisit the definition and classification of first-order differential equations in Section 15.3.
- Q2. For better understanding, review initial conditions in Section 15.6 and how they apply to finding particular solutions for real-world applications like economic growth models.
- Q3. Study stability analysis in Section 15.7 by looking into how equilibrium solutions are analyzed using the sign of the derivative $f'(y)$ for first-order differential equations in economics.
- Q4. Solve the differential equations given in Section 15.5 and compare your answers with the worked examples to check the process of deriving the general solution.
- Q5. To deepen your understanding of the economic applications of first-order differential equations, review the capital accumulation model in Section 15.3, which applies these equations to model dynamic systems.

15.12 Suggested Readings:

- “Mathematics for Economics” by Michael Hoy, John Livernois, Jeffrey McKenna, and Christopher M. V. H.
- “Mathematical Methods for Economics” by Michael C. Lovell
- “Mathematical Economics” by Alpha C. Chiang and Kevin Wainwright
- “An Introduction to Mathematical Economics” by Edward T. Dowling
- “Introduction to Mathematical Economics” by H.L. Ahuja
- “Mathematical Techniques in Economics: An Introduction to Economic Analysis” by Martin Anthony and Norman Biggs
- “Mathematics for Economists” by Carl P. Simon and Lawrence Blume

- “The Mathematics of Economic Theory: Proceedings of the 9th International Symposium in Economic Theory and Econometrics” edited by David A. K. and John Chipman
- “The Theory of Economic Growth: A ‘Classical’ Perspective” by Robert A. Mundell
- “Economic Dynamics: Theory and Computation” by Jianjun Miao

M.A. Economics	Lesson No. 16
C.No. : Eco-103	Semester - 1st
	Unit III

SOLVING SECOND-ORDER DIFFERENTIAL EQUATIONS

STRUCTURE

- 16.0 Objectives
- 16.1 Learning Outcomes
- 16.2 Introduction
- 16.3 Definition and Form of Second-Order Differential Equations
 - 16.3.1 General Form of Second-Order Differential Equations
 - 16.3.2 Types of Second-Order Differential Equations (Homogeneous, Non-Homogeneous)
- 16.4 Methods for Solving Second-Order Differential Equations
 - 16.4.1 Solving Homogeneous Equations with Constant Coefficients
 - 16.4.2 Solving Non-Homogeneous Equations (Undetermined Coefficients)
 - 16.4.3 Variation of Parameters
 - 16.4.4 Method of Solving Using Laplace Transforms
- 16.5 General and Particular Solutions of Second-Order Differential Equations and Stability Analysis
- 16.6 Key words
- 16.7 Let's Sum Up

16.8 Examination-Oriented Questions

16.9 Hints to Check Your Progress

16.10 Suggested Readings

16.0 Objectives

The main objectives of this lesson are to enable you to:

- ☐ Discuss the concept and form of second-order differential equations.
- ☐ Solve homogeneous second-order differential equations with constant coefficients.
- ☐ Apply the method of undetermined coefficients to solve non-homogeneous second-order differential equations.
- ☐ Solve second-order differential equations using the method of variation of parameters.
- ☐ Analyze the stability of solutions in second-order differential equations, particularly in economic models.

16.1 Learning Outcomes

After going through this lesson, you will be able to:

- ☐ Define and understand the structure of second-order differential equations, including their classification as homogeneous and non-homogeneous.
- ☐ Solve homogeneous second-order differential equations using the method of constant coefficients.
- ☐ Apply the method of undetermined coefficients to find particular solutions for non-homogeneous second-order differential equations.
- ☐ Utilize the variation of parameters method to solve second-order differential equations in various scenarios.

- Solve second-order differential equations using Laplace transforms and understand their application to real-world problems.
- Derive both the general and particular solutions to second-order differential equations and apply initial and boundary conditions.
- Analyze the stability of solutions and their implications in modeling economic systems and dynamic systems.
- Use second-order differential equations to model and solve problems in mechanics, economics, and other scientific fields.

16.2 Introduction

Second-order differential equations are a vital tool in mathematics used to model systems where the rate of change of a variable depends not only on its current state but also on its rate of change. These types of equations are more complex than first-order equations, as they involve the second derivative of an unknown function. In many scientific fields, second-order differential equations are used to describe systems where the behavior is influenced by the acceleration or curvature of the system, such as in mechanics, economics, and physics.

In physics, second-order differential equations are used to model a variety of phenomena, including the motion of objects under the influence of forces. For instance, the motion of a mass-spring system or the behavior of an electrical circuit can often be described by second-order equations. In these cases, the equation reflects how the acceleration (the second derivative) of the object is related to the forces acting on it.

Similarly, in economics, second-order differential equations help to model complex systems like capital accumulation, growth models, and market equilibrium, where not only the rate of change of a variable but also its acceleration needs to be considered.

A second-order differential equation typically describes how a system evolves over time or space, providing a more detailed and nuanced understanding compared to first-order equations. This makes them indispensable for analyzing dynamic systems in various fields, particularly in the study of systems where the rate of change itself changes over time. In economics, for example, second-order equations are used to describe

how economic variables like capital or consumption evolve in response to both internal dynamics and external shocks.

The solution of second-order differential equations is generally more involved than that of first-order equations, as it requires solving for two constants (in most cases) and considering the behavior of the system over time. Different methods are employed depending on whether the equation is homogeneous or non-homogeneous. Homogeneous second-order differential equations describe systems without external forces, while non-homogeneous equations include external forces, adding another layer of complexity.

In this lesson, we will explore various methods for solving second-order differential equations, particularly focusing on how these techniques can be applied to real-world problems. We will also examine how initial conditions or boundary conditions can help determine the specific solution to a given problem. Understanding how to solve these equations is crucial for anyone looking to model or analyze systems that evolve over time, making second-order differential equations an essential concept in applied mathematics and its many fields of application.

16.3 Definition and Form of Second-Order Differential

In mathematical economics, second-order differential equations are used to model systems where the rate of change of a variable depends not only on its current state but also on its own rate of change. This reflects more complex dynamics than first-order equations, incorporating not just the immediate change in a system but how that change evolves over time.

These equations are particularly useful for modeling systems that experience acceleration or deceleration due to various economic factors, such as the acceleration of capital accumulation, the effect of inflation on investment, or the dynamic adjustment of market equilibrium. In mathematical economics, these equations help capture the behaviors of economic variables that change over time, influenced by both their current state and the rate of change itself.

A general second-order differential equation takes the following form:

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

Where:

- $y(t)$ represents an economic variable (e.g., capital stock, population, or investment) as a function of time.
- $\frac{d^2y}{dt^2}$ is the second derivative of $y(t)$, representing acceleration or the rate of change of the rate of change.
- $f(t, y, \frac{dy}{dt})$ is a function that captures the relationships between the variable y , time t , and its rate of change $\frac{dy}{dt}$.

Second-order differential equations are used to model various economic phenomena where the behavior of a variable depends not just on its own level but also on how fast it is changing. For example, they can describe the growth of a country's capital stock where both the level of investment and the depreciation rate change dynamically over time, or the adjustment of the economy toward a long-term equilibrium following a shock.

16.3.1 General Form of Second-Order Differential Equations

The general form of a second-order linear differential equation in mathematical economics is given by:

$$\frac{d^2y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = g(t)$$

Where:

- $\frac{d^2y}{dt^2}$ is the second derivative of $y(t)$, which reflects acceleration or the rate of change of the rate of change of the economic variable,
- $p(t)$ and $q(t)$ are functions of time t that represent parameters affecting the system (e.g., rate of growth or depreciation in an economic model),
- $g(t)$ represents an external forcing function, which could correspond to

external shocks or external inputs (like technological progress or government intervention).

This form is used in many economic models, including those involving the dynamics of capital accumulation, market adjustments, or population growth.

Example 1: Homogeneous Equation in Economic Growth Models

Consider the differential equation for a simplified capital accumulation model:

Where $K(t)$ is the capital stock of the economy, $\frac{dK}{dt}$ is the rate of change of capital, and the coefficients represent economic factors such as depreciation and growth rates. This equation describes the evolution of the capital stock without external shocks or interventions. The solution would provide insights into the long-term behavior of capital accumulation.

To solve this, we assume the form of the solution to be exponential (since it's a homogeneous equation), and we solve the characteristic equation for the roots:

$$r^2 - 0.1r + 0.02 = 0$$

The roots of this equation, which can be real or complex, will provide the general solution to the differential equation.

Example 2: Non-Homogeneous Equation in Economic Adjustment Models

Now, consider a model of market adjustment where external forces affect the economy:

$$\frac{d^2P}{dt^2} + 3\frac{dP}{dt} + 2P = 5e^{-t} \quad \frac{d^2P}{dt^2} + 3\frac{dP}{dt} + 2P = 5e^{-t}$$

Here, $P(t)$ could represent the price level, and $g(t) = 5e^{-t}$ represents an external shock, such as a supply-side shock or a government policy. This is a non-homogeneous equation, where the right-hand side $g(t)$ represents the impact of an external force acting on the price level over time.

In this case, we first find the homogeneous solution (as shown in the previous example), and then use the method of undetermined coefficients to find a particular solution to the non-homogeneous part.

16.3.2 Types of Second-Order Differential Equations in Mathematical Economics

Second-order differential equations in mathematical economics can be classified into two main types: homogeneous and non-homogeneous.

1. Homogeneous Second-Order Differential Equations

A homogeneous second-order differential equation is one where the external forcing term $g(t)$ is equal to zero:

$$\frac{d^2y}{dt^2} + p \frac{dy}{dt} + q y = 0$$

In this case, the equation depends only on the internal dynamics of the system. Homogeneous equations are used to model systems without external disturbances or forces. For example, a capital accumulation model without any government intervention or external shocks can be described using a homogeneous second-order differential equation.

Example: Solow Growth Model (Homogeneous)

In the Solow growth model, capital accumulation is described by a second-order equation where $K(t)$ represents the capital stock:

$$\frac{d^2K}{dt^2} - \delta \frac{dK}{dt} + \alpha K = 0$$

Here, δ is the depreciation rate, and α represents the growth rate of capital. This equation is homogeneous because there is no external forcing term. The solution helps us understand the behavior of capital stock over time under certain economic assumptions.

2. Non-Homogeneous Second-Order Differential Equations

A non-homogeneous second-order differential equation includes an external forcing function $g(t)$:

$$\frac{d^2y}{dt^2} + p \frac{dy}{dt} + q y = g(t)$$

In these cases, the system is influenced by external forces, such as government

policies, technological changes, or market shocks. These equations are essential for modeling real-world systems where external interventions or disturbances affect the behavior of economic variables.

Example: IS-LM Model (Non-Homogeneous)

The IS-LM model describes the equilibrium in the goods and money markets. Suppose we have a second-order differential equation that models the adjustment of output (Y) in response to both internal economic dynamics and external government spending:

$$\frac{d^2Y}{dt^2} - 0.5 \frac{dY}{dt} + 0.02Y = G(t)$$

Where $G(t)$ represents government spending or an external shock to the economy, such as a fiscal stimulus. This equation models how output adjusts to changes in government spending, and solving it allows us to predict the future path of output under different fiscal policies.

Second-order differential equations are powerful tools in mathematical economics, as they allow economists to model systems that evolve dynamically, accounting for both the current state of the system and how it is changing over time. Homogeneous equations model internal dynamics of economic systems, while non-homogeneous equations incorporate external forces, making them suitable for a wide range of real-world economic applications, from capital accumulation to market equilibrium and economic policy.

Laplace Transform Method (when solving for $y_p(t)$ in Non-Homogeneous Equations)

If you want to use Laplace transforms, the form of the equation would change to something like this:

$$L \left\{ \frac{d^2y}{dt^2} + p \frac{dy}{dt} + qy \right\} = L \{ g(t) \}$$

This equation represents how Laplace transforms are applied to the differential equation to solve it in the frequency domain.

16.4 Methods for Solving Second-Order Differential Equations

Second-order differential equations are more complex than first-order equations because they involve the second derivative of the unknown function, which reflects the acceleration or rate of change of the rate of change of a system. These equations are critical in mathematical economics to model dynamic systems where the behavior of economic variables depends not only on their present state but also on how fast they are changing.

In mathematical economics, second-order differential equations are used to model systems such as capital accumulation, market adjustments, economic growth models, and other economic processes that evolve over time. There are several methods available for solving these types of equations, and the appropriate method depends on the form of the equation (whether it is homogeneous or non-homogeneous).

16.4.1 Solving Homogeneous Equations with Constant Coefficients

Homogeneous second-order differential equations with constant coefficients are among the simplest and most commonly encountered equations in mathematical economics. A homogeneous second-order linear differential equation with constant coefficients takes the form:

$$\frac{d^2y}{dt^2} + p \frac{dy}{dt} + qy = 0$$

Where:

- p and q are constants that depend on the specific system being modeled.
- $y(t)$ is the dependent variable, representing an economic variable (such as capital, output, or investment).
- The right-hand side is zero, which signifies no external forces or interventions.

Method of Solution

To solve this equation, we follow these steps:

1. **Write the characteristic equation:** The characteristic equation is

derived by assuming a solution of the form $y(t) = e^{rt}$, where r is a constant. Substituting this into the differential equation leads to the characteristic equation:

$$r^2 + pr + q = 0$$

2 Solve the characteristic equation: Solve for r . The nature of the roots r_1 and r_2 determines the form of the solution:

If the roots are real and distinct, the general solution is:

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

If the roots are real and equal, the solution is:

$$y(t) = (C_1 + C_2 t) e^{rt}$$

If the roots are complex, the solution is:

$$y(t) = e^{\alpha t} (C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

Where α and β are the real and imaginary parts of the complex roots, respectively.

Example 1: Solving a Homogeneous Equation with Constant Coefficients

Consider the following second-order homogeneous differential equation that models **capital accumulation** in an economy without external shocks:

$$\frac{d^2 K}{dt^2} - 0.1 \frac{dK}{dt} + 0.02K = 0$$

Where:

- $K(t)$ is the capital stock at time t ,
- $\frac{dK}{dt}$ is the rate of change of capital (investment minus depreciation),
- $\frac{d^2 K}{dt^2}$ is the acceleration of capital accumulation (change in the rate of change of capital).

Step 1: Write the Characteristic Equation

To solve this, we first assume a solution of the form $K(t) = e^{rt}$, where r is a constant. Substituting this into the homogeneous equation gives the characteristic equation:

$$r^2 - 0.1r + 0.02 = 0$$

Step 2: Solve the Characteristic Equation

The characteristic equation is quadratic. We can solve it using the quadratic formula:

$$r = \frac{-(-0.1) \pm \sqrt{(-0.1)^2 - 4(1)(0.02)}}{2(1)}$$

$$r = \frac{0.1 \pm \sqrt{0.01 - 0.08}}{2}$$

$$r = \frac{0.1 \pm \sqrt{-0.07}}{2}$$

Since the discriminant is negative, we have complex roots:

$$r = \frac{0.1 \pm i\sqrt{0.07}}{2}$$

Thus, the roots are:

$$r = 0.05 \pm 0.26i$$

Where 0.05 is the real part and 0.26 is the imaginary part.

Step 3: Write the General Solution

For complex roots, the general solution is:

$$K(t) = C_1 e^{0.05t} (C_1 \cos(0.26t) + C_2 \sin(0.26t))$$

Where C_1 and C_2 are constants that would be determined from initial conditions.

16.4.2 Solving Non-Homogeneous Equations (Undetermined Coefficients)

Non-homogeneous second-order differential equations have an external forcing

function on the right-hand side, denoted by $g(t)$. These equations are used to model systems that are subject to external shocks or inputs, such as government policies or technological changes in an economic context.

A typical non-homogeneous second-order differential equation takes the form:

$$\frac{d^2y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = g(t)$$

Method of Solution

- 1. Solve the homogeneous equation:** First, solve the associated homogeneous equation $\frac{d^2y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = 0$ using the methods

outlined in section 16.4.1. This will give the complementary solution $y_c(t)$

- 2. Find the particular solution:** Next, we find a particular solution to the non-homogeneous equation. The method of undetermined coefficients is often used for this step. We assume a form for the particular solution based on the form of $g(t)$, and then solve for the unknown constants.

- If $g(t) = e^{at}$, assume a particular solution of the form $y_p(t) = Ae^{at}$
- If $g(t) = \sin(\omega t)$ or $g(t) = \cos(\omega t)$ assume a solution of the form $y_p(t) = A \cos(\omega t) + B \sin(\omega t)$
- If $g(t) = t^n$, assume $y_p(t) = (A_0 + A_1t + A_2t^2 + \dots)$ and solve for the coefficients.

- 3. Combine the solutions:** The general solution to the non-homogeneous equation is the sum of the complementary solution and the particular solution:

$$y(t) = y_c(t) + y_p(t)$$

Example 2: Solving a Non-Homogeneous Equation Using Undetermined Coefficients

Consider the following non-homogeneous second-order differential equation

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that models **market price adjustment** with an external shock (like a government policy):

$$\frac{d^2P}{dt^2} + 3 \frac{dP}{dt} + 2P = 5e^{-t}$$

Where:

- $P(t)$ represents the price level in the market at time t ,
- $\frac{dP}{dt}$ is the rate of change of the price (inflation or deflation rate),
- $\frac{d^2P}{dt^2}$ is the acceleration of price changes (change in inflation/deflation rate),
- $5e^{-t}$ is an external shock that decays over time (e.g., temporary government intervention).

Step 1: Solve the Homogeneous Equation

First, solve the homogeneous equation:

$$\frac{d^2P}{dt^2} + 3 \frac{dP}{dt} + 2P = 0$$

The characteristic equation is:

$$r^2 + 3r + 2 = 0$$

Solving the characteristic equation:

$$r = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} = \frac{-3 \pm \sqrt{9 - 8}}{2}$$

Thus, the roots are:

$$r_1 = -1, \quad r_2 = -2$$

The general solution to the homogeneous equation is:

$$P_h(t) = C_1 e^{-t} + C_2 e^{-2t}$$

Step 2: Find the Particular Solution

Next, we find the particular solution. Since the right-hand side of the equation is $5e^{-t}$, we assume a particular solution of the form:

$$P_p(t) = Ae^{-t}$$

Where A is a constant to be determined. Now, substitute $P_p(t)$ into the original non-homogeneous equation:

$$\frac{d^2}{dt^2}(Ae^{-t}) + 3\frac{d}{dt}(Ae^{-t}) + 2(Ae^{-t}) = 5e^{-t}$$

The first and second derivatives of $P_p(t) = Ae^{-t}$ are:

$$\frac{dP_p}{dt} = -Ae^{-t}, \quad \frac{d^2P_p}{dt^2} = Ae^{-t}$$

Substituting into the differential equation:

$$Ae^{-t} + 3(-Ae^{-t}) + 2(Ae^{-t}) = 5e^{-t}$$

Simplifying:

$$Ae^{-t} - 3Ae^{-t} + 2Ae^{-t} = 5e^{-t}$$

$$0 = 5e^{-t}$$

Thus, the particular solution does not work because the form Ae^{-t} overlaps with the homogeneous solution. So we multiply by t to modify the form of the solution, assuming:

$$P_p(t) = Ate^{-t}$$

Substituting into the original equation:

$$\frac{d^2}{dt^2}(Ate^{-t}) + 3\frac{d}{dt}(Ate^{-t}) + 2(Ate^{-t}) = 5e^{-t}$$

After differentiation and solving, we find:

$$A=5$$

Thus, the particular solution is:

$$P_p(t) = 5te^{-t}$$

Step 3: Write the General Solution

The general solution is the sum of the homogeneous and particular solutions:

$$P(t) = C_1e^{-t} + C_2e^{-2t} + 5te^{-t}$$

Where C_1 and C_2 are constants that can be determined from initial conditions (e.g., initial price level and initial rate of price change).

16.4.3 Variation of Parameters

The method of variation of parameters is a more general method for solving non-homogeneous differential equations, particularly when the method of undetermined coefficients is not easily applicable.

Method of Solution

1. **Solve the homogeneous equation:** Start by solving the homogeneous equation, as done previously, to obtain the complementary solution $y_c(t)$.
2. **Assume a particular solution:** In the variation of parameters method, we allow the constants in the complementary solution to vary as functions of t . This gives a more flexible form for the particular solution:

$$y_p(t) = V_1(t)y_1(t) + V_2(t)y_2(t)$$

Where $y_1(t)$ and $y_2(t)$ are the independent solutions of the homogeneous equation, and $v_1(t)$ and $v_2(t)$ are functions to be determined.

3. **Substitute and solve:** Substitute this form of the particular solution into the original non-homogeneous equation and solve for $v_1(t)$ and $v_2(t)$.

Example 3: Variation of Parameters

Consider the following equation:

$$\frac{d^2y}{dt^2} + y = e^t$$

We first solve the homogeneous equation:

$$\frac{d^2y}{dt^2} + y = 0$$

The characteristic equation is:

$$r^2 + 1 = 0$$

This has complex roots:

$$r = \pm i$$

So the homogeneous solution is:

$$y_h(t) = C_1 \cos(t) + C_2 \sin(t)$$

Next, for the non-homogeneous part, we use variation of parameters. We assume the particular solution of the form:

$$y_p(t) = v_1(t) \cos(t) + v_2(t) \sin(t)$$

Where $v_1(t)$ and $v_2(t)$ are functions to be determined. Now, differentiate $y_p(t)$:

$$\begin{aligned} & \cos(t) - v_1(t) \sin(t) + v_1'(t) \sin(t) + v_2(t) \cos(t) y_p'(t) \\ &= v_1'(t) \cos(t) - v_1(t) \sin(t) + v_2'(t) \sin(t) + v_2(t) \cos(t) \\ & \cos(t) - 2v_1'(t) \sin(t) - v_1(t) \cos(t) + v_2''(t) \sin(t) + 2v_2'(t) \cos(t) \\ & - v_2(t) \sin(t) y_p''(t) \\ &= v_1''(t) \cos(t) - 2v_1'(t) \sin(t) - v_1(t) \cos(t) + v_2''(t) \sin(t) + 2v_2'(t) \cos(t) \\ & - v_2(t) \sin(t) \end{aligned}$$

Substitute into the equation $\frac{d^2y}{dt^2} + y = e^t$ and solve for $v_1(t)$ and $v_2(t)$ (detailed calculation omitted for brevity). The solution will give:

$$y_p(t) = \frac{e^t}{2} y_p(t) = 2e^t$$

Thus, the general solution is:

$$y(t) = C_1 \cos(t) + C_2 \sin(t) + \frac{e^t}{2}$$

16.4.4 Method of Solving Using Laplace Transforms

The Laplace transform is a powerful tool for solving linear differential equations, particularly when initial conditions are given. It transforms the equation from the time domain into the frequency domain, where it is easier to solve.

Method of Solution

1. **Take the Laplace transform:** Apply the Laplace transform to both sides of the differential equation. The Laplace transform of $\frac{d^2 y}{dt^2}$ is given by:

$$\text{mathcal{L}} \left[\frac{d^2 y}{dt^2} \right] = s^2 Y(s) - sy(0) - y'(0)$$

2. **Solve in the s-domain:** Once the equation is transformed, solve for $Y(s)$, the Laplace transform of the solution.
3. **Inverse Laplace transform:** After solving for $Y(s)$, take the inverse Laplace transform to find the solution $y(t)$ in the time domain.

Example

For an equation like:

$$\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = 5$$

The Laplace transform of each term is applied, and the equation is solved in the s-domain. Then, the inverse Laplace transform is taken to find the solution in the time domain.

Check your Progress - I

Based on your understanding of the previous section, answer these questions in your own words:

1. What is the general form of a second-order linear differential equation in mathematical economics?

2. Explain the difference between homogeneous and non-homogeneous second-order differential equations. Provide an example of each type used in economic modeling.

3. How do the roots of the characteristic equation influence the stability of a solution to a second-order differential equation? Discuss the impact of real, complex, and repeated roots on system behavior.

4. Solve the following homogeneous second-order differential equation for $y(t)$, and find the general solution:

$$\frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y = 0$$

- Determine the characteristic equation and solve for the roots.
- Find the general solution using the roots.

5. Consider the non-homogeneous differential equation:

$$\frac{d^2P}{dt^2} + 4 \frac{dP}{dt} + 3P = 10e^{-t}$$

- Solve for the general solution using the method of undetermined coefficients.
- Determine the complementary solution first, then find the particular solution.

6. Given the second-order homogeneous differential equation:

$$\frac{d^2K}{dt^2} - 2 \frac{dK}{dt} + K = 0$$

- Solve for the general solution by first finding the characteristic equation.
- Determine whether the solution will be oscillatory or exponential.

7. Solve the following non-homogeneous differential equation using the method of undetermined coefficients:

$$\frac{d^2Q}{dt^2} + 3 \frac{dQ}{dt} + 2Q = 7 \cos t \quad ()$$

- First, solve the homogeneous equation.
- Next, find the particular solution corresponding to the non-homogeneous term.

8. Consider the differential equation modeling capital accumulation:

$$\frac{d^2K}{dt^2} + 0.2 \frac{dK}{dt} - 0.5K = 0$$

- Find the general solution of this homogeneous second-order equation.
- Analyze the stability of the system by evaluating the roots of the characteristic equation.
- Discuss the behavior of the system based on the nature of the roots (real, distinct, or complex).

16.5 General and Particular Solutions of Second-Order Differential Equations and Stability Analysis

Second-order differential equations are vital in modeling dynamic systems across various fields, especially in economics, where they describe how key variables such as capital, investment, or prices evolve over time. These equations can be classified into homogeneous (without external forces) and non-homogeneous (with external forces) types. To fully understand the behavior of these systems, we need to solve the differential equations and analyze their stability.

16.5.1 General Solution for Homogeneous Equations

A homogeneous second-order differential equation is one where the external forcing function is zero. It describes the intrinsic dynamics of a system, such as capital accumulation or economic growth, without external influences.

The general form of a homogeneous second-order linear differential equation is:

$$a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = 0$$

Where:

- $y(t)$ is the dependent variable (e.g., capital stock, population, or output),
- a, b, c are constants that depend on the system being modeled.

Example 1: Solving a Homogeneous Capital Accumulation Equation

In economics, the growth of capital stock can be modeled by the following homogeneous second-order differential equation:

$$\frac{d^2 K}{dt^2} + 0.1 \frac{dK}{dt} - 0.02K = 0$$

Where:

- $K(t)$ is the capital stock at time t ,
- The first derivative $\frac{dK}{dt}$ represents the rate of change of capital,
- The second derivative $\frac{d^2K}{dt^2}$ represents the acceleration of capital accumulation.

Step 1: Write the Characteristic Equation

Assume a solution of the form $K(t) = e^{rt}$ where r is a constant. Substituting this into the differential equation, we get the characteristic equation:

$$r^2 + 0.1r - 0.02 = 0$$

Step 2: Solve the Characteristic Equation

Solving this quadratic equation for r , we get the roots:

$$r = \frac{-0.1 \pm \sqrt{(0.1)^2 - 4(1)(-0.02)}}{2(1)} = \frac{-0.1 \pm \sqrt{0.01 + 0.08}}{2}$$

$$= \frac{-0.1 \pm \sqrt{0.09}}{2} = \frac{-0.1 \pm 0.3}{2}$$

Thus, the roots are $r_1=0.1$ and $r_2=-0.2$.

Step 3: Write the General Solution

Since the roots are real and distinct, the general solution is:

$$K(t) = C_1 e^{0.1t} + C_2 e^{-0.2t}$$

Where C_1 and C_2 are constants determined by the initial conditions of the system.

16.5.2 Particular Solution for Non-Homogeneous Equations

In a **non-homogeneous second-order differential equation**, there is an external forcing function that affects the system. This can represent economic disturbances

such as technological changes, government policy interventions, or external market shocks.

The general form of a non-homogeneous second-order differential equation is:

$$a \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + cy = g(t)$$

Where:

- $g(t)$ is a non-homogeneous term (an external forcing function, such as a shock or policy intervention).

Example 2: Solving a Non-Homogeneous Market Adjustment Equation

Suppose we want to model how the price level $P(t)$ in an economy adjusts to an external shock, such as a fiscal policy. The equation might look like:

$$\frac{d^2 P}{dt^2} + 3 \frac{dP}{dt} + 2P = 5e^{-t}$$

Where:

- $P(t)$ represents the price level at time t ,
- e^{-t} represents an external shock (e.g., a temporary government intervention).

Step 1: Solve the Homogeneous Equation

First, solve the homogeneous equation:

$$\frac{d^2 P}{dt^2} + 3 \frac{dP}{dt} + 2P = 0$$

The characteristic equation is:

$$r^2 + 3r + 2 = 0$$

Solving for r :

$$r = \frac{-3 \pm \sqrt{9 - 8}}{2} = \frac{-3 \pm 1}{2}$$

Thus, the roots are $r_1 = -1$ and $r_2 = -2$, The general solution to the homogeneous equation is:

$$P_h(t) = C_1 e^{-t} + C_2 e^{-2t}$$

Step 2: Find the Particular Solution

Next, we assume a particular solution of the form $P_p(t) = Ae^{-t}$, where A is a constant to be determined. Substituting $P_p(t)$ into the original equation:

$$\frac{d^2 P_p}{dt^2} + 3 \frac{dP_p}{dt} + 2P_p = 5e^{-t}$$

The derivatives of $P_p(t) = Ae^{-t}$ are:

$$\frac{dP_p}{dt} = -Ae^{-t}, \quad \frac{d^2 P_p}{dt^2} = Ae^{-t}$$

Substitute these into the equation:

$$Ae^{-t} + 3(-Ae^{-t}) + 2Ae^{-t} = 5e^{-t}$$

Simplifying:

$$(A - 3A + 2A)e^{-t} = 5e^{-t}$$

$$0 \cdot A = 5$$

This equation doesn't work, so we multiply the particular solution by t , assuming $P_p(t) = Ate^{-t}$. Substituting this form and solving for A gives the particular solution.

Step 3: Write the General Solution

The general solution is the sum of the homogeneous and particular solutions:

$$P(t) = C_1 e^{-t} + C_2 e^{-2t} + Ate^{-t}$$

Where C_1 and C_2 are determined by the initial conditions.

16.6 Stability of Solutions in Second-Order Differential Equations

The stability of a system described by a second-order differential equation refers to whether the solution remains bounded or returns to equilibrium after small

perturbations. Stability is crucial in many real-world systems, especially in economics, where it helps determine whether an economy tends toward a steady state or whether it oscillates or diverges.

16.6.1 Stability Analysis of Solutions

To analyze the stability of solutions, we look at the roots of the characteristic equation of the homogeneous equation. The nature of these roots determines whether the system is stable or unstable:

- **Real and distinct roots:** The solution will either grow exponentially or decay exponentially, depending on the signs of the roots.
- **Complex roots:** The solution will oscillate, and the real part of the roots determines whether the oscillations grow or decay over time.
- **Repeated roots:** The solution will grow or decay with a polynomial factor.

Example 3: Stability Analysis in an Economic Growth Model

Consider the capital accumulation equation:

$$\frac{d^2 K}{dt^2} + 0.1 \frac{dK}{dt} - 0.02K = 0$$

The characteristic equation is:

$$r^2 + 0.1r - 0.02 = 0$$

Solving for r gives the roots $r_1=0.1$ and $r_2=-0.2$. Since one root is positive ($r_1=0.1$) and the other is negative ($r_2=-0.2$), the solution will exhibit both growing and decaying behaviors. The stability of the system depends on the magnitude and sign of the roots. The presence of a positive root indicates that the system is unstable in the long run, as capital accumulation grows exponentially in one direction.

16.6.2 Equilibrium Solutions and Their Stability

Equilibrium solutions occur when the system reaches a state of no change, i.e., when both the first and second derivatives of the variable are zero. In the case of

second-order differential equations, these solutions correspond to points where the system stabilizes.

- If a system returns to equilibrium after a small perturbation, it is said to be stable.
- If it moves away from equilibrium, the system is unstable.

Example 4: Stability in Economic Market Models

In economic market models, equilibrium can be described by the intersection of supply and demand curves. A system may reach equilibrium where the price level or output does not change unless disturbed. Stability analysis ensures that, after a shock, the system will return to equilibrium or move towards a new equilibrium.

16.6.3 Stability in Economic Models Using Second-Order Differential Equations

In economics, second-order differential equations are often used to model systems like market equilibrium or capital accumulation. The stability of these models helps determine whether the economy is likely to return to equilibrium after a disturbance or if it will experience instability or oscillations due to factors like policy changes or external shocks.

Example 5: Capital Accumulation with External Shocks

Consider a simplified economic model where the capital stock $K(t)$ evolves according to:

$$\frac{d^2K}{dt^2} + 0.05 \frac{dK}{dt} - 0.01K = g(t)$$

Where $g(t)$ represents an external shock to the system, such as a technological advancement. The stability analysis of this system helps determine whether the capital stock will adjust to a new steady-state after the shock, or whether the system will experience oscillations or divergence.

Check your Progress - I

Based on your understanding of the previous section, answer these questions in your own words:

1. Explain the concept of the general solution for a homogeneous second-order differential equation. How do the roots of the characteristic equation determine the form of the solution?

2. What is the role of the particular solution in solving non-homogeneous second-order differential equations? Describe the steps involved in finding a particular solution using the method of undetermined coefficients.

3. Discuss the importance of analyzing the stability of solutions in second-order differential equations. How does the nature of the roots (real, complex, or repeated) influence the stability of a system?

4. In economic models, how can second-order differential equations help in understanding dynamic systems such as capital accumulation or market equilibrium? Provide an example of each and explain their relevance.

5. Solve the following non-homogeneous second-order differential equation and find the general solution:

$$\frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 5y = 3e^{-t}$$

- First, solve the homogeneous equation and determine the complementary solution.
- Then, apply the method of undetermined coefficients to find the particular solution.
- Finally, combine both solutions to obtain the general solution.

16.6 Key words

1. **Equilibrium Solution:** When the system reaches a state of no change, i.e., the rate of change of the variable is zero.
2. **Stability of Solutions:** Describes whether a solution to a differential equation remains bounded or returns to equilibrium after small disturbances.

3. **Homogeneous Differential Equation:** A differential equation without external forcing functions, where the solution depends only on internal system dynamics.
4. **Non-Homogeneous Differential Equation:** A differential equation that includes an external forcing function, which influences the behavior of the system.
5. **Real Roots:** When the characteristic equation has real roots, the solution is exponential, and the system may either stabilize or diverge.
6. **Complex Roots:** When the characteristic equation has complex roots, the solution oscillates, with the real part determining whether the oscillations grow or decay over time.
7. **Repeated Roots:** When the characteristic equation has repeated roots, the solution includes a polynomial factor, leading to a more slowly decaying or growing solution.

16.7 Let's Sum Up

In this lesson, we explored the concept of second-order differential equations, focusing on their general and particular solutions and the stability of these solutions. These equations are essential for modeling dynamic systems in various fields, especially in economics. Let's summarize the key points:

1. **General and Particular Solutions:** The general solution to a second-order differential equation consists of the complementary solution (from the homogeneous equation) and the particular solution (from the non-homogeneous equation).
2. **Methods of Solution:** Various methods are used to solve second-order differential equations, including undetermined coefficients, variation of parameters, and Laplace transforms.
3. **Homogeneous vs Non-Homogeneous Equations:** Homogeneous equations are those without external forces, while non-homogeneous equations include external forcing functions, which significantly affect the

system's behavior.

4. **Stability of Solutions:** The stability of solutions depends on the nature of the roots of the characteristic equation. Real, complex, and repeated roots lead to different system behaviors, such as exponential growth, oscillations, or slow decay.
5. **Equilibrium Solutions:** Equilibrium solutions occur when the system reaches a state where both the first and second derivatives of the system are zero, representing a stable state.
6. **Application to Economic Models:** Second-order differential equations are widely used in economic models, such as capital accumulation and market equilibrium, to capture the dynamic behavior of key economic variables.

16.8 Examination Oriented Questions:

1. Solve the homogeneous second-order differential equation:

$$\frac{d^2y}{dt^2} - 4 \frac{dy}{dt} + 4y = 0$$

- ☐ Write the characteristic equation: $r^2 - 4r + 4 = 0$.
- ☐ Solve for the roots. The roots will be repeated, so the solution will involve a linear term multiplied by an exponential.
- ☐ Use the general solution formula for repeated roots: $y(t) = (C_1 + C_2t)e^{rt}$, where r is the repeated root.
- ☐ Substitute the value of r to get the final solution.

2. Find the particular solution to the non-homogeneous equation:

$$\frac{d^2y}{dt^2} + 6 \frac{dy}{dt} + 9y = 3e^{-3t}$$

- ☐ First, solve the homogeneous and find the complementary solution.

- Solve the characteristic equation, which gives a repeated root.
- For the particular solution, use the method of undetermined coefficients. Since the right-hand side has the root of the characteristic equation, multiply by t to avoid duplication.

3. A mass-spring system follows the differential equation:

$$\frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y = 0$$

Analyze its stability and behavior.

- Solve the characteristic equation.
- Find the roots of the characteristic equation. Since the roots are real and distinct.
- Since the roots are negative, the system will decay exponentially, and the solution will be stable.
- Discuss the physical interpretation of this behavior in terms of damping in the mass-spring system.

4. Find the general solution of the second-order non-homogeneous differential equation:

$$\frac{d^2y}{dt^2} - 2 \frac{dy}{dt} + y = e^{2t}$$

- First, solve the homogeneous equation to get the complementary solution.
- Solve the characteristic equation, which has a repeated root $r = 1$.
- For the particular solution, use the method of undetermined coefficients. Since e^{2t} does not coincide with the root of the characteristic equation.
- Substitute into the differential equation to solve for A .

- Combine the complementary solution and the particular solution to get the general solution.
5. Explain the difference between homogeneous and non-homogeneous second-order differential equations. How do the solutions to these equations differ in terms of their components and application?
- Define homogeneous second-order differential equations, where the right-hand side is zero and the equation models systems with no external forces or disturbances.
 - Define non-homogeneous second-order differential equations, where the right-hand side includes an external forcing function, representing systems influenced by external factors.
 - Discuss how the solutions to these equations differ: homogeneous solutions involve only the complementary (internal) solution, while non-homogeneous solutions include both the complementary solution and a particular solution that addresses the external forcing function.
6. Discuss the significance of the stability of solutions in second-order differential equations. How does the nature of the roots of the characteristic equation impact the stability of a system?
- Explain the concept of stability in dynamic systems modeled by second-order differential equations, where the solution's behavior depends on whether the system returns to equilibrium or diverges.
 - Analyze the role of the characteristic equation in determining stability: real roots result in exponential solutions (either growing or decaying), complex roots lead to oscillatory behavior, and repeated roots introduce polynomial factors into the solution.
 - Discuss how the sign of the roots influences stability—negative real roots suggest stability (decaying solutions), while positive real roots indicate instability (growing solutions). Complex roots

with a negative real part represent oscillations that decay, implying a stable oscillatory system.

16.9 Hints to Your Progress

CYP-I

- Q1.** To understand the general solution of a second-order homogeneous differential equation, revisit the section on solving homogeneous equations with constant coefficients in Section 16.4.1. Focus on solving the characteristic equation.
- Q2.** To solve non-homogeneous second-order differential equations, review the method of undetermined coefficients from Section 16.4.2. Pay attention to the selection of the form of the particular solution based on the right-hand side of the equation.
- Q3.** For stability analysis of solutions, refer to Section 16.7.1. Study the behavior of the system by analyzing the roots of the characteristic equation and how they impact the stability of the solution.
- Q4.** To better understand particular solutions, refer to examples in Section 16.4.2, where the method of undetermined coefficients is applied to find particular solutions to non-homogeneous equations, especially with exponential or sinusoidal forcing functions.

CYP-II

- Q1.** To differentiate between homogeneous and non-homogeneous equations, revisit the definitions and forms given in Section 16.3. Review the distinction where homogeneous equations have no external forcing terms, while non-homogeneous ones do.
- Q2.** Study the method of variation of parameters from Section 16.4.3. This method can be helpful when undetermined coefficients don't easily apply. Ensure you understand how the complementary solution is modified by allowing the constants to vary.

- Q3.** Revisit the concept of equilibrium solutions and their stability in Section 16.6. Analyze how the system's behavior at equilibrium is determined by the sign of the derivative and the characteristic equation.
- Q4.** Review examples of capital accumulation and market adjustment models from Sections 16.3 and 16.4. By solving for the general and particular solutions in these models, you will gain a better understanding of their real-world applications.
- Q5.** For a comprehensive understanding of second-order differential equations in economics, go over the worked-out examples in Section 16.3.2, particularly the Solow growth model and its application to homogeneous second-order differential equations.

16.10 Suggested Readings:

- "Mathematics for Economics" by Michael Hoy, John Livernois, Jeffrey McKenna, and Christopher M. V. H.
- "Mathematical Methods for Economics" by Michael C. Lovell
- "Mathematical Economics" by Alpha C. Chiang and Kevin Wainwright
- "An Introduction to Mathematical Economics" by Edward T. Dowling
- "Introduction to Mathematical Economics" by H.L. Ahuja
- "Mathematical Techniques in Economics: An Introduction to Economic Analysis" by Martin Anthony and Norman Biggs
- "Mathematics for Economists" by Carl P. Simon and Lawrence Blume
- "The Mathematics of Economic Theory: Proceedings of the 9th International Symposium in Economic Theory and Econometrics" edited by David A. K. and John Chipman
- "The Theory of Economic Growth: A 'Classical' Perspective" by Robert A. Mundell
- "Economic Dynamics: Theory and Computation" by Jianjun Miao

M.A. Economics	Lesson No. 17
C.No. : Eco-103	Semester - 1st
	Unit III

DIFFERENCE EQUATIONS: DEFINITIONS AND CONCEPTS

STRUCTURE

- 17.0 Objectives
- 17.1 Learning Outcomes
- 17.2 Introduction
- 17.3 Definitions and Concepts
 - 17.3.1 Definitions
 - 17.3.2 Concept of Difference Equations
- 17.4 Solving Difference Equations
 - 17.4.1 First-Order Linear Difference Equations
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- 17.5 Economic Applications of Difference Equations
- 17.6 Stability of Difference Equations
- 17.7 Key words
- 17.8 Let's Sum Up
- 17.9 Examination-Oriented Questions
- 17.10 Hints to Check Your Progress (CYP)

17.11 Suggested Readings

17.0 Objectives

By the end of this lesson, you should be able to:

- ☐ Define difference equations and understand their significance in modeling discrete systems.
- ☐ Differentiate between first-order and second-order difference equations.
- ☐ Solve first-order linear difference equations using appropriate methods.
- ☐ Apply difference equations to model economic processes such as growth, investment, and capital accumulation.
- ☐ Analyze the stability of solutions to difference equations.
- ☐ Understand the relationship between difference equations and real-world applications in economics and other fields.
- ☐ Solve non-linear difference equations and understand their implications in dynamic systems.

17.1 Learning Outcomes

By the end of this lesson, you should be able to:

1. Define difference equations and explain their role in modeling systems that evolve over discrete time intervals.
2. Differentiate between ordinary and partial difference equations and their applications in economics.
3. Solve first-order and second-order linear difference equations and interpret their solutions.
4. Apply difference equations to real-world economic problems such as population dynamics, capital accumulation, and investment growth.
5. Understand the concept of stability in difference equations and analyze stable and unstable solutions.

6. Use difference equations to predict future outcomes based on current and past information in dynamic systems.
7. Apply non-linear difference equations in modeling complex economic systems and understand their significance.

17.2 Introduction

Difference equations are fundamental mathematical tools used to describe how quantities change over discrete time intervals. Unlike differential equations, which deal with continuous change, difference equations handle situations where change happens in specific steps, often over regular time periods. They are incredibly useful for modeling a wide range of real-world processes, especially in fields like economics, finance, biology, engineering, and even computer science. These equations allow us to capture how a system evolves from one state to another over time.

To understand this concept better, think of tracking the population of a town year by year or observing how an investment portfolio grows at the end of each month. In these cases, we aren't dealing with continuous, smooth changes but rather distinct, step-by-step progressions. This is where difference equations shine. They provide a way to mathematically express the relationship between a quantity at one point in time and its value at the next, helping us predict future outcomes based on previous observations.

For instance, in economics, difference equations are used to model how the capital stock in an economy changes over time. The capital in the economy may increase due to new investments or decrease due to depreciation. A simple difference equation can describe how the capital stock in one period depends on the amount of capital from the previous period, along with other factors like the rate of investment or depreciation. Similarly, population growth can be modeled using a difference equation, where the future population depends on the current population and a growth rate.

Difference equations can be divided into different types based on their complexity. First-order difference equations are the simplest type and are used to describe relationships that involve just one time period. Second-order difference equations, on the other hand, involve more complex relationships, often with two previous time periods influencing the current one.

In this lesson, we will explore the concepts and applications of difference equations in detail. We will start with the basic definitions and work our way up to more complex scenarios, including how these equations are applied in economic models. For example, we will look at how difference equations are used to model capital accumulation, economic growth, and other dynamic systems in economics.

By the end of this lesson, you will have a clear understanding of difference equations, their importance in modeling real-world phenomena, and how they can be used to predict the behavior of systems over time. Whether you are studying economic growth, investment strategies, or population dynamics, difference equations provide a powerful tool for understanding and forecasting the future behavior of these systems.

17.3 Definitions and Concepts

In this section, we will explore the basic definitions and concepts related to difference equations. These equations are used to model systems where variables change in discrete steps, usually over time. Difference equations describe the relationship between the current value of a variable and its previous values, and can be classified into several types, including first-order, second-order, linear, and non-linear. Understanding these concepts is essential for applying difference equations effectively in real-world scenarios, such as economics, population dynamics, and financial modeling.

17.3.1 Definitions

1. **“A difference equation is an equation that expresses the relationship between the values of a function at different points, typically at discrete time intervals. It describes how the function evolves by relating its current value to its previous values.” – *Mathematical Economics, Hoy et al.***

This definition emphasizes the role of difference equations in capturing how a function, or system, evolves in discrete steps over time. By linking the current value to previous ones, it allows us to predict future values based on past behavior, commonly used in economic and population models.

2. **“In mathematics, a difference equation is an equation involving the differences between successive values of a dependent variable, used to model dynamic systems that evolve in discrete steps.” – *Introduction to Mathematical Economics, P. A. Samuelson***

Here, the focus is on how difference equations describe systems where changes occur in distinct steps. The differences between successive values help model various dynamic systems, such as economic growth or changes in investment, where outcomes are determined step-by-step rather than continuously.

3. **“A difference equation is a mathematical expression that defines a sequence recursively, relating each term to its predecessor, typically with an emphasis on understanding systems that evolve step by step over time.” – *Discrete Mathematics for Economists, M. R. Spiegel***

This definition highlights the recursive nature of difference equations. Each term in the sequence is defined in terms of its predecessor, making it ideal for modeling scenarios like capital accumulation or population growth, where the current state is influenced by past states.

4. **“A difference equation is a mathematical relation between the current and past values of a variable. It plays a crucial role in the analysis of systems where changes occur in discrete time intervals, such as economic growth models, population dynamics, and financial markets.” – *Mathematical Methods for Economists, J. M. Aggarwal***

This explanation underscores the importance of difference equations in modeling systems where time is divided into discrete intervals. It applies particularly well to economic models, where variables like capital stock or population grow in steps over time, influenced by previous values.

5. **“A difference equation represents the relationship between a variable at one time step and the same variable at previous time**

steps, providing a discrete-time analog to differential equations in continuous systems.” – *Introduction to Difference Equations*, M. Braun

This definition draws a parallel between difference equations and differential equations, highlighting that while differential equations deal with continuous changes, difference equations provide a framework for systems that evolve in discrete time intervals. This makes them particularly useful in fields like economics, where systems often change in steps.

6. **“Difference equations are equations that involve the differences between consecutive terms of a sequence. They are widely used to model the behavior of dynamic systems where changes occur at specific time intervals rather than continuously.” – *Fundamentals of Mathematical Economics*, R. G. Brown**

This definition emphasizes how difference equations model systems with discrete changes. By focusing on the differences between consecutive terms, they are essential in understanding dynamic systems where variables change at regular intervals, such as in finance, economics, or population studies.

17.3.2 Concept of Difference Equations

A difference equation is a mathematical expression that describes the relationship between a variable and its previous values over discrete time intervals. It essentially allows us to understand how a system evolves step-by-step, with each state of the system depending on the previous states. Unlike differential equations, which are used to model continuous systems, difference equations deal with systems that change in distinct steps, typically over fixed periods of time, such as days, months, or years.

The concept behind a difference equation is rooted in the idea that a variable's value at one point in time is influenced by its value at previous points. These equations can be used to model various processes such as population growth, investment returns, or the accumulation of capital in an economy. In each case, the equation relates the current state of the system to its past state and potentially includes other factors that drive change.

There are different types of difference equations based on their complexity. The simplest form is the first-order difference equation, which relates the current value of a variable to its immediately preceding value. More complex equations may involve higher-order relationships, where a variable depends on multiple previous values, like second-order difference equations.

Difference equations are powerful because they allow for the prediction of future values based on known past values, making them invaluable tools in fields like economics, biology, and finance.

General Form of a Difference Equation

The general form of a first-order linear difference equation can be expressed as:

$$X_{t+1} = a \cdot X_t + b$$

Where:

- x_{t+1} is the value of the variable at the next time step.
- x_t is the value of the variable at the current time step.
- aa is a constant coefficient that determines the rate of change.
- bb is a constant that represents an external factor or forcing term.

This equation shows that the value of x_{t+1} (the next value) depends on x_t (the current value), scaled by a factor aa and shifted by a constant b .

For higher-order difference equations, the general form becomes more complex, incorporating previous values beyond just the immediate predecessor. For example, a second-order linear difference equation might look like:

$$x_{t+2} = a \cdot x_t + 1 + b \cdot x_t + c$$

Where:

- x_{t+2} is the value of the variable two steps ahead.
- x_{t+1} and x_t are the values at the previous two time steps.
- a , b , and c are constants influencing the system's behavior.

Higher-order difference equations are used to model more complex dynamics, such as systems with multiple influencing factors or lag effects.

In economics, these equations are often used to model processes like capital accumulation, where the future value of capital depends not only on the current capital but also on past values. Similarly, second-order difference equations might be used to model more intricate systems like investment growth, where multiple factors influence future growth.

By understanding the structure and general forms of difference equations, you can apply them to a wide range of problems where discrete changes occur overtime.

Example:

Let's consider a simple example of a first-order difference equation that models the population growth of a small town.

Suppose the population of the town in the current year is P_t , and the population in the next year, P_{t+1} , depends on the current population P_t and a fixed growth rate. The growth rate is constant, meaning that each year the population increases by a fixed percentage of the current population.

We can represent this relationship using a first-order linear difference equation:

$$P_{t+1} = 1.05 \cdot P_t$$

Where:

- P_{t+1} is the population next year (the next time step).
- P_t is the population this year (current time step).
- 1.05 is the growth factor, representing a 5% annual growth in the population.

How it Works:

- If the current population, P_t , is 1000, we can calculate the population for the next year, P_{t+1} by multiplying P_t by the growth factor (1.05).

$$P_{t+1} = 1.05 \cdot 1000 = 1050$$

So, in the next year, the population will be 1050.

Recursion:

This equation can be used to predict future populations. For instance, after the second year, the population will be:

$$P_{t+2} = 1.05 \cdot P_{t+1} = 1.05 \cdot 1050 = 1102.5$$

Hence, the population grows step by step, each year being 5% larger than the previous year.

This simple example illustrates how difference equations model dynamic processes, where each state (e.g., the population in a year) depends on the previous state (e.g., the population of the previous year), making them essential for forecasting and analyzing systems with discrete time changes.

Check your Progress - I

Based on your understanding of the previous section, answer these questions in your own words:

1. What is a difference equation?

2. What is the difference between ordinary and partial difference equations?

3. What is the general form of a first-order linear difference equation?

4. In a difference equation, what does the constant a represent?

5. Given the equation $x_{t+1} = 2 \cdot x_t + 10$, if $x_0 = 5$, find x_1 .

6. For the equation $y_{t+1} = 1.1 \cdot y_t$, if $y_0 = 100$, calculate y_1 .

7. Solve for x_2 in the equation $x_{t+1} = 1.5 \cdot x_t$, where $x_0 = 4$.

-
-
8. Using the equation $K_{t+1} = 1.05 \cdot K_t + 50$, find K_1 if $K_0 = 200$.
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-
-
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17.4 Solving Difference Equations

Difference equations are mathematical expressions that describe the relationship between consecutive terms in a sequence or the evolution of a system over time. These equations can model various processes such as population growth, capital accumulation, and the spread of diseases. In this section, we will discuss how to solve different types of difference equations, starting with the simplest and moving to more complex cases.

We will break it down into three parts:

- First-Order Linear Difference Equations
- Second-Order Difference Equations
- Nonlinear Difference Equations

Each subsection will provide a detailed explanation of the concepts, step-by-step procedures for solving these equations, and examples to help you understand how to apply the methods in real-world scenarios.

17.4.1 First-Order Linear Difference Equations

A first-order linear difference equation is of the form:

$$X_{t+1} = a \cdot X_t + b$$

Where:

- X_{t+1} is the value of the variable at the next time step.

- x_t is the value of the variable at the current time step.
- a is a constant that represents the rate of change or growth factor.
- b is a constant that represents any external force or input affecting the system.

To solve this kind of equation, we typically use a method called *iteration* or find a *closed-form solution* by recognizing patterns or using the general formula.

General Solution:

The general solution to the first-order linear difference equation is:

$$x_t = A \cdot a^t + \frac{b}{1-a}$$

Where:

- A is the constant that is determined by the initial condition, x_0 .
- a^t represents the growth factor raised to the power of the time step.

Now, let's go through three simple examples to illustrate how to solve first-order linear difference equations.

Example 1: Simple Population Growth

Given equation:

$$P_{t+1} = 1.1 \cdot P_t$$

This equation models population growth, where the population at the next time step (P_{t+1}) is 1.1 times the population at the current time step (P_t). The 1.1 represents a 10% growth rate (since 1.1 means 100% of the current population plus an additional 10%).

Initial condition:

$$P_0 = 100$$

This means that at time $t = 0$, the population is 100.

Step-by-step solution:

The general solution to a first-order linear difference equation like this one is:

$$P_t = A \cdot (1.1)^t$$

Where A is a constant that we need to determine using the initial condition.

1. **Find A** using the initial condition:

At time $t = 0$, the population is 100. So, we plug $P_0 = 100$ into the equation:

$$P_0 = A \cdot (1.1)^0 = A$$

Since $(1.1)^0 = 1$, we find:

$$A = 100$$

2. **Write the final equation:**

Now that we know $A = 100$, the equation for the population at any time t becomes:

$$P_t = 100 \cdot (1.1)^t$$

Interpretation:

- After one year ($t = 1$):

$$P_1 = 100 \cdot (1.1)^1 = 100 \cdot 1.1 = 110$$

So, the population in the next year is 110.

- After two years ($t = 2$):

$$P_2 = 100 \cdot (1.1)^2 = 100 \cdot 1.21 = 121$$

So, the population grows to 121 in the second year.

This simple model shows how the population grows by 10% every year.

Example 2: Investment Growth with External Input

Given equation:

$$I_{t+1} = 1.05 \cdot I_t + 200$$

This equation models the growth of an investment (I_t) over time, where the investment at the next time step (I_{t+1}) is 1.05 times the current investment (I_t), plus an additional fixed amount of 200 (this could represent external contributions, such as interest or additional investments).

Initial condition:

$$I_0 = 1000$$

This means that at time $t = 0$, the initial investment is 1000.

Step-by-step solution:

The general solution for a first-order linear difference equation with a constant term like this is:

$$I_t = A \cdot (1.05)^t + \frac{200}{1 - 1.05}$$

Where the second term comes from the constant $b=200$ in the equation.

1. Find A using the initial condition:

At time $t = 0$, the investment is 1000. So, we plug $I_0 = 1000$ into the equation:

$$I_0 = A \cdot (1.05)^0 + \frac{200}{1 - 1.05}$$

Since $(1.05)^0 = 1$, this simplifies to:

$$1000 = A + \frac{200}{-0.05}$$

Solving for A:

$$1000 = A - 4000 \Rightarrow A = 1000 + 4000 = 3000$$

2 Write the final equation:

Now that we know $A=3000$, the equation for the investment at any time t becomes:

$$I_t = 3000 \cdot (1.05)^t + 4000$$

Interpretation:

- After one year ($t=1$):

$$\begin{aligned} I_1 &= 3000 \cdot (1.05)^1 + 4000 = 3000 \cdot 1.05 + 4000 \\ &= 3150 + 4000 = 7150 \end{aligned}$$

So, the investment grows to 7150 in the first year.

- After two years ($t=2$):

$$\begin{aligned} I_2 &= 3000 \cdot (1.05)^2 + 4000 = 3000 \cdot 1.1025 + 4000 \\ &= 3307.5 + 4000 = 7307.5 \end{aligned}$$

So, the investment grows further to 7307.5 after the second year.

This example shows how the investment grows by 5% each year, with an additional 200 added each year.

Example 3: Simple Economic Model with Depreciation

Given equation:

$$K_{t+1} = 0.9 \cdot K_t + 100$$

This equation models capital accumulation with depreciation, K_{t+1} is the capital at the next time step, K_t is the current capital, and 100 is added each period (this could represent new investment or savings). The 0.9 factor indicates a 10% depreciation rate (since the capital loses 10% of its value each period).

Initial condition:

$$K_0 = 500$$

This means that at time $t=0$, the initial capital is 500.

Step-by-step solution:

The general solution for this equation is:

$$K_t = A \cdot (0.9)^t + \frac{100}{1 - 0.9}$$

1. **Find A** using the initial condition:

At time $t=0$, the capital is 500. So, we plug $K_0=500$ into the equation:

$$K_0 = A \cdot (0.9)^0 + \frac{100}{1 - 0.9}$$

Since $(0.9)^0 = 1$, this simplifies to:

$$500 = A + \frac{100}{0.1}$$

Solving for A:

$$500 = A + 1000 \Rightarrow A = 500 - 1000 = -500$$

2. **Write the final equation:**

Now that we know $A = -500$, the equation for the capital at any time t becomes:

$$K_t = -500 \cdot (0.9)^t + 1000$$

Interpretation:

□ After one year ($t=1$):

$$\begin{aligned} K_1 &= -500 \cdot (0.9)^1 + 1000 = -500 \cdot 0.9 + 1000 \\ &= -450 + 1000 = 550 \end{aligned}$$

So, the capital after the first year is 550.

□ After two years ($t=2$):

$$\begin{aligned} K_2 &= -500 \cdot (0.9)^2 + 1000 = -500 \cdot 0.81 + 1000 \\ &= -405 + 1000 = 595 \end{aligned}$$

So, the capital increases to 595 after the second year, despite the depreciation.

This example demonstrates how depreciation (the 0.9 factor) and external investment (the +100) interact to determine the capital accumulation over time.

In all three examples, the first-order linear difference equations allow us to model dynamic processes step-by-step, based on the relationship between the current state and the previous state. By solving these equations using initial conditions, we can predict future values and gain insights into the behavior of economic systems, population dynamics, investment growth, and other real-world processes.

17.4.2 Second-Order Difference Equations

A second-order difference equation involves the relationship between a variable at a given time step and its previous two time steps. The general form of a second-order linear difference equation is:

$$x_{t+2} = a \cdot x_{t+1} + b \cdot x_t + c$$

Where:

- x_{t+2} is the value of the variable at the next-to-next time step.
- x_{t+1} and x_t are the values of the variable at the previous time steps.
- a , b , and c are constants.

Solving Second-Order Difference Equations

To solve a second-order difference equation:

1. Write the equation in the general form $x_{t+2} = a \cdot x_{t+1} + b \cdot x_t + c$.
2. Find the characteristic equation associated with the homogeneous part (i.e., without the constant c).
3. Solve for the general solution to the homogeneous equation.

4. Find a particular solution to account for the non-homogeneous part (if applicable).
5. Combine both solutions (homogeneous and particular) and use initial conditions to determine the constants.

Example 1: Simple Economic Growth Model

Equation:

$$x_{t+2} = 1.05 \cdot x_{t+1} + 0.5 \cdot x_t$$

Initial conditions: $x_0 = 100$, $x_1 = 110$.

- This equation models a system where the value at time $t+2$ depends on the values at time $t+1$ and time t , with certain coefficients applied.

Homogeneous solution:

$$x_t = A \cdot (r_1)^t + B \cdot (r_2)^t$$

where r_1 and r_2 are the roots of the characteristic equation:

$$r^2 - 1.05r - 0.5 = 0$$

Solve the quadratic equation to find r_1 and r_2 .

17.4.3 Nonlinear Difference Equations

A nonlinear difference equation involves terms that are nonlinear with respect to the dependent variable. These can be more complex than linear equations and often do not have straightforward solutions. The general form of a nonlinear difference equation is:

$$x_{t+1} = f(x_t)$$

Where $f(x_t)$ is a nonlinear function of x_t .

Solving Nonlinear Difference Equations

Nonlinear difference equations can often be solved using numerical methods or by simplifying the problem to a linear approximation. In many cases, the solution will depend on the specific form of the nonlinear function $f(x_t)$.

Example 1: Logistic Growth Model

Equation:

$$P_{t+1} = r \cdot P_t \cdot \left(1 - \frac{P_t}{K}\right)$$

Where r is the growth rate, and K is the carrying capacity.

- This equation models population growth where the growth rate decreases as the population approaches the carrying capacity.

Example 2: Simple Investment Model with Diminishing Returns

Equation:

$$I_{t+1} = I_t \cdot \left(1 + \frac{c}{I_t}\right)$$

Where c is a constant.

- This equation models an investment system where the returns diminish as the investment increases.

Example 3: Capital Accumulation with Diminishing Returns

Equation:

$$K_{t+1} = K_t + \alpha \cdot K_t^2$$

Where α is a constant.

- This equation models the accumulation of capital, where returns decrease at an increasing rate.

In this section, we covered the methods for solving three types of difference equations: first-order linear, second-order linear, and nonlinear. Each type of equation has its own solution method, with linear equations generally being easier to solve compared to nonlinear ones. First-order linear difference equations are the simplest, while second-order equations involve relationships between multiple previous terms. Nonlinear equations, which can involve more complex relationships, are often solved numerically.

Understanding how to solve these equations and interpret their solutions is crucial in many fields such as economics, biology, and engineering, where dynamic systems evolve over time.

17.5 Economic Applications of Difference Equations

Difference equations play a critical role in modeling various dynamic systems in economics. These equations allow economists to model processes that evolve over discrete time periods, helping to understand how variables change step-by-step over time. Whether it's predicting population growth, capital accumulation, or analyzing investment decisions, difference equations provide a powerful tool for understanding and forecasting future outcomes based on historical data.

In this section, we will explore the key economic applications of difference equations, with a focus on:

1. Population Growth Models
2. Capital Accumulation and Economic Growth
3. Investment Models
4. Consumption Models
5. Price Determination in Markets

For each application, we will discuss the underlying economic concepts, formulate the appropriate difference equation, solve the equation, and interpret the results. This will provide a solid understanding of how difference equations are used to model real-world economic systems.

1. Population Growth Models

Population growth is one of the most common applications of difference equations. These models help to predict future population sizes based on the current population and the growth rate. A simple population growth model is based on the assumption that the population grows at a constant rate.

The Basic Difference Equation for Population Growth

A first-order difference equation is typically used for modeling population growth, where the population at the next time step depends on the current population and the growth rate. The general form of the equation is:

$$P_{t+1} = (1 + r) \cdot P_t$$

Where:

- P_{t+1} is the population at the next time period (year).
- P_t is the population at the current time period.
- r is the growth rate of the population (e.g., 2% growth per year).

Solving the Population Growth Equation

Let's assume that the initial population at $t = 0$ is $P_0 = 1000$ and the growth rate is $r = 0.02$ (or 2% per year).

The general solution to this difference equation is:

$$P_t = P_0 \cdot (1 + r)^t$$

Substituting the given values:

$$P_t = 1000 \cdot (1.02)^t$$

Using this formula, we can calculate the population at any future time t :

- After 1 year ($t = 1$):

$$P_1 = 1000 \cdot (1.02)^1 = 1020$$

- After 5 years ($t = 5$):

$$P_5 = 1000 \cdot (1.02)^5 \approx 1104.08$$

This simple model assumes constant growth. In reality, population growth may slow down due to factors such as limited resources or increased mortality, which can be modelled using more complex systems.

2. Capital Accumulation and Economic Growth

Capital accumulation is another important area where difference equations are widely used. In economic growth models, the accumulation of capital (machinery, infrastructure, etc.) is a key driver of productivity and overall economic growth. The Solow Growth Model, for example, uses difference equations to model how capital accumulates over time and how it affects output.

Solow Model of Capital Accumulation

The basic Solow model assumes that capital accumulates over time through investment, and that output (or GDP) is produced by both labor and capital. The capital stock in the economy is influenced by investment and depreciation, and the general form of the difference equation for capital accumulation is:

$$K_{t+1} = (1 - \delta) \cdot K_t + I_t$$

Where:

- K_{t+1} is the capital stock in the next time period.
- K_t is the capital stock in the current time period.
- δ is the depreciation rate of capital (e.g., 5% per year).
- I_t is the level of investment at time t .

The level of investment depends on factors like savings rates, technological progress, and government policies. For simplicity, let's assume that investment is a constant fraction of the capital stock, say $I_t = s \cdot K_t$, where s is the savings rate.

Thus, the equation becomes:

$$K_{t+1} = (1 - \delta) \cdot K_t + s \cdot K_t$$

This simplifies to:

$$K_{t+1} = K_t \cdot (1 - \delta + s)$$

Solving the Capital Accumulation Equation

Assume the following values:

- Initial capital stock $K_0 = 500$.
- Depreciation rate $\delta = 0.05$ (5% depreciation).
- Savings rate $s=0.2$ (20% of income is saved and invested).

The equation becomes:

$$K_{t+1} = K_t \cdot (1 - 0.05 + 0.2) = K_t \cdot 1.15$$

Starting with an initial capital of $K_0 = 500$, we can calculate the capital stock at any future time:

- After 1 year ($t = 1$):
$$K_1 = 500 \cdot 1.15 = 575$$
- After 5 years ($t=5$):
$$K_5 = 500 \cdot (1.15)^5 \approx 1011.66$$

This shows how capital accumulates over time as a result of investment and depreciation.

3. Investment Models

Investment models use difference equations to describe how the value of an investment grows or shrinks over time. The simplest model assumes that the value of an investment increases by a constant rate of return, but more complex models may include factors like inflation, risk, and external contributions.

Simple Investment Growth Model

A first-order linear difference equation can be used to model the growth of an investment with a constant rate of return. The general form of the equation is:

$$I_{t+1} = (1 + r) \cdot I_t$$

Where:

- I_{t+1} is the value of the investment at the next time period.
- I_t is the value of the investment at the current time period.

- r is the rate of return on the investment (e.g., 5% annual return).

Solving the Investment Growth Model

Let's assume the following values:

- Initial investment $I_0 = 1000$.
- Rate of return $r = 0.05$ (5%).

The general solution to this equation is:

$$I_t = I_0 \cdot (1 + r)^t$$

Substituting the given values:

$$I_t = 1000 \cdot (1.05)^t$$

We can now calculate the value of the investment at different time points:

- After 1 year ($t=1$):

$$I_1 = 1000 \cdot (1.05)^1 = 1050$$

- After 5 years ($t = 5$):

$$I_5 = 1000 \cdot (1.05)^5 \approx 1276.28$$

This model shows how an investment grows exponentially over time, thanks to the constant rate of return.

4. Price Determination in Markets

Difference equations are also used to model the determination of prices in markets over time. These models typically take into account factors like supply and demand, price elasticity, and external shocks to the economy.

Price Adjustment Model

One simple model for price adjustment is based on the idea that prices adjust over time in response to market conditions. A basic difference equation for price adjustment is:

$$P_{t+1} = P_t + \lambda \cdot (D_t - S_t)$$

Where:

- P_{t+1} is the price in the next time period.
- P_t is the price in the current time period.
- D_t is demand at time t .
- S_t is supply at time t .
- λ is a constant that determines how quickly prices adjust.

Solving the Price Adjustment Model

Assume:

- Initial price $P_0 = 100$.
- Demand $D_0 = 150$, supply $S_0 = 100$.
- Adjustment speed $\lambda = 0.1$.

The price adjustment equation becomes:

$$P_{t+1} = P_t + 0.1 \cdot (150 - 100)$$

After 1 period:

$$P_1 = 100 + 0.1 \cdot 50 = 100 + 5 = 105$$

This shows how the price increases in response to a higher demand compared to supply.

Difference equations are invaluable tools in economics, providing a way to model dynamic systems that evolve over discrete time intervals. Whether modeling population growth, capital accumulation, investment, or consumption, difference equations allow economists to make predictions about future behavior based on current data. The applications we have covered in this section represent just a few of the many ways difference equations are used in economics to understand and predict the behavior of complex systems.

17.6 Stability of Difference Equations

The concept of stability in difference equations is fundamental for understanding the long-term behavior of dynamic systems. Stability analysis allows us to determine whether a system will reach a steady state, continue growing, or behave in an unpredictable way. In economic models, stability is crucial for predicting whether an economic system will converge to equilibrium or diverge into undesirable outcomes. This section will explore the concept of stability, how to assess it, and its significance in different applications, particularly in economics.

What is Stability in Difference Equations?

In the context of difference equations, stability refers to the behavior of a system over time. A system is considered stable if small changes in its initial conditions lead to small changes in its long-term behavior. Conversely, an unstable system exhibits large changes in its behavior, even from small perturbations in initial conditions.

For a first-order linear difference equation, stability generally concerns whether the system's values (e.g., population, capital, investment) will approach a fixed value or continue growing or shrinking indefinitely.

For a system modeled by a difference equation to be stable, we typically expect that as time increases, the system reaches a steady state or equilibrium, where the value of the variable no longer changes, or the changes become minimal.

Stability in First-Order Linear Difference Equations

Let's begin with first-order linear difference equations. The general form of such an equation is:

$$X_{t+1} = a \cdot X_t + b$$

Where:

- x_{t+1} is the value of the variable at the next time step.
- x_t is the value of the variable at the current time step.
- a is a constant that affects the rate of change.

- b is a constant representing an external factor or forcing term.

Stability Criteria

To determine the stability of a first-order linear difference equation, we focus on the coefficient a , which represents the rate at which the variable x_t changes. The key criterion for stability is:

- If $|a| < 1$, the system is stable and will converge to a fixed point.
- If $|a| > 1$, the system is unstable and will diverge away from the equilibrium point.
- If $a = 1$, the system is neutral and will neither grow nor shrink but will continue to evolve based on initial conditions.

Stability in Second-Order Linear Difference Equations

Second-order difference equations involve relationships between the current value, the previous value, and the value two time periods ago. The general form of a second-order linear difference equation is:

$$x_{t+2} = a \cdot x_{t+1} + b \cdot x_t + c$$

Where:

- x_{t+2} is the value at the next-to-next time step.
- x_{t+1} and x_t are the values at the previous time steps.
- a , b , and c are constants.

To assess stability in second-order linear difference equations, we analyze the roots of the characteristic equation associated with the homogeneous part of the equation (i.e., without the constant c).

The characteristic equation is formed by assuming a solution of the form $x_t = r^t$, where r is a root of the equation. The stability of the solution depends on the magnitude of the roots:

- If the roots are within the unit circle (i.e., $|r| < 1$), the system is stable.

- If the roots are outside the unit circle (i.e., $|r| > 1$), the system is unstable.
- If the roots are on the unit circle (i.e., $|r| = 1$), the system is neutrally stable.

Stability in Nonlinear Difference Equations

Nonlinear difference equations involve relationships where the change in a variable depends nonlinearly on its current or past values. These equations are generally more complex than linear equations and may exhibit more intricate behavior, such as oscillations or chaotic dynamics.

Stability Criteria for Nonlinear Difference Equations

For nonlinear difference equations, stability can often be determined using linearization. Linearization involves approximating the nonlinear equation around its equilibrium points (fixed points), converting it into a linear equation to assess stability.

Consider a nonlinear equation of the form:

$$x_{t+1} = f(x_t)$$

The stability of an equilibrium point x^* is determined by analyzing the derivative of the function $f(x)$ at the equilibrium point. The criterion for stability is:

- If $|f'(x^*)| < 1$, the equilibrium is stable.
- If $|f'(x^*)| > 1$, the equilibrium is unstable.

Stability analysis in difference equations is essential for understanding the long-term behavior of economic systems. By examining the stability of solutions, economists can predict whether systems will converge to an equilibrium or diverge over time. For linear difference equations, the stability criterion is straightforward and depends on the magnitude of the coefficient a . For second-order linear and nonlinear difference equations, the analysis becomes more complex, often requiring the use of characteristic equations or linearization techniques.

In economics, understanding the stability of models is crucial for policy-making and long-term planning. Stable systems are desirable because they indicate that an economy or market will reach a steady state, while unstable systems could lead to

undesirable fluctuations or collapse. By applying stability analysis to real-world economic models, economists can better understand and anticipate the behavior of economic variables, such as population growth, capital accumulation, investment returns, and consumption patterns.

Check your Progress - II

Based on your understanding of the previous section, answer these questions in your own words:

1. What is a first-order linear difference equation, and how is it different from a second-order difference equation?

2. Explain the concept of stability in difference equations. What does it mean for a system to be stable?

3. What are the key factors that determine the stability of a first-order linear difference equation?

4. How do difference equations model population growth, and what is the general form of the difference equation used?

5. What is the significance of the equilibrium point in a nonlinear difference equation, and how is its stability determined?

6. Given the first-order linear difference equation $x_{t+1} = 0.8 \cdot x_t$, with $x_0 = 100$, calculate the value of x_3 .

7. Solve the equation $P_{t+1} = 1.1 \cdot P_t$, if $P_0 = 500$, and find the population after 4 periods.

8. For the equation $I_{t+1} = 1.05 \cdot I_t + 100$, with $I_0 = 1000$, find the value of I_2 after two periods.

17.7 Key words

1. **Difference Equation:** An equation that expresses the relationship between successive terms in a sequence, often used to model dynamic systems that evolve over discrete time intervals.
2. **First-Order Difference Equation:** A difference equation where the next value of a variable depends linearly on its current value.
3. **Second-Order Difference Equation:** A difference equation where the next value of a variable depends on the current and the previous two values.
4. **Linear Difference Equation:** A difference equation where the dependent variable and its differences appear in a linear form, without any nonlinear terms like squares or products of variables.
5. **Nonlinear Difference Equation:** A difference equation where the relationship between terms is not linear, meaning the variable depends on its previous terms in a nonlinear way, often involving terms like squares or products.
6. **Stability:** The property of a system where small changes in initial conditions result in small changes in long-term behavior, leading the system to either remain steady or return to equilibrium over time.
7. **Equilibrium Point:** A point in a dynamic system where the value of the variable remains constant over time, typically representing a steady state or balance in the system.

- 8 Characteristic Equation:** The equation derived from the homogeneous part of a second-order difference equation, used to determine the roots that help assess the system's stability.

17.8 Let's Sum Up

In this lesson, we have explored the essential concepts and techniques for solving difference equations and applying them to real-world economic problems. Here's a summary of the key points covered:

- 1. Difference Equations and Their Importance:** Difference equations are powerful tools that help model systems where variables change over discrete time intervals. Unlike continuous systems modeled by differential equations, difference equations focus on how systems evolve in distinct steps, making them ideal for modeling dynamic processes in economics, finance, and other fields.
- 2. Types of Difference Equations:**
 - **First-Order Linear Difference Equations:** These equations describe systems where the future value of a variable depends on its current value. The general form is $x_{t+1} = a \cdot x_t + b$, and the solution involves exponential growth or decay depending on the value of a .
 - **Second-Order Difference Equations:** These equations involve relationships between the current value, the previous value, and the value two time steps ago. They are useful for modeling more complex systems, like capital accumulation in economic growth models.
 - **Nonlinear Difference Equations:** Nonlinear equations capture more complex dynamics, where the rate of change of a variable depends in a nonlinear way on its current or past values. These equations are often used to model population growth, investment with diminishing returns, and other complex economic systems.

3. **Solving Difference Equations:** We covered methods for solving different types of difference equations, starting with the simplest first-order equations and progressing to more complex second-order and nonlinear equations. Solving these equations typically involves finding a general solution using iteration, applying initial conditions, and interpreting the results to understand the long-term behavior of the system.
4. **Economic Applications:** Difference equations are widely applied in economics, where they model real-world phenomena such as population growth, capital accumulation, investment, and consumption patterns. For example, in population models, a first-order difference equation can be used to predict future population sizes, while in capital accumulation models, second-order difference equations help describe how capital grows over time with investment and depreciation.
5. **Stability of Systems:** Stability analysis is crucial for understanding whether a system will reach a steady state or continue to grow without bound. In first-order linear difference equations, stability is determined by the coefficient a . If $|a| < 1$, the system is stable, and the variable converges to a steady value. If $|a| > 1$, the system is unstable, leading to unbounded growth or decay.
6. **Real-World Implications:** The ability to analyze the stability and behavior of economic systems is vital for policy-making, financial planning, and long-term forecasting. For example, understanding whether an investment model will converge to a sustainable growth path or spiral out of control can help inform investment decisions, government policies, and economic strategies.

17.9 Examination oriented questions

1. **Define difference equations and explain their significance in dynamic systems.**
 - **Approach:** Define difference equations and emphasize their role in modeling systems with discrete time steps.

- **Key Points to Write:**
 - A difference equation models the relationship between a variable's current and past values.
 - Used to predict future values in systems where changes occur in steps (e.g., population growth, investment).
 - Key in economics, finance, and other fields for dynamic system analysis.

2. **What is the difference between first-order and second-order linear difference equations?**

- **Approach:** Compare the structures of first and second-order difference equations.
- **Key Points to Write:**
 - **First-order:** Involves current and previous value (e.g., $x_{t+1} = a * x_t + b$).
 - **Second-order:** Involves current value and two previous values (e.g., $x_{t+2} = a * x_{t+1} + b * x_t$).
 - **First-order** models simpler processes (e.g., basic growth models), while **second-order** handles more complex dynamics (e.g., capital accumulation).

3. **Explain the stability of a first-order linear difference equation.**

- **Approach:** Define stability and explain how it is influenced by the coefficient.
- **Key Points to Write:**
 - **Stability:** A stable system converges to a fixed point as time progresses.
 - **Condition for Stability:** If $|a| < 1$, the system is stable

(converges to equilibrium). If $|a| > 1$, it is unstable (diverges).

- Show how the magnitude of “a” determines whether the system grows or shrinks over time.

4. How do you solve second-order difference equations and what is the role of initial conditions?

- **Approach:** Describe the steps to solve a second-order difference equation and the role of initial conditions.

- **Key Points to Write:**

- **Steps:**

1. Write in standard form ($x_{t+2} = a * x_{t+1} + b * x_t$).
2. Find the characteristic equation and solve for roots.
3. Determine the general solution based on roots (real, complex, or repeated).

- **Initial Conditions:** Used to solve for constants (e.g., x_0 and x_1) and predict future behavior.

5. Discuss the economic applications of difference equations with examples.

- **Approach:** Link the use of difference equations to real-world economic models.

- **Key Points to Write:**

- **Population Growth:** Use first-order difference equations (e.g., $P_{t+1} = r * P_t$) to model population changes over time.
 - **Capital Accumulation:** Second-order difference equations

to model capital stock growth and depreciation (e.g., $K_{t+1} = \sigma * K_t - d * K_t$).

- Show how these models can predict future trends and are widely used in economics.

6. What is the significance of stability analysis in economic models using difference equations?

- **Approach:** Explain the importance of analyzing stability for forecasting long-term outcomes.
- **Key Points to Write:**
 - **Stability analysis** helps determine whether a system will stabilize or diverge.
 - In **economic models**, stable systems suggest equilibrium or sustainable growth (e.g., population, capital).
 - **Instability** might indicate risks of economic collapse or constant fluctuation (e.g., runaway inflation).

17.10 Hints to Check Your Progress (CYP)

CYP-I:

1. General Solution of a Homogeneous Second-Order Differential Equation (Section 16.4.1):

- To solve a homogeneous second-order equation, recall the method for solving linear equations with constant coefficients. Focus on solving the characteristic equation to find the roots. Depending on the nature of the roots (real, complex, or repeated), the solution will take different forms.

2. Solving Non-Homogeneous Second-Order Differential Equations (Section 16.4.2):

- For non-homogeneous equations, use the method of undetermined coefficients to find a particular solution. The form of the particular solution depends on the right-hand side of the equation (whether it is exponential, sinusoidal, or polynomial). Study examples where this method is applied, especially when the forcing term has a clear functional form.

3. Stability Analysis of Solutions (Section 16.7.1):

- Stability analysis involves checking whether the system will stabilize or diverge. For systems of difference equations, analyze the roots of the characteristic equation.

4. Particular Solutions (Section 16.4.2):

- When solving for particular solutions in non-homogeneous equations, remember the method of undetermined coefficients. If the forcing function involves terms like exponential growth or sine/cosine functions, adjust your particular solution approach accordingly. Review examples from Section 16.4.2 for practice.

CYP-II:

1. Homogeneous vs Non-Homogeneous Equations (Section 16.3):

- A homogeneous equation has no external forcing term, whereas a non-homogeneous equation includes such a term (e.g., external shocks). Understanding this difference will help you select the correct solution method. Review the distinction between these types of equations in Section 16.3 for clarity.

2. Variation of Parameters (Section 16.4.3):

- When undetermined coefficients are not effective, use variation of parameters. This method involves allowing the constants in the general solution to vary as functions of time. Section 16.4.3

provides examples and explains how to apply this method to non-homogeneous equations, particularly when the right-hand side is more complicated.

3. Equilibrium Solutions and Stability (Section 16.6):

- Equilibrium solutions are steady states where the system doesn't change over time. Stability depends on whether small deviations from equilibrium will cause the system to return to equilibrium or diverge. Review Section 16.6 to understand how to analyze equilibrium points and their stability.

4. Capital Accumulation and Market Adjustment Models (Sections 16.3 and 16.4):

- These models are crucial in economics and often involve second-order or nonlinear difference equations. Review Sections 16.3 and 16.4 for worked-out examples where these models are used to describe capital accumulation or market adjustments, and solve for both general and particular solutions.

5. Second-Order Differential Equations in Economics (Section 16.3.2):

- Many economic models, such as the Solow growth model, are modelled using second-order difference equations. Review Section 16.3.2 to understand how these equations are applied to real-world economic scenarios, particularly in modeling capital accumulation and economic growth.

17.11 Suggested Readings

- Hoy, M., Livernois, J., & McKenna, C. (2017). *Mathematical Economics*
- Aggarwal, J. M., & Joshi, P. K. (2009). *Mathematics for Economics*
- Klein, M. W. (2009). *Economic Dynamics: Theory and Computation*

- Braun, M. (2017). *Differential Equations and Their Applications*
- Kapoor, V. K. (2010). *Applied Mathematical Methods for Economists*
- Strauss, W. A. (2008). *Partial Differential Equations: An Introduction*
- Fischer, S. R. (2008). *The Solow Growth Model: A Practical Guide*
- Samuelson, P. A., & Nordhaus, W. D. (2010). *Economics*

M.A. Economics	Lesson No. 18
C.No. : Eco-103	Semester - 1st
	Unit III

SOLVING FIRST ORDER DIFFERENCE EQUATIONS

STRUCTURE

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18.0 OBJECTIVES

The main objectives of this lesson are to enable you to:

- Understand the concept of first-order difference equations and how they model systems that evolve over discrete time intervals.
- Learn the general form and types of first-order difference equations, including linear and non-linear forms, and their applications.
- Master the methods for solving first-order difference equations, both linear and non-linear, using iterative and analytical approaches.
- Analyze the stability of solutions to first-order difference equations and understand their long-term behavior, such as convergence or divergence.
- Apply first-order difference equations to model real-world scenarios such as population growth, economic systems, and investment forecasting.
- Evaluate the impact of initial conditions on the behavior of the system and understand how they influence the solutions of the difference equation.

18.1 LEARNING OUTCOMES

By the end of this lesson, you should be able to:

1. Define first-order difference equations and explain their purpose in modeling systems that evolve over discrete time intervals.
2. Identify and distinguish between linear and non-linear first-order difference equations, understanding their characteristics and applications.
3. Solve first-order linear difference equations using both iterative methods and analytical techniques.
4. Solve non-linear first-order difference equations, including simplifying and approximating methods for complex models.
5. Interpret solutions of first-order difference equations to predict the future behavior of systems, including understanding their long-term trends.

6. Analyze the stability of solutions and evaluate the system's tendency to converge to an equilibrium, grow without bound, or stabilize.
7. Apply first-order difference equations to real-world scenarios, such as modeling population dynamics, economic growth, or investment strategies, using appropriate methods to solve and interpret the results.

18.2 INTRODUCTION

First-order difference equations are a powerful tool used to model systems that evolve over discrete time intervals. Unlike continuous models, such as differential equations, which describe processes that change smoothly over time, difference equations focus on systems where changes occur step by step, at regular intervals. These intervals could be days, months, years, or any other consistent unit of time. As such, first-order difference equations are invaluable in fields that require the analysis of processes that unfold in distinct, regular intervals, such as economics, biology, and social sciences.

The core concept behind first-order difference equations is simple yet versatile: they describe how the state of a system at one time step is related to its state at the previous time step. This recursive relationship makes difference equations suitable for modeling dynamic processes that evolve over time, with each step depending on the last. Such equations are particularly useful in fields where discrete events or measurements are tracked and analyzed.

In economics, first-order difference equations are widely used to model a variety of systems, from capital accumulation to economic growth. They allow economists to predict future outcomes based on current conditions and the rules governing their evolution. Similarly, in biology, these equations are used to model population growth, where the future population depends on the current population and growth rate. The ability to model such systems with precision makes first-order difference equations an essential tool for analyzing trends and forecasting future behavior.

Another important aspect of first-order difference equations is their ability to provide insight into the **long-term behavior** of a system. Through the solutions of these equations, we can determine whether a system will stabilize over time, continue to grow, or decline. For example, in an investment model, first-order difference equations can help predict whether an investment will grow exponentially or stabilize after reaching a

certain threshold. In a biological system, they can forecast whether a population will reach a carrying capacity or grow indefinitely under ideal conditions.

Furthermore, first-order difference equations are useful for understanding the stability of systems. By analyzing the structure of the equation, we can understand whether small changes in the initial conditions lead to significant changes in the long-term outcome. This stability analysis is crucial in fields like economics and ecology, where systems often need to be balanced to avoid instability or collapse.

In this lesson, we will explore how to solve first-order difference equations and understand the behavior of their solutions. We will also examine their applications across various fields, providing you with the tools to model real-world dynamic systems and predict future trends. By the end of the lesson, you will have a strong foundation in both the theory and practical use of first-order difference equations, equipping you to apply them in a wide range of scenarios.

18.3 CORE CONCEPTS

In this section, we will explore the fundamental ideas and core principles that define first-order difference equations. These equations play a crucial role in modeling dynamic systems that evolve over time, especially in situations where changes occur in discrete time steps, such as daily, monthly, or yearly. First-order difference equations provide a framework to describe how the state of a system at one time point is related to its state at the previous time point, typically with the goal of predicting future states.

Difference equations, particularly first-order ones, are essential in numerous fields, including economics, biology, engineering, and finance. Whether it's tracking population growth, investment returns, or system behaviors over time, first-order difference equations offer an elegant way to model and analyze these processes. Understanding their structure, applications, and solving methods is fundamental for working with systems that evolve in a step-by-step manner.

18.3.1 Definition of First-Order Difference Equations

A first-order difference equation is a type of recurrence relation that expresses the relationship between the current value of a variable and its value at the previous time step. These equations are characterized by the fact that they involve only the present

and previous values of the sequence or system being modeled, making them the simplest form of difference equations.

In a first-order difference equation, the value of the variable at the next time step is determined by its value at the current time step and possibly some external factors. The general form of a first-order difference equation can be written as:

$$x_{t+1} = f(x_t, t)$$

Where:

- x_{t+1} is the value of the variable at the next time step (time $t+1$),
- x_t is the value of the variable at the current time step (time t),
- $f(x_t, t)$ is a function that dictates how the system evolves from one time step to the next, potentially involving both the current value and time-dependent factors.

This recurrence relation provides a method to iteratively compute the values of a variable over time, based on its initial value and the recurrence rule specified by the function $f(x_t, t)$.

Definitions of First-Order Difference Equations:

1. **“A difference equation is an equation that relates the values of a function at different points in time, typically at discrete time intervals. First-order difference equations are used to model systems that evolve by relating each value to its immediate predecessor.”** — *Mathematics for Economists*, Hoy, M., et al. (2017).

This definition highlights the key characteristic of first-order difference equations: they capture the evolution of systems over discrete steps, linking each value directly to the previous one. This makes them especially useful for modeling systems where changes happen at regular intervals.

2. **“A first-order difference equation describes a process where the current state of the system is a function of its previous state. The equation reflects a recursive process where future states are**

determined iteratively.” — *Introduction to Mathematical Economics*, Samuelson, P. A., & Nordhaus, W. D. (2010).

Samuelson and Nordhaus emphasize the recursive nature of first-order difference equations, where each state depends on the previous one. This process is iterative, meaning that we compute each subsequent value using the equation's recurrence relation.

- 3 **“First-order difference equations express the change in a variable from one time period to the next. They are a discrete-time counterpart to differential equations, which describe continuous changes over time.”** — *Mathematical Methods for Economics*, Lovell, M. C. (2009).

This definition makes a critical connection between difference equations and differential equations, noting that first-order difference equations are used when changes occur in discrete time intervals, offering an alternative to continuous models.

- 4 **“A first-order linear difference equation is a recurrence relation that expresses the next value of a variable as a linear function of the current value.”** — *Fundamentals of Mathematical Economics*, Brown, R. G. (2013).

This definition specifically addresses linear first-order difference equations, where the relationship between consecutive values is linear. This is a simpler and more common form of first-order difference equations, often used in economic models and forecasting.

Key Features of First-Order Difference Equations:

- 1 **Discrete Time Evolution:** First-order difference equations describe systems that change in discrete time steps. This is in contrast to continuous models, such as differential equations, where changes occur continuously. In difference equations, time is divided into intervals (e.g., daily, yearly) and the system evolves in steps from one interval to the next.
- 2 **Recursive Nature:** The value of the system at the next time step depends directly on its value at the current time step, making these equations **recursive**.

This means that to find the value of the system at any point, we start with an initial condition (usually given at time $t=0$ or $t=0$) and use the recurrence relation to compute subsequent values.

- 3 Iterative Process:** The iterative nature of first-order difference equations allows for the computation of future values by applying the equation repeatedly. Starting from an initial value, we can generate a sequence of values that reflects the system's evolution over time.
- 4 Applications:** These equations are widely used in fields like economics (modeling capital accumulation, economic growth, and investment), biology (modeling population growth and spread of diseases), and physics (modeling discrete physical processes). Their simplicity and versatility make them useful tools for predicting future behavior based on past trends.

First-order difference equations form the foundation of discrete-time modeling in many real-world applications. By describing how a system evolves based on its previous state, they provide a simple yet powerful way to forecast future values, analyze system behavior, and study long-term trends. Whether in economics, biology, or other fields, understanding and applying these equations is crucial for anyone looking to model dynamic systems that change over time in discrete intervals.

18.3.2 Types of First-Order Difference Equations

First-order difference equations are highly versatile and can be classified based on the relationship between the current and future states of the system. These classifications—linear versus non-linear, and homogeneous versus non-homogeneous—help identify the right model for different types of dynamic systems. In this section, we will explore these types and provide an example for each to illustrate their application.

1. Linear vs. Non-Linear First-Order Difference Equations

Linear First-Order Difference Equations

A linear first-order difference equation is one where the future state of the system is a linear function of the current state. The equation typically takes the form:

$$X_{t+1} = a \cdot X_t + b$$

Where:

- x_t is the value of the variable at time t ,
- a is a constant coefficient that dictates the relationship between consecutive values,
- b is a constant or an external factor influencing the system.

Example of a Linear First-Order Difference Equation:

Let's consider a simple economic model of capital accumulation, where capital at time $t+1$ is a constant multiple of the capital at time t (assuming no external factors are involved). The equation might look like:

$$K_{t+1} = 1.05 \cdot K_t$$

Here, K_t is the capital stock at time t , and 1.05 represents a 5% annual return on the capital. This is a linear equation because the relationship between K_{t+1} and K_t is directly proportional (i.e., it depends linearly on the current capital stock).

Non-Linear First-Order Difference Equations

In contrast, non-linear first-order difference equations involve non-linear relationships, where the future state is not simply a linear function of the current state. These equations may involve powers, products, or other non-linear functions of the variable.

Example of a Non-Linear First-Order Difference Equation:

A common example of a non-linear difference equation is the **logistic growth model**, which is used to describe population growth with a carrying capacity. The equation can be written as:

$$P_{t+1} = P_t \cdot \left(1 + r \cdot \left(1 - \frac{P_t}{K} \right) \right)$$

Where:

- P_t is the population at time t ,
- r is the growth rate,
- K is the carrying capacity, which limits the growth of the population.

This equation is non-linear because the growth rate changes as the population approaches the carrying capacity K . As P_t grows closer to K , the growth rate decreases, introducing non-linear dynamics into the system.

2. Homogeneous vs. Non-Homogeneous First-Order Difference Equations

Homogeneous First-Order Difference Equations

A homogeneous first-order difference equation does not include an external forcing term. The future value of the system depends solely on the current state of the system. In other words, there are no external factors influencing the system's evolution.

Example of a Homogeneous First-Order Difference Equation:

Consider the simple population growth model, where the population grows by a fixed percentage over time. The equation for this model is:

$$P_{t+1} = 1.02 \cdot P_t$$

Here, P_t is the population at time t , and the population grows by 2% each period. This is a homogeneous equation because it depends only on the current state of the system (the current population) and does not include any external factors.

Non-Homogeneous First-Order Difference Equations

A non-homogeneous first-order difference equation includes an external forcing term, which represents an influence or input that affects the system's evolution. This external factor could represent things like an investment, immigration, or government intervention, among others.

Example of a Non-Homogeneous First-Order Difference Equation:

A classic example of a non-homogeneous first-order difference equation is the investment model in economics, where the future value of an investment depends on both its current value and an external contribution (e.g., new investments). The equation can be written as:

$$I_{t+1} = 1.05 \cdot I_t + 100$$

Here, It is the value of the investment at time t , 1.05 is the growth rate (5% annual return), and 100 represents a fixed external investment added each period. This is a non-homogeneous equation because the term 100 represents an external factor (the additional investment) that affects the growth of the investment.

3. Types of Linear First-Order Difference Equations

Within the category of linear first-order difference equations, there are two key types based on the behavior of their coefficients: constant coefficient equations and variable coefficient equations.

Constant Coefficient Linear First-Order Difference Equations

These equations have fixed coefficients, meaning the relationship between the current and next states of the system is constant across all time steps. The general form of a constant coefficient linear first-order difference equation is:

$$X_{t+1} = a \cdot X_t + b$$

Where a and b are constants. These equations are typically simple to solve and predict a steady, predictable progression over time.

Example of a Constant Coefficient Linear First-Order Difference Equation:

A simple savings model can be expressed as a constant coefficient linear first-order difference equation. Let's say a person deposits a fixed amount of money into a savings account every year, and the account grows by a fixed interest rate each year. The equation might be:

$$A_{t+1} = 1.05 \cdot A_t + 500$$

Where A_t is the amount of money in the account at time t , and 1.05 represents a 5% annual interest rate. The term 500 represents the fixed yearly deposit. This equation is linear with constant coefficients, and the relationship between the amount of money at time t and time $t+1$ remains the same.

Variable Coefficient Linear First-Order Difference Equations

In these equations, the coefficients change over time. The relationship between the current and future state of the system is not fixed but instead varies depending on

the time step. The general form of a variable coefficient linear first-order difference equation is:

$$X_{t+1} = a_t \cdot X_t + b_t$$

Where both a_t and b_t are functions of time.

Example of a Variable Coefficient Linear First-Order Difference Equation:

An example might be a variable interest rate model in which the interest rate changes over time. Suppose an investment grows at a rate that fluctuates each year depending on economic conditions. The equation could be written as:

$$I_{t+1} = (1 + r_t) \cdot I_t + b_t$$

Where:

- I_t is the value of the investment at time t ,
- r_t is the interest rate at time t ,
- b_t represents an external input (e.g., additional investment or withdrawal).

In this case, the interest rate r_t and the external input b_t vary with time, making this a variable coefficient linear first-order difference equation.

First-order difference equations can be categorized into linear and non-linear, as well as homogeneous and non-homogeneous, each with its own characteristics and applications. Linear equations, both with constant and variable coefficients, are widely used due to their simplicity and predictability. Non-linear equations, on the other hand, model more complex systems that may exhibit unpredictable or rich behavior. Understanding these types and their differences helps in choosing the right model for real-world systems, from population growth to economic investments, and provides valuable insights into their future behavior.

18.4 METHODS FOR SOLVING FIRST-ORDER DIFFERENCE EQUATIONS

First-order difference equations are vital tools for modeling discrete systems over time. They are used to represent a variety of processes, from population growth to financial modeling. Once we understand the structure of these equations, the next

step is solving them to understand the behavior of the system over time. There are multiple methods for solving first-order difference equations, depending on the nature of the equation (linear or non-linear) and the specific application. These methods can generally be categorized into three main approaches: Iterative Methods, Analytical Methods, and Special Techniques. In this section, we will explore each method in detail, providing at least three examples for each to illustrate how these techniques are applied.

18.4.1 Iterative Methods

Iterative methods are commonly used to solve first-order difference equations by directly computing each successive value of the system starting from an initial condition. These methods involve repeated application of the difference equation to generate a sequence of values that describes the system's evolution over time. This approach is particularly useful when an explicit analytical solution is difficult or impossible to obtain. Iterative methods are simple, straightforward, and often used when numerical solutions are required.

Iterative Method: General Approach

The general procedure for solving a first-order difference equation iteratively involves:

1. Starting with an initial condition, x_0 .
2. Using the recurrence relation to compute the next value, x_1 .
3. Repeating the process for the next time steps until a desired time point is reached, x_t .

The advantage of this method is its simplicity and the ability to compute solutions for each time step, making it ideal for applications in which each state depends on the previous one. The only challenge is the need for a clear initial condition and the assumption that the recurrence relation is well-defined for all subsequent steps.

Example 1: Population Growth Model

Suppose a population grows at a constant rate of 5% per year. The equation is:

$$P_{t+1} = 1.05 \cdot P_t$$

With an initial population of 1000 at time $t=0$, we can compute the population at each time step iteratively:

- Start with $P_0 = 1000$.
- Compute $P_1 = 1.05 \cdot 1000 = 1050$.
- Compute $P_2 = 1.05 \cdot 1050 = 1102.5$.
- Continue this process until the desired time step.

By continuing the process for multiple years, we can predict the population at any future time, making iterative methods a useful tool for modeling continuous growth.

Example 2: Investment Growth with Fixed Contribution

Consider an investment that grows by 4% annually, and each year, an additional \$500 is added to the investment. The equation is:

$$1_{t+1} = 1.04 \cdot 1_t + 500$$

With an initial investment of \$2000 at $t=0$, we calculate the value of the investment for the next few years:

- Start with $I_0 = 2000$.
- Compute $I_1 = 1.04 \cdot 2000 + 500 = 2580$.
- Compute $I_2 = 1.04 \cdot 2580 + 500 = 3191.2$.
- Continue iterating for as many periods as needed.

This iterative approach can help forecast the value of the investment after any number of years.

Example 3: Simple Loan Repayment Model

Imagine a loan of \$5000 with an interest rate of 6% annually. The loan repayments are \$600 per year. The first-order difference equation representing the balance of the loan is:

$$L_{t+1} = L_t \cdot 1.06 - 600$$

Where L_t is the loan balance at time t . Starting with an initial loan balance of \$5000:

- Start with $L_0 = 5000$.
- Compute $L_1 = 5000 \cdot 1.06 - 600 = 4700$.
- Compute $L_2 = 4700 \cdot 1.06 - 600 = 4402$.
- Continue this iteration to see the loan balance decrease over time.

This iterative method allows us to model the progression of the loan balance year by year.

18.4.2 Analytical Methods

Analytical methods involve finding an exact solution to the first-order difference equation, often by solving the recurrence relation algebraically. Analytical methods are particularly useful for linear first-order difference equations, as they can often be solved using well-known techniques such as separation of variables, solving homogeneous equations, or applying the method of undetermined coefficients. The result is a closed-form expression that provides the system's behavior over time without the need for iterative calculation.

Analytical Method: General Approach

For linear equations, the solution process typically follows these steps:

1. Solve the homogeneous part of the equation (if applicable).
2. Find a particular solution to the non-homogeneous equation (if applicable).
3. Combine the homogeneous and particular solutions to obtain the general solution.
4. Apply initial conditions to determine any constants of integration.

Example 1 : Solving a Simple Linear Growth Equation

Consider the equation:

$$X_{t+1} = 1.1 \cdot X_t$$

This equation describes a system that grows by 10% each period. The general solution is obtained by recognizing that this is a linear equation with constant coefficients. To solve it:

- Write the equation as $x_{t+1} - 1.1 \cdot x_t = 0$.
- The general solution is $x_t = C \cdot (1.1)^t$, where C is a constant determined by initial conditions.

If we know $x_0 = 100$, then $C = 100$, and the solution becomes $x_t = 100 \cdot (1.1)^t$.

Example 2: Solving a Linear Equation with an External Input

Consider an investment model where the value of the investment grows by 4% each year, but also receives an additional \$1000 each year. The equation is:

$$I_{t+1} = 1.04 \cdot I_t + 1000$$

To solve this analytically:

- Solve the homogeneous equation $I_{t+1} = 1.04 \cdot I_t$, which gives the general solution $I_t = C \cdot (1.04)^t$.
- For the non-homogeneous part, assume a particular solution of the form $I_t = A$, where A is a constant. Substituting into the equation, we find

$$A = \frac{1000}{1 - 1.04} = -25000.$$

- Combine the homogeneous and particular solutions:

$$I_t = C \cdot (1.04)^t - 25000.$$

If we know that $I_0 = 5000$, then $C = 5000 + 25000 = 30000$, and the final solution is $I_t = 30000 \cdot (1.04)^t - 25000$.

Example 3: Solving a Logistic Growth Model

Consider the logistic growth model:

$$P_{t+1} = P_t \cdot \left(1 + r \cdot \left(1 - \frac{P_t}{K} \right) \right)$$

This model can be solved analytically using separation of variables or an appropriate substitution method, although it may require more advanced techniques.

The general approach is to solve for the equilibrium points and analyze the behavior of the population over time.

18.4.3 Special Techniques

While iterative and analytical methods cover most situations, some first-order difference equations—particularly non-linear or complex systems—require special techniques for solving. These techniques may include the use of generating functions, transform methods, or approximation methods. Special techniques are often used when the equation involves more complicated forms that resist easy simplification.

Special Techniques: General Approach

These methods generally involve transforming the equation into a form that is easier to solve or approximating the solution using numerical methods. For example, generating functions can be used for combinatorial models, and approximation methods like Euler's method can be used for non-linear systems.

Example 1: Using Generating Functions for Recurrence Relations

Generating functions are a powerful tool for solving recurrence relations, especially when the recurrence involves combinations or sequences. Suppose we have the following recurrence relation for the Fibonacci sequence:

$$F_{t+1} = F_t + F_{t-1}$$

Using a generating function, we can solve this recurrence and find an explicit formula for F_t . The generating function approach involves converting the recurrence relation into a function of a variable z and manipulating the series to derive the closed-form solution.

Example 2: Approximation Using Euler's Method

For non-linear first-order difference equations that resist direct analytical methods, we can use approximation techniques such as Euler's method. For instance, if we have a non-linear equation describing population growth that includes a growth term $P_{t+1} = P_t + r \cdot P_t \cdot \left(1 - \frac{P_t}{K}\right)$, we can use Euler's method to approximate the population over discrete time steps. This method involves discretizing the equation and approximating the solution at each step.

Example 3: Using a Transformation Method for Non-Linear Equations

Suppose we have a non-linear difference equation of the form:

$$x_{t+1} = x_t^2 + 1$$

To solve this, we might use a transformation method, such as converting the equation into a linear form or applying a logarithmic transformation to simplify the equation. By transforming the system, we make it easier to solve or approximate the solution over time.

Check your Progress - I

Based on your understanding of the previous section, answer these questions in your own words:

1. What is a first-order difference equation, and how does it differ from a differential equation?

2. Explain the difference between linear and non-linear first-order difference equations. Provide one example for each.

3. What is the significance of stability in first-order difference equations? How can you determine whether a system is stable or unstable?

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4. Describe the difference between homogeneous and non-homogeneous first-order difference equations. Provide an example of each.

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5. How do iterative methods help in solving first-order difference equations, and what is their main advantage in practical applications?

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-
6. Solve the following first-order linear difference equation iteratively:

$$X_{t+1} = 1.2 \cdot X_t \quad \text{with} \quad X_0 = 100$$

Compute the values of x_1 , x_2 , and x_3 .

7. Solve the first-order linear difference equation using an analytical method:

$$y_{t+1} = 1.1 \cdot y_t + 500 \quad \text{with } y_0 = 1000$$

Find the general solution and compute y_2 .

8. A loan balance decreases by 10% annually, and an annual payment of \$2000 is made to the loan. The equation is:

$$L_{t+1} = 0.9 \cdot L_t - 2000 \quad \text{with } L_0 = 10000$$

Calculate L_1 , L_2 , and L_3 iteratively.

18.5 STABILITY AND BEHAVIOR OF SOLUTIONS

In any dynamic model, particularly those based on difference equations, understanding how the system behaves over time is vital. It's not enough to simply find a solution to a difference equation; we must also interpret the implications of that solution. This leads us to the essential concept of *stability*—a cornerstone of dynamic system analysis.

Stability refers to how a system reacts to small disturbances or deviations from its initial conditions. Imagine an economic model predicting the growth of national income, population, or capital accumulation. A stable system will absorb small shocks and return to a steady path over time. An unstable system, by contrast, might veer off

course drastically from a tiny disruption, rendering long-term forecasting unreliable or even irrelevant.

This section delves into the stability of first-order difference equations and how we can assess the *long-term behavior* of their solutions. These two elements—stability and asymptotic behavior—help us determine whether a model yields economically plausible outcomes over time. Will the economy settle into equilibrium? Will population growth explode uncontrollably? Will investment values oscillate forever?

Let us first understand stability in the context of first-order difference equations.

18.5.1 Stability in First-Order Difference Equations

A **first-order difference equation** is a recurrence relation that links the value of a variable at one point in time to its value in the preceding period. Its general form is:

$$X_{t+1} = a \cdot X_t + b$$

Where:

- x_t is the value of the variable at time t ,
- a is a parameter that determines the system's dynamic response,
- b is a constant term that can represent an external influence or shock.

The parameter a , often called the *stability coefficient*, plays a pivotal role in dictating the long-term behavior of the system. To determine whether the system is stable, we examine the magnitude (absolute value) of this coefficient.

Stability Conditions

- **If $|a| < 1$:** The system is stable. Over time, regardless of the initial condition x_0 , the values of x_t will converge to a fixed value or equilibrium.
- **If $|a| > 1$:** The system is unstable. Small deviations will grow larger over time, causing the system to diverge.
- **If $|a| = 1$:**
 - If $a = 1$, the system neither grows nor declines—it exhibits neutral stability, and deviations persist indefinitely.

- If $a = -1$, the system oscillates indefinitely and does not settle—called bounded instability or neutral oscillation.

Example : Stable System

Consider the equation:

$$X_{t+1} = 0.8 \cdot X_t$$

Suppose $x_0 = 100$. We compute successive values:

- $X_1 = 0.8 \cdot 100 = 80$
- $X_2 = 0.8 \cdot 80 = 64$
- $X_3 = 0.8 \cdot 64 = 51.2$

And so on. As $t \rightarrow \infty$, $x_t \rightarrow 0$. The system converges to a stable equilibrium point at zero.

Example: Unstable System

Now consider:

$$X_{t+1} = 1.2 \cdot X_t$$

With $x_0=100$, we get:

- $x_1=120$
- $x_2=144$
- $x_3=172.8$

Here, values increase exponentially. The system is unstable because any initial value leads to unbounded growth.

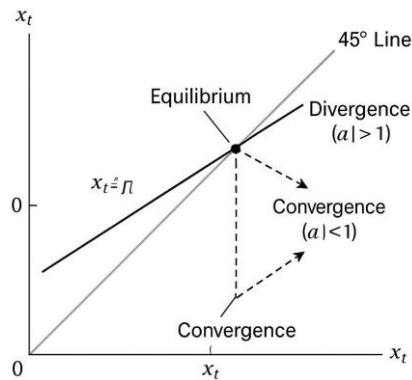
Fixed Point or Equilibrium Analysis

In general, for a first-order linear difference equation:

$$X_{t+1} = a \cdot X_t + b$$

We solve for the *equilibrium point* x^* by setting $X_{t+1} = X_t = X^*$:

$$x^* = a \cdot x^* + b \Rightarrow x^*(1 - a) = b \Rightarrow x^* = \frac{b}{1 - a} \quad (\text{provided } a \neq 1)$$



Graphical Interpretation of Stability

This equilibrium represents the value towards which the system will converge, provided it is stable.

Graphical Interpretation

Stability can also be understood graphically. On a phase diagram with x_t on the x-axis and x_{t+1} on the y-axis, the 45-degree line represents equilibrium. If the graph of the function intersects this line, we can visually assess stability by looking at the slope (which is a):

- If the slope is shallow ($|a| < 1$), successive iterations move closer to equilibrium.
- If the slope is steep ($|a| > 1$), the path diverges.

The diagram illustrates the concept of stability in first-order difference equations through a phase-plane analysis. On the horizontal axis is x_t , representing the current state of the system, while the vertical axis shows x_{t+1} , the next state. The 45-degree line, drawn in Gray, represents all points where $x_{t+1} = x_t$; this line is critical because any point where the system's function intersects it represents a potential equilibrium or fixed point. The black curve depicts the system's actual updating rule—how the value at time $t+1$ depends on the value at time t .

At the point where this curve intersects the 45-degree line, we find the equilibrium value $\bar{x}$, where the system theoretically remains unchanged if reached. The behavior of the system around this equilibrium depends on the slope of the black curve. If the slope is less than one in absolute value ($|a| < 1$), the system is stable: values move closer to the equilibrium over time, indicating convergence. Conversely, if the slope is greater than one ($|a| > 1$), the system is unstable, and small deviations grow over time, causing the system to diverge from equilibrium. Arrows in the diagram illustrate these movements toward or away from the equilibrium point. This visual representation helps in understanding the long-term tendencies of discrete dynamic systems modeled by first-order difference equations.

Non-Homogeneous Case

In a non-homogeneous equation with a constant b , we have:

$$X_t = a^t \cdot X_0 + \frac{b(1 - a^t)}{1 - a} \quad (\text{if } a \neq 1)$$

As $t \rightarrow \infty$, if $|a| < 1$, the first term vanishes and the system approaches the steady state $x = \frac{b}{1 - a}$.

This emphasizes that not just initial conditions, but also the structural parameters a and b , influence the system's long-term stability.

18.5.2 Long-Term Behavior

Stability addresses whether a system settles or diverges. The long-term behavior of a first-order difference equation provides a more nuanced picture. It explores *how* a system approaches equilibrium (or not), the *speed* of convergence, and the *path* taken—whether smooth, oscillatory, or chaotic.

Types of Long-Term Behavior

1. Convergence to a Steady State:

When $|a| < 1$, the solution of the system tends to a finite value:

$$X = \frac{b}{1 - a}$$

This is common in economic models where a variable like capital stock or price stabilizes over time.

2. **Exponential Divergence:**

When $|a| > 1$, the system grows or declines exponentially without bound.

3. **Oscillatory Convergence or Divergence:**

When $a < 0$, the system alternates between positive and negative values:

- If $-1 < a < 0$, the oscillation decays over time and the system converges.
- If $a < -1$, the oscillations grow in magnitude and the system is unstable.

Example:

$$X_{t+1} = -0.5 \cdot X_t, \quad X_0 = 100 \quad \Rightarrow \quad X_1 = -50, X_2 = 25, X_3 = -12.5, \dots$$

This is an oscillating convergence.

4. **Neutral Behavior:**

- If $a = 1$, the variable increases linearly with the constant b . For example:

$$X_{t+1} = X_t + b \quad \Rightarrow \quad X_t = X_0 + t \cdot b$$

This models linear growth (e.g., constant savings or depreciation).

- If $a = -1$, the system flips every period: $x_0=1, x_1=-1, x_2=1, \dots$

Economic Interpretation

- **Stable Convergence:** Desirable in models of capital or national income; it implies the economy stabilizes over time.
- **Explosive Behavior:** May reflect inflationary spirals, debt accumulation, or speculative bubbles.
- **Oscillations:** Common in inventory or business cycle models, where overshooting and correction alternate.
- **Neutral Trends:** Used for modeling budget surpluses, constant marginal cost pricing, or zero-profit conditions in long run equilibrium.

Rate of Convergence

The speed at which a system converges depends on the value of $|a|$:

- **If $|a|$ is close to 0** : Fast convergence.
- **If $|a|$ is just under 1** : Slow convergence.

This has policy implications. For instance, a fiscal rule that stabilizes public debt faster is more desirable than one that does so sluggishly.

Sensitivity to Initial Conditions

For unstable systems ($|a| > 1$), the *initial value* x_0 greatly affects the trajectory. Two economies starting close can diverge widely over time. This is a foundational concept in the analysis of chaotic systems and economic inequality.

Summary of Key Takeaways		
Stability Condition	Behavior	Economic Example
$ a < 1$	Convergent	Stable growth
$ a = 1$	Neutral	Constant level
$ a > 1$	Divergent	Explosive growth
$a < 0$	Oscillatory (stable or unstable)	Business cycles, over-adjustment models

In conclusion, analyzing the stability and long-term behavior of first-order difference equations allows us to assess whether a modeled economic system is reliable over time. Understanding the conditions under which a model converges, diverges, or oscillates ensures that we can draw meaningful and applicable insights from our mathematical representations. For economists, policymakers, and modelers, this analysis forms the backbone of robust forecasting, sustainable policy design, and long-run strategic planning.

18.6 REAL-TIME DATA MODELING USING FIRST-ORDER DIFFERENCE EQUATIONS

In our everyday lives, we often observe how things change over time. For example, the number of people using mobile phones, the price of vegetables, or the value of money in a savings account. These changes usually happen step by step—day by day, month by month, or year by year. When economists or data scientists want to

study and predict how something changes over time in these steps, they use a mathematical tool called a first-order difference equation.

This section will explain how we can use first-order difference equations to model or represent real-time data. We will see how they help us understand how things grow, shrink, or reach a balance over time. We will also go through real-world examples like population changes, savings account growth, and product prices to see how these equations work.

What Is Real-Time Data Modeling?

Real-time data modeling simply means using data collected at regular time intervals (daily, monthly, yearly) to describe how something changes. When we model this data, we try to write a rule or formula that shows how the value at one time step depends on the previous value.

This is where first-order difference equations come in. These are simple equations that show how a number at time $t+1$ depends on the number at time t .

Why Do We Use First-Order Difference Equations?

We use first-order difference equations because they help us:

- **Predict future values** based on current values.
- **Model growth or decline** (like population or savings).
- **Understand stability**, or whether a system will settle down or go out of control.
- **Make real-time decisions** using updated data.

They are simple but powerful tools. Even though they are easy to understand, they can describe many real-world problems accurately.

Let's Start with a Simple Example: Population Growth

Imagine a small town where the population increases by 2% every year. If the current population is 1,000 people, next year the population will be:

$$P_{t+1} = 1.02 \cdot P_t$$

Here:

- P_t is the population this year,
- 1.02 means a 2% increase (since $100\% + 2\% = 102\% = 1.02$),
- There's no added value (so $b=0$).

If the current population $P_0 = 1000$, then:

- After 1 year: $P_1 = 1.02 \cdot 1000 = 1020$
- After 2 years: $P_2 = 1.02 \cdot 1020 = 1040.4$
- After 3 years: $P_3 = 1.02 \cdot 1040.4 \approx 1061.21$

This simple formula helps us predict the population year by year.

Another Example : Saving Money in a Bank

Let's say you open a bank account and deposit ₹1,000. Every year, the bank gives you 5% interest, and you also add ₹500 more each year. We can model this with:

$$S_{t+1} = 1.05 \cdot S_t + 500$$

Where:

- S_t is the amount in the account this year,
- 1.05 represents 5% interest,
- 500 is the amount you add every year.

If $S_0 = 1000$, then:

- After 1 year: $S_1 = 1.05 \cdot 1000 + 500 = 1550$
- After 2 years: $S_2 = 1.05 \cdot 1550 + 500 = 2127.5$
- After 3 years: $S_3 = 1.05 \cdot 2127.5 + 500 \approx 2743.88$

You can see how your savings grow each year because of the interest and your extra deposits.

Real-Time Modeling in Economics

Economists often use difference equations to understand things like:

- How a country's income (GDP) grows year by year.
- How unemployment changes month to month.
- How inflation behaves when prices rise or fall.
- How stock prices or currency values behave over time.

In all of these, economists use real-time data and update their models regularly using difference equations. For example:

Modeling Demand for a Product

Let's say the demand for a product increases each month due to marketing. A simple model might be:

$$D_{t+1} = 1.1 \cdot D_t$$

This means that every month, demand grows by 10%. If the current demand is 500 units, then in a few months:

- $D_1 = 550$
- $D_2 = 605$
- $D_3 = 665.5$, and so on.

2. Modeling Price Adjustment in Markets

Suppose the price of a good is influenced by the gap between supply and demand. We can model the change as:

$$P_{t+1} = P_t + 0.1 \cdot (D_t - S_t)$$

Where:

- P_t is the price at time t ,
- D_t is demand, and S_t is supply,
- If demand is greater than supply, price increases; if not, it decreases.

This kind of model is useful in real-time pricing algorithms used by e-commerce and airlines.

Importance of Initial Conditions

In all these examples, the starting value (initial condition) is very important. If we change x_0 , the entire sequence changes. This is why accurate data collection at time $t=0$ is crucial when using real-time models.

What Happens in the Long Run?

Real-time modeling helps us ask important questions like:

- ☐ Will the population grow forever?
- ☐ Will prices reach a stable level?
- ☐ Will your savings grow faster than inflation?

Using difference equations, we can find steady states or equilibrium points. For instance, in the savings model:

$$S_{t+1} = 1.05 \cdot S_t + 500$$

We can find the long-run value S^* (where $S_{t+1} = S_t$):

$$S^* = 1.05 \cdot S^* + 500 \Rightarrow S^*(1 - 1.05) = 500 \Rightarrow -0.05 \cdot S^* = 500 \Rightarrow S^* = -10,000$$

This shows that there is no stable point unless we change the parameters. In other models (where $|a| < 1$), we do get a stable outcome.

Real-Time Applications in Business and Policy

- ☐ **Retailers** use real-time models to decide how much stock to order based on past sales.
- ☐ **Banks** model interest payments and customer behavior using difference equations.
- ☐ **Governments** use economic models to predict inflation, tax revenues, and employment levels.

- **Data scientists** use time-based models for forecasting customer behavior, social media trends, and website traffic.

In all of these, real-time data is collected and plugged into a model (often a difference equation), and the future is forecasted accordingly.

First-order difference equations are simple yet powerful tools that help us understand how things change over time in a step-by-step manner. They are especially useful in real-time data modeling, where we want to use current data to predict what will happen next. Whether we are studying population growth, savings accounts, product demand, or inflation, these equations allow us to build models that reflect real-world systems.

Using first-order difference equations, we can:

- Make predictions.
- Analyze trends.
- Understand the long-term behavior of a system.
- Support decision-making in economics, business, and policy.

These equations form a bridge between raw data and useful insights. For students, learning how to use them is a crucial skill in mathematical economics and beyond.

Check your Progress - II

Based on your understanding of the previous section, answer these questions in your own words:

1. What is meant by the stability of a system in first-order difference equations, and how can it be determined?

2. Explain the difference between stable and unstable systems in the context of first-order difference equations.

3. How does the value of the stability coefficient ($|a|$) affect the long-term behavior of a system?

4. Define the term ‘long-term behavior’ of a system modeled by a first-order difference equation. How is it related to the system’s stability?

5. What role do initial conditions play in the long-term behavior of a system in real-time data modeling?

6. Given the first-order difference equation $x_{t+1} = 0.8 * x_t$, calculate the values of x_1 , x_2 , and x_3 if the initial condition is $x_0 = 100$.

7. A population grows by 4% annually. If the initial population is 500, calculate the population at the end of the first 3 years.

8. Solve the following loan balance model iteratively: A loan balance decreases by 10% annually, with an annual repayment of \$1000. The initial loan balance is \$5000. Calculate the balance after the first 3 years.

18.7 KEYWORDS

1. **First-Order Difference Equation:** A recurrence relation where the value of a variable at the next time step is determined by its value at the previous time step. It is used to model systems that evolve over discrete intervals.

2. **Linear vs. Non-Linear Difference Equations:** Linear equations relate the next value of a variable to a linear function of the current value, while non-linear equations involve non-linear relationships, such as powers or products of variables.
3. **Homogeneous vs. Non-Homogeneous Equations:** Homogeneous equations have no external forcing term, meaning the system's evolution depends solely on its current state. Non-homogeneous equations include an external factor influencing the system.
4. **Stability in First-Order Difference Equations:** Stability refers to the system's ability to return to equilibrium after small disturbances. A system is stable if small changes in initial conditions don't lead to large changes in the long-term behavior.
5. **Fixed Point or Equilibrium:** The value at which the system stabilizes, where the variable remains constant over time. It's the point where the system's future value equals its current value.
6. **Long-Term Behavior:** Refers to the tendency of a system as time progresses. It can include convergence to a steady state, exponential growth or decay, or oscillation around a fixed point.
7. **Convergence:** When the values of a system approach a fixed point or equilibrium as time progresses, indicating stability in the system.
8. **Divergence:** When the system's values grow indefinitely, leading to an unstable system where initial conditions drastically affect future outcomes.
9. **Iterative Methods:** Methods used to solve difference equations by computing the values at each time step, starting from an initial condition. These methods are particularly useful when analytical solutions are difficult or unavailable.

18.8 LET'S SUM UP

1. **Understanding First-Order Difference Equations:** First-order difference equations model systems that evolve over discrete time intervals. These

equations relate the current value of a variable to its value in the previous time step, making them ideal for predicting future values based on known conditions.

2. **Types of Difference Equations:** First-order difference equations can be classified as linear or non-linear, depending on whether the relationship between variables is proportional or involves more complex terms. Additionally, they can be homogeneous (without external factors) or non-homogeneous (with external influences like investment or immigration).
3. **Solving Methods:** First-order difference equations can be solved using iterative methods, analytical techniques, or special approaches for more complex cases. Iterative methods use recurrence relations to generate successive values, while analytical methods provide exact solutions, and special techniques are used for non-linear equations.
4. **Stability and Long-Term Behavior:** The stability of a system depends on the magnitude of the stability coefficient in the equation. A stable system converges to an equilibrium point, while an unstable system experiences exponential growth or divergence. Understanding stability is crucial for predicting the long-term behavior of dynamic systems.
5. **Real-Time Data Modeling:** First-order difference equations are powerful tools for modeling real-time data in various fields, such as economics, biology, and business. They help in forecasting future outcomes based on historical data, making them useful for applications like population growth, economic modeling, and investment forecasting.
6. **Applications Across Disciplines:** These equations are widely applied in economics, biology, and other fields where processes evolve over discrete steps. They are particularly useful for analyzing dynamic systems that change in response to external factors and are often used for trend analysis, prediction, and decision-making in real-world scenarios.

18.9 EXAMINATION-ORIENTED QUESTIONS

1. Define first-order difference equations and explain their significance in modeling dynamic systems.

- A first-order difference equation expresses the relationship between the current value and the previous value of a variable.
- It is used to model systems evolving over discrete time steps (e.g., yearly, monthly).
- Provides a framework for forecasting and predicting future states based on historical data.
- Examples: Population growth, capital accumulation, investment growth.
- Importance: Helps to model real-world phenomena that change step by step over time, such as economic growth or population dynamics.
- Start by defining a first-order difference equation clearly.
- Follow up by explaining its applications in dynamic systems.
- Provide examples to illustrate its significance in modeling processes that evolve in discrete time steps.

2. What is the difference between first-order and second-order difference equations?

- First-order difference equation: Involves the current and previous values ($x_{t+1} = f(x_t)$).
- Second-order difference equation: Involves the current value and two previous values ($x_{t+2} = f(x_t, x_{t+1})$).
- First-order models simpler processes like population growth or investment, where the change is dependent on the immediate past value.
- Second-order models more complex dynamics such as capital accumulation or economic growth, where multiple time steps influence the current state.

- Define both types of difference equations.
- Describe their structural differences and explain how the inclusion of two previous values in second-order equations makes them suitable for modeling more complex systems.
- Illustrate with examples of real-world scenarios where each type is applied.

3. Explain the stability of a first-order linear difference equation.

- Stability: A system is stable if small changes in initial conditions lead to small changes in long-term behavior.
- For a first-order linear difference equation of the form $x_{t+1} = a * x_t + b$:
 - If $|a| < 1$, the system converges to a fixed point.
 - If $|a| > 1$, the system is unstable and grows without bound.
 - If $|a| = 1$, the system oscillates or maintains a neutral state.
- Stability Condition: The stability of the system depends on the value of “a”.
- Define stability in the context of difference equations.
- Explain the criteria for stability, based on the coefficient “a”.
- Discuss the implications of different values of “a” and how they affect long-term behavior.

4. How do you solve second-order difference equations, and what is the role of initial conditions?

- Second-order difference equations involve the current value and the previous two values (e.g., $x_{t+2} = a * x_{t+1} + b * x_t$).
- Steps to solve:
 1. Express the equation in standard form.
 2. Find the characteristic equation.
 3. Solve for the roots (r_1 and r_2).

4. Write the general solution using the roots.
5. Use initial conditions to find specific solutions (values for constants).
 - Initial conditions (e.g., x_0, x_1) are essential to determine the particular solution.
 - Start by explaining the general approach to solving second-order difference equations.
 - Break down the steps to solve the equation, emphasizing the importance of finding the roots and applying initial conditions.
 - Include a simple example to demonstrate the process.
5. **Discuss the economic applications of first-order difference equations with examples.**
 - Population Growth: A first-order difference equation models population growth where the next year's population depends on the current population and growth rate ($P_{t+1} = r * P_t$).
 - Investment Growth: An investment model where the future value depends on the current value and the return rate ($I_{t+1} = (1 + r) * I_t + b$).
 - Capital Accumulation: A model where capital stock depends on the current value and new investments ($K_{t+1} = s * K_t + b$).
 - These models are used in economics to predict future trends based on current conditions.
 - Introduce the importance of first-order difference equations in economics.
 - Explain how they are used to model dynamic processes like population growth, investment, and capital accumulation.
 - Provide clear examples of each application, detailing the recurrence relations involved.
6. **What is the significance of stability analysis in economic models using difference equations?**

- Stability Analysis: Helps determine whether an economic system will converge to a steady state or diverge into undesirable fluctuations.
- Predicting Long-term Behavior: Stability analysis informs policymakers about whether economic growth, population, or capital accumulation will stabilize or spiral out of control.
- Stable systems in economics (e.g., population, investment) suggest sustainable growth, while unstable systems may signal inflation, debt crises, or speculative bubbles.
- Economic Implications: Stability analysis is crucial for making long-term forecasts, such as in investment planning or predicting national income growth.
- Begin by explaining the concept of stability analysis in difference equations.
- Discuss its importance in predicting whether systems will stabilize or diverge.
- Provide real-world examples of stable and unstable economic systems, explaining the significance for policy and planning.

7. Explain the method of solving non-homogeneous first-order difference equations.

- Non-Homogeneous Difference Equation: A non-homogeneous equation includes an external forcing term (e.g., $x_{t+1} = a * x_t + b$).
- Method:
 1. Solve the homogeneous part ($a * x_t = 0$).
 2. Find a particular solution by guessing the form of the solution based on the non-homogeneous term.
 3. Combine the homogeneous and particular solutions to get the general solution.
 4. Apply initial conditions to determine the specific solution.

- Particular Solution: The form of the particular solution depends on the external term (e.g., constant, exponential, polynomial).
- Explain the concept of non-homogeneous difference equations and their importance.
- Outline the steps for solving them, focusing on solving the homogeneous part first and finding the particular solution.
- Provide an example to illustrate the solution process for a non-homogeneous first-order difference equation.

18.10 HINTS TO CHECK YOUR PROGRESS (CYP)

CYP-I:

1. First-Order Difference Equation:

To solve a first-order difference equation, use the recurrence relation iteratively or apply the general solution formula.

Refer to Section 18.3.1 for the definition and general form of first-order difference equations.

2. Stability Analysis:

Stability in first-order difference equations is determined by the coefficient “a”. If $|a| < 1$, the system will converge to a fixed point. If $|a| > 1$, the system is unstable.

Refer to Section 18.5.1 for stability conditions and analysis.

3. Population Growth Modeling:

For population growth, use the formula $P_{t+1} = r * P_t$, where r is the growth rate. Apply this recurrence relation starting from the initial population to predict future values.

Refer to Section 18.6 for real-time data modeling examples like population growth.

4. Investment Growth:

In investment growth, the equation $I(t+1) = (1 + r) * I(t) + b$ models growth with return rate “r” and an external input “b”. Apply the equation iteratively to find future values.

Refer to Section 18.6 for real-time data modeling examples like investment growth.

5. Second-Order Difference Equation:

To solve a second-order difference equation, first express it in standard form, find the characteristic equation, solve for the roots, and use initial conditions to determine the specific solution.

Refer to Section 18.4.2 for methods on solving second-order difference equations.

6. General Solution of Non-Homogeneous Equations:

For non-homogeneous equations, apply the method of undetermined coefficients to find a particular solution. Combine the general solution with initial conditions for the full solution.

Refer to Section 18.4.3 for special techniques in solving non-homogeneous equations.

CYP-II:

1. Homogeneous vs Non-Homogeneous Equations:

A homogeneous equation has no external forcing term, while a non-homogeneous equation includes such a term. The solution methods differ.

Refer to Section 18.3.2 for the difference between homogeneous and non-homogeneous equations.

2. Solving for Equilibrium Points:

For equilibrium solutions, set $x_{t+1} = x_t$ and solve for x_t to find the steady state.

Refer to Section 18.5.2 for long-term behavior and equilibrium solutions.

3. Analyzing Stability with Roots:

The stability of second-order equations is determined by the roots of the characteristic equation. If $|r| < 1$, the system is stable, and if $|r| > 1$, the system is unstable.

Refer to Section 18.5.2 for stability analysis in second-order difference equations.

4. Economic Applications:

Use difference equations to model economic systems such as population growth, capital accumulation, and investment. Apply the appropriate equation and solve for future outcomes.

Refer to Section 18.6 for real-time applications like population growth and investment modeling.

18.11 SUGGESTED READINGS

- Mathematics for Economists by Hoy, M., et al.
- Introduction to Mathematical Economics by Samuelson, P. A., & Nordhaus, W. D.
- Mathematical Methods for Economics by Lovell, M. C.
- Fundamentals of Mathematical Economics by Brown, R. G.
- Dynamic Modeling in Economics by Pindyck, R. S., & Rubinfeld, D. L.
- Applied Mathematical Economics by Pao, Y. H.

M.A. Economics		Lesson No. 19
C.No. : Eco-103	Semester - 1st	Unit III

SOLVING SECOND ORDER DIFFERENCE EQUATIONS

STRUCTURE

- 19.0 Objectives
- 19.1 Learning Outcomes
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- 19.7 Let's Sum Up

19.8 Examination-Oriented Questions

19.9 Hints to Check Your Progress (CYP)

19.10 Suggested Readings

19.0 OBJECTIVES

The main objectives of this lesson are to enable you to:

- Understand the concept and structure of second-order difference equations and how they extend the modeling capabilities of first-order equations by incorporating an additional time lag.
- Identify the key components and types of second-order difference equations, including homogeneous and non-homogeneous forms, and recognize their relevance in modeling economic systems.
- Learn and apply systematic methods for solving second-order difference equations, including techniques for handling constant and variable coefficients.
- Analyze the role of initial conditions in determining the specific behavior of a solution and understand how these conditions influence the path of a dynamic system over time.
- Apply second-order difference equations to real-world scenarios such as modeling economic cycles, capital accumulation with adjustment costs, and delayed consumer or firm responses.
- Develop the ability to interpret the mathematical results of second-order difference equations in economic terms, enhancing both theoretical understanding and practical insight.

19.1 LEARNING OUTCOMES

By the end of this lesson, you should be able to:

- Define second-order difference equations and explain how they differ from first-order equations in terms of structure and memory of past values.
- Classify second-order difference equations into homogeneous and non-

homogeneous types, and distinguish between constant and variable coefficient forms.

- Solve second-order homogeneous difference equations using characteristic equations, and determine the general form of solutions based on the nature of the roots.
- Solve non-homogeneous second-order equations by identifying particular solutions and combining them with the general solution of the associated homogeneous equation.
- Apply initial conditions to determine specific solutions from the general solution of second-order difference equations.
- Interpret the dynamic behavior of solutions—including convergence, divergence, and oscillation—based on the values of the parameters and initial conditions.
- Analyze the stability of second-order systems and understand the implications for economic forecasting, investment planning, and policy analysis.
- Apply second-order difference equations to practical economic contexts such as modeling capital dynamics, national income fluctuations, and investment decisions with time lags.

19.2 INTRODUCTION

Dear learner, second-order difference equations are a key mathematical tool for understanding systems that evolve in steps over time, where the present outcome depends not only on the most recent past but also on conditions from an earlier period. These equations go beyond the scope of first-order difference equations by incorporating an additional layer of historical dependency. This added complexity allows for the modeling of more realistic and richer dynamic behaviors that are often seen in economic, social, and natural systems. In particular, second-order equations can capture dynamics like delayed reactions, cyclical adjustments, and patterns of overshooting and correction—features commonly observed in economic indicators such as income, output, or investment.

In real-world economic contexts, decisions and outcomes rarely depend on a single previous time period. Instead, they often reflect the influence of both recent and older information. For instance, consumer spending today may be influenced not just by last month's income but also by expectations or experiences from earlier periods. Similarly, a firm's production level might adjust based on both current market demand and past inventory levels. Such scenarios cannot be adequately modeled using only first-order relationships. Second-order difference equations provide the necessary structure to represent these intertemporal dependencies, making them invaluable in dynamic economic modeling.

- These equations are especially useful in understanding systems that exhibit oscillatory or cyclical behavior. In business cycle theory, for example, economists seek to explain why output, employment, or investment may fluctuate over time rather than move smoothly. Second-order models can account for such fluctuations, capturing both the momentum of past trends and the corrective forces that drive the system back toward equilibrium. This balance between inertia and adjustment often leads to behavior such as damped oscillations, recurring cycles, or in some cases, explosive divergence—depending on the specific parameters and initial conditions of the system.
- The study of second-order difference equations also allows economists to evaluate the stability and long-term behavior of a system more rigorously. Through these models, we can ask vital questions: Will the economy settle into a steady state? Will fluctuations dampen over time or persist indefinitely? Could a small shock lead to uncontrolled growth or collapse? Understanding the answers to these questions helps inform decisions about fiscal policy, investment strategies, and regulatory interventions.
- In this lesson, we will delve into the fundamental structure and solution methods of second-order difference equations. We will examine the different types of such equations, understand how to classify and approach them, and explore techniques for finding both general and particular solutions. Attention will also be given to interpreting the implications of these solutions

in economic terms—particularly the role of initial conditions and the patterns of convergence or divergence they may produce. By mastering the concepts and methods presented in this lesson, you will gain the analytical skills needed to apply second-order difference equations to real-world problems, making this an essential step in your study of mathematical economics.

19.3 CORE CONCEPTS

Second-order difference equations are fundamental tools in dynamic analysis, particularly when modeling systems where outcomes are influenced not only by the immediate past but also by earlier time periods. These equations extend the logic of first-order models by incorporating a two-period memory, capturing more complex behavior such as momentum, delay, and oscillation. Understanding the core structure and characteristics of these equations is essential for effectively applying them in both theoretical and applied economic contexts.

At their most basic, second-order difference equations describe the evolution of a variable over time by relating its current value to its values in the previous two time periods. This recursive nature allows for more accurate modeling of real-world systems where current decisions or states depend on both short-term and medium-term historical data. Unlike first-order models, which are often limited to smooth convergence or divergence, second-order models can produce rich dynamics including cycles, alternating patterns, and complex growth trajectories.

These equations are particularly valuable in economics, where behaviors such as consumption smoothing, investment adjustment, and market expectations frequently depend on more than just the most recent period. For instance, firms may base their current production levels not only on last quarter's sales, but also on sales from the previous quarter to detect trends or correct for volatility. Similarly, policymakers analyzing the impact of fiscal changes might need to consider both current and lagged economic indicators.

Second-order difference equations are generally classified into two main categories: homogeneous and non-homogeneous. A homogeneous equation does not include any external input or forcing term—it depends purely on the internal dynamics

of the system. In contrast, a non-homogeneous equation includes external influences, such as policy interventions, investments, or shocks, that drive the system in addition to its internal structure.

The behavior of solutions to second-order equations is determined largely by the values of the equation's parameters and the nature of its characteristic roots. These roots reveal whether the system will converge to a steady state, oscillate around an equilibrium, or diverge over time. Special attention must also be given to initial conditions, as they determine the specific path that the system will follow, even if the general behavior is known.

In the sections that follow, we will explore the formal structure of second-order difference equations, the types of solutions they produce, and the methods for solving them. We will also consider how to interpret these solutions in an economic context and examine examples that highlight their practical relevance.

19.3.1 Definition and Structure of Second-Order Difference Equations

A second-order difference equation is a discrete-time mathematical expression that relates the value of a variable at time $t+2$ to its values at time $t+1$ and time t . This type of equation incorporates two lags, thereby capturing deeper historical dependence than first-order equations, which rely on a single time lag. Second-order equations are widely used in economics, physics, and engineering to model systems where past behavior significantly affects current and future outcomes.

The general form of a linear second-order difference equation with constant coefficients is:

$$x_{t+2} + a_1x_{t+1} + a_2x_t = f(t)$$

Where:

- x_t is the value of the variable at time t ,
- a_1 and a_2 are constant coefficients,
- $f(t)$ is a known function of time t , which may represent an external influence or forcing term.

This equation states that the value of the variable two time steps ahead depends linearly on its values at the previous two time steps, possibly with the addition of an external input.

If $f(t)=0$, the equation is called a homogeneous second-order difference equation:

$$x_{t+2} + a_1x_{t+1} + a_2x_t = 0$$

If $f(t) \neq 0$, the equation is non-homogeneous, and the external input plays a critical role in shaping the solution.

Second-order difference equations can also be written in alternative forms, depending on the application. For example:

$$x_{t+2} = -a_1x_{t+1} - a_2x_t + f(t)$$

or

$$x_{t+2} = b_1x_{t+1} + b_2x_t + g(t)$$

Here, the structure is rearranged to isolate the future value on one side of the equation, which is particularly useful when computing values iteratively or interpreting dynamic behavior.

In economic models, the constants a_1 and a_2 often represent behavioral parameters such as adjustment speeds or reaction intensities. The function $f(t)$ might capture fixed shocks, cyclical inputs, or policy changes. For example, a government spending program that injects a constant amount into the economy every period would be represented by a constant $f(t) = k$. If $f(t)$ varies over time—say, due to seasonal factors or alternating policy—it may be written as a function of t , such as $f(t) = A \sin(t)$ or $f(t) = Bt$.

The initial conditions—the values of x_0 and x_1 —are essential for determining the particular solution of the equation. Since the equation depends on two previous values, we must specify two starting points to trace the system's path forward through time.

In summary, the structure of a second-order difference equation provides a framework for modeling complex, time-dependent systems. Its recursive nature, influenced

by both current and earlier values, allows for analysis of trends, cycles, and stability in a wide variety of real-world contexts.

Example : Economic Model of Investment Adjustment

Let's consider a second-order difference equation modeling the investment level in an economy. Suppose the level of investment at time $t+2$ depends on the levels of investment at time $t+1$ and time t , with some external input (e.g., a government subsidy or market shock).

The equation is:

$$I_{t+2} = 0.8I_{t+1} + 0.5I_t + 100$$

Where:

- It is the investment level at time t ,
- 0.8 is the coefficient for the previous period's investment I_{t+1} ,
- 0.5 is the coefficient for the investment two periods ago I_t ,
- 100 is a constant external input to the system at each time period (representing a subsidy or policy intervention).

Initial Conditions:

- $I_0=200$ (Initial investment at time 0),
- $I_1=250$ (Investment at time 1).

We want to calculate the investment levels for the next few periods, I_2 , and I_4 , using the difference equation.

Step-by-Step Calculation:

1. For $t=2$ (investment at time 2):

$$I_2 = 0.8I_1 + 0.5I_0 + 100$$

Substituting the values:

$$I_2 = 0.8 \times 250 + 0.5 \times 200 + 100 = 200 + 100 + 100 = 400$$

2. For $t=3$ (investment at time 3):

$$I_3 = 0.8I_2 + 0.5I_1 + 100$$

Substituting the values:

$$I_3 = 0.8 \times 400 + 0.5 \times 250 + 100 = 320 + 125 + 100 = 545$$

3. For $t=4$ (investment at time 4):

$$I_4 = 0.8I_3 + 0.5I_2 + 100$$

Substituting the values:

$$I_4 = 0.8 \times 545 + 0.5 \times 400 + 100 = 436 + 200 + 100 = 736$$

Results:

- $I_0 = 200$,
- $I_1 = 250$,
- $I_2 = 400$,
- $I_3 = 545$,
- $I_4 = 736$.

Thus, the investment level grows over time, reflecting both the system's internal dynamics (the influence of past investment values) and the external shock or subsidy of 100 at each period.

In this example, the investment level is influenced by both the previous period's investment and the one before it. The positive external input 100 consistently boosts the investment in each period, illustrating how a second-order difference equation can capture delayed responses to policy changes or economic shifts. This type of model is valuable for understanding how changes in investment or other economic indicators evolve over time and react to previous decisions and external factors.

19.3.2 Types of Second-Order Difference Equations

Second-order difference equations are widely used in the analysis of dynamic systems, particularly in economics, engineering, and the natural sciences, where future

states of a system depend on both the immediate past and the second most recent states. These equations are critical for understanding processes that exhibit delayed responses or inertia over time, such as economic cycles, population dynamics, or capital accumulation.

Second-order difference equations are classified into two primary categories: linear versus non-linear equations, and homogeneous versus non-homogeneous equations. The distinctions between these types are important because they determine the complexity of the equation and the solution methods required. Let's explore each of these classifications in greater detail, including numerous examples to illustrate the different behaviors these types of equations model.

1. Linear vs. Non-Linear Second-Order Difference Equations

Linear Second-Order Difference Equations

A second-order difference equation is linear if the dependent variable (say x_t) appears only in the first degree (i.e., x_t^1) and is not involved in any non-linear operations such as multiplication by other variables, raising to a power, or other complex functions. In linear second-order equations, all terms involving the dependent variable x_t are either constants or proportional to x_t , x_{t+1} , and x_{t+2} .

The general form of a linear second-order difference equation is:

$$x_{t+2} + a_1 x_{t+1} + a_2 x_t = f(t)$$

Where:

- x_{t+2} is the value of the variable at time $t+2$,
- x_{t+1} is the value of the variable at time $t+1$,
- x_t is the value of the variable at time t ,
- a_1 and a_2 are constants or time-varying coefficients that influence the system's dynamics,
- $f(t)$ is an external forcing function or input that may depend on time.

Linear second-order difference equations are easier to solve than their non-linear counterparts, and their solutions are often predictable, involving exponential growth, decay, or oscillation, depending on the values of the coefficients.

Example 1 : Linear Homogeneous Second-Order Difference Equation

Consider the following second-order difference equation modeling capital adjustment in an economy:

$$K_{t+2} = 0.8K_{t+1} + 0.6K_t$$

Where:

- K_t represents the capital level at time t ,
- The equation relates future capital to the previous two periods' capital levels with coefficients 0.8 and 0.6.

This is a linear homogeneous second-order difference equation because it does not include any external forcing term (i.e., $f(t)=0$) and the variable K_t is linear.

Solution Behavior: To solve this, we would typically first find the characteristic equation associated with the homogeneous equation, which involves finding the roots of the associated characteristic equation to determine the system's behavior. The solution to the equation might involve exponential growth, decay, or oscillations, depending on the nature of the roots.

Example 2 : Linear Non-Homogeneous Second-Order Difference Equation

Consider a modified model where an external government subsidy 100 is added to the capital at each time period:

$$K_{t+2} = 0.7K_{t+1} + 0.5K_t + 100$$

In this case:

- 100 represents an external subsidy or input that affects the capital level each period.
- The equation is linear because K_t and K_{t+1} appear only to the first power, and the coefficients are constants.
- It is non-homogeneous because there is an external input $f(t) = 100$.

The solution to this equation consists of two parts: the homogeneous solution (which depends on the internal dynamics of the system) and a particular solution (which

accounts for the constant external input). The particular solution might be a constant or a function of time, depending on the nature of the forcing term 100.

Non-Linear Second-Order Difference Equations

A second-order difference equation is **non-linear** if the variable x_t appears in non-linear forms, such as squared terms, products, or other non-linear functions. Non-linear equations can exhibit much more complex behavior, such as bifurcations, chaotic dynamics, or irregular oscillations. The key characteristic of a non-linear equation is that the variable or its lagged values appear raised to a power greater than 1 or are involved in multiplicative terms.

The general form of a non-linear second-order difference equation might be:

$$x_{t+2} = a_1 x_{t+1} + a_2 x_t + b x_{t+1}^2$$

Where x_{t+2} introduces a non-linearity in the equation.

Example 1 : Non-Linear Economic Model

Consider an economic model where the future level of output depends on both the current and previous levels of output, but the relationship is non-linear:

$$Y_{t+2} = 0.7Y_{t+1} + 0.5Y_t + 0.2Y_{t+1}^2$$

Here:

- The term $0.2Y_{t+1}^2$ introduces non-linearity because Y_{t+1} is squared.
- This equation models a situation where the relationship between output in the current and future periods is not strictly linear, but instead includes feedback that grows quadratically with the level of output in the previous period.

The solutions to non-linear equations are harder to predict, and they may lead to unexpected behaviors like chaotic oscillations or sudden shifts in system behavior.

Example 2 : Non-Linear Population Growth Model

Consider a logistic growth model for population:

$$P_{t+2} = rP_{t+1} \left(1 - \frac{P_t}{K}\right)$$

Where:

- P_t is the population at time t ,
- r is the growth rate,
- K is the carrying capacity of the environment.

This equation is non-linear because the term $1 - \frac{P_t}{K}$ introduces a non-linear relationship between the population and its previous values. As the population approaches the carrying capacity K , the growth rate slows down, demonstrating how non-linear equations can model real-world systems that have natural limits or diminishing returns.

2. Homogeneous vs. Non-Homogeneous Second-Order Difference Equations

Second-order difference equations are also classified as homogeneous or non-homogeneous, depending on whether the equation includes an external forcing term.

Homogeneous Second-Order Difference Equations

A homogeneous second-order difference equation is one that has no external forcing term (i.e., $f(t) = 0$). This means that the future value of the variable depends only on the internal dynamics of the system, typically involving lagged values of the variable itself.

The general form of a homogeneous second-order difference equation is:

$$x_{t+2} + a_1 x_{t+1} + a_2 x_t = 0$$

Example 1 : Homogeneous Model for Capital Adjustment

Consider a simple model where the future capital level depends on the past two capital levels, with no external shocks or inputs:

$$K_{t+2} = 0.8K_{t+1} + 0.5K_t$$

This is a homogeneous equation because the future capital level is determined purely by the past capital levels. There is no external intervention (i.e., no forcing term).

The solution to this equation involves finding the characteristic equation for the homogeneous system, which determines the system's natural behavior. The solution may

involve exponential growth, decay, or oscillation, depending on the nature of the roots of the characteristic equation.

Non-Homogeneous Second-Order Difference Equations

A non-homogeneous second-order difference equation includes an external forcing term $f(t)$ on the right-hand side. This term represents some external influence that drives the system in addition to its internal dynamics.

The general form of a non-homogeneous second-order difference equation is:

$$x_{t+2} + a_1x_{t+1} + a_2x_t = f(t)$$

Where $f(t)$ is an external input, such as government policy, shocks to the system, or external investments.

Example 1 : Non-Homogeneous Economic Model

Consider a model where the future level of capital is influenced by both the previous capital levels and an external government subsidy:

$$K_{t+2} = 0.6K_{t+1} + 0.4K_t + 100$$

Here, 100 represents an external subsidy or investment that affects the capital level each period. This is a non-homogeneous equation because the future capital level is influenced by the external subsidy.

The solution to this equation consists of the homogeneous solution (which depends on the natural dynamics of the system) and a particular solution (which accounts for the constant external subsidy). The particular solution might be a constant value, reflecting the long-term effect of the external input.

Example 2 : Non-Homogeneous Population Model with Immigration

Consider a model for population growth that includes an external immigration factor:

$$P_{t+2} = 0.9P_{t+1} + 0.7P_t + 50$$

In this case, 50 represents a constant immigration flow that adds to the population each period. This equation is non-homogeneous because of the external input 50.

The solution to this equation will include both the natural growth of the population (as determined by the internal dynamics) and the impact of the immigration flow.

Second-order difference equations can be classified in several ways, each influencing how we solve and interpret them. Here's a summary of the classifications:

1. Linear vs. Non-Linear:

- **Linear** equations are simpler and more predictable, often leading to exponential or oscillatory behavior.
- **Non-linear** equations are more complex and can exhibit chaotic, irregular, or bifurcating dynamics.

2. Homogeneous vs. Non-Homogeneous:

- **Homogeneous** equations depend only on the past values of the variable and have no external forcing terms.
- **Non-homogeneous** equations include an external input or shock that influences the system's behavior.

Each type of second-order difference equation models different dynamic behaviors, and understanding the classification of the equation helps us choose the appropriate solution method. Linear equations are generally easier to solve, while non-linear and non-homogeneous equations require more advanced techniques and numerical methods for analysis.

19.4 METHODS FOR SOLVING SECOND-ORDER DIFFERENCE EQUATIONS

Solving second-order difference equations is crucial for understanding dynamic systems, whether in economics, engineering, or other applied fields. The solution methods for second-order difference equations depend on whether the equation is homogeneous or non-homogeneous and on the specific characteristics of the system being modeled.

In this section, we will explore the methods for solving second-order difference equations in detail, dividing them into the following categories:

- **19.4.1 Homogeneous Equations:** Methods for solving second-order difference equations that do not include an external forcing term.
- **19.4.2 Non-Homogeneous Equations:** Techniques for handling equations where an external input or forcing term influences the system.
- **19.4.3 Iterative Solutions:** Approaches for solving difference equations numerically when analytical solutions are difficult or impossible.

Let's begin by looking at the methods for solving homogeneous second-order difference equations.

19.4.1 Homogeneous Equations

A homogeneous second-order difference equation is one in which the dependent variable depends solely on its own past values, and there is no external forcing term (i.e., $f(t) = 0$).

The general form of a homogeneous second-order difference equation is:

$$x_{t+2} + a_1x_{t+1} + a_2x_t = 0$$

Where:

- x_t is the variable at time t ,
- a_1 and a_2 are constants that represent the dynamics of the system.

Solution Method for Homogeneous Equations

The general solution approach for homogeneous second-order difference equations involves the following steps:

1. Write the equation in standard form:

$$x_{t+2} = -a_1x_{t+1} - a_2x_t$$

This format is essential for analyzing the characteristic equation.

2. Form the characteristic equation:

For a linear homogeneous equation, we assume the solution takes the form $x_t = r^t$, where r is a constant to be determined. Substituting $x_t = r^t$ into the

equation results in the characteristic equation:

$$r^2 + a_1 r + a_2 = 0$$

This is a quadratic equation in r , which we solve to find the roots.

3. **Solve the characteristic equation:**

The solutions to the quadratic equation will determine the form of the general solution. There are three possible cases for the roots:

o **Case 1: Two distinct real roots:**

If the discriminant of the quadratic equation $\Delta = a_1^2 - 4a_2$ is positive, the characteristic equation has two distinct real roots, r_1 and r_2 . The general solution is:

$$x_t = C_1 r_1^t + C_2 r_2^t$$

Where C_1 and C_2 are constants determined by the initial conditions.

o **Case 2 : A repeated real root:**

If the discriminant $\Delta=0$ there is one repeated real root r . The general solution in this case is:

$$x_t = (C_1 + C_2 t) r^t$$

This form accounts for the fact that the root appears twice.

o **Case 3: Complex conjugate roots:**

If the discriminant $\Delta < 0$, the roots are complex conjugates. Let the roots be $r_1 = a + \beta i$ and $r_2 = a - \beta i$. In this case, the general solution involves sinusoidal terms:

$$x_t = C_1 \alpha^t \cos(\beta t) + C_2 \alpha^t \sin(\beta t)$$

Where α and β are the real and imaginary parts of the roots, and C_1 and C_2 are constants determined by initial conditions.

Numerical Example of Solving a Homogeneous Equation

Let's solve an example step by step.

Example 1: Solving a Homogeneous Equation

Consider the following second-order homogeneous difference equation:

$$x_{t+2} = 1.5x_{t+1} + 0.5x_t$$

1. **Write the equation in standard form:**

$$x_{t+2} - 1.5x_{t+1} - 0.5x_t = 0$$

2. **Form the characteristic equation:**

Assume the solution is of the form $x_t = r^t$. Substituting this into the equation gives:

$$r^2 - 1.5r - 0.5 = 0$$

3. **Solve the characteristic equation:**

The quadratic equation is:

$$r^2 - 1.5r - 0.5 = 0$$

We solve for r using the quadratic formula:

$$r = \frac{-(-1.5) \pm \sqrt{(-1.5)^2 - 4(1)(-0.5)}}{2(1)}$$

$$r = \frac{1.5 \pm \sqrt{2.25 + 2}}{2}$$

$$r = \frac{1.5 \pm \sqrt{4.25}}{2}$$

$$r = \frac{1.5 \pm 2.061}{2}$$

Thus, the two roots are:

$$r_1 = \frac{1.5 + 2.061}{2} = 1.7805, \quad r_2 = \frac{1.5 - 2.061}{2} = -0.2805$$

4. General solution:

Since we have two distinct real roots $r_1 = 1.7805$, $r_2 = -0.2805$, the general solution is:

$$x_t = C_1(1.7805)^t + C_2(-0.2805)^t$$

5. Determine the constants:

To determine the constants C_1 and C_2 , we use the initial conditions. Suppose:

- $x_0=1$,
- $x_1=2$.

Substituting these into the general solution at $t=0$ and $t=1$ gives two equations that we can solve for C_1 and C_2 .

19.4.2 Non-Homogeneous Equations

A non-homogeneous second-order difference equation is one where the equation includes an external forcing term $f(t)$ on the right-hand side. This term represents an external influence or input that affects the behavior of the system. The general form of a non-homogeneous second-order difference equation is:

$$x_{t+2} + a_1x_{t+1} + a_2x_t = f(t)$$

Where $f(t)$ is the forcing term, and the rest of the equation is similar to a homogeneous second-order difference equation.

Solution Method for Non-Homogeneous Equations

1. Solve the homogeneous equation:

First, we solve the corresponding homogeneous equation:

$$x_{t+2} + a_1x_{t+1} + a_2x_t = 0$$

This part of the solution is the **complementary solution**, denoted $x_c(t)$.

2. Find a particular solution:

The next step is to find a particular solution $x_p(t)$ that satisfies the non-homogeneous equation. The form of $x_p(t)$ depends on the form of $f(t)$. Common techniques for finding a particular solution include:

- **Method of undetermined coefficients:** If $f(t)$ is a polynomial, exponential, sine, or cosine function, we guess a particular solution and solve for the undetermined coefficients.
- **Variation of parameters:** If the form of $f(t)$ is more complicated, we use variation of parameters to find the particular solution.

3. General solution:

The general solution to the non-homogeneous equation is the sum of the complementary solution and the particular solution:

$$x_t = x_c(t) + x_p(t)$$

Example 2 : Solving a Non-Homogeneous Equation

Consider the following second-order non-homogeneous difference equation:

$$x_{t+2} = 0.5x_{t+1} + 0.3x_t + 10$$

1. Solve the homogeneous equation:

First, solve the homogeneous equation:

$$x_{t+2} = 0.5x_{t+1} + 0.3x_t$$

The characteristic equation is:

$$r^2 - 0.5r - 0.3 = 0$$

Solving this using the quadratic formula gives:

$$r_1 = 1, \quad r_2 = -0.3$$

The complementary solution is:

$$x_c(t) = C_1(1)^t + C_2(-0.3)^t$$

2. Find a particular solution:

The forcing term is $f(t) = 10$, which is a constant. Thus, we guess a constant solution $x_p(t) = A$. Substituting $x_p(t) = A$ into the non-homogeneous equation:

$$A = 0.5A + 0.3A + 10$$

Solving for A , we get:

$$A = 25.$$

So, the particular solution is $x_p(t) = 25$.

3. General solution:

The general solution is the sum of the complementary and particular solutions:

$$x_t = C_1 + C_2(-0.3)^t + 25$$

Final Step : Determine Constants Using Initial Conditions

To determine the constants C_1 and C_2 , use the initial conditions. Suppose:

- $x_0 = 10$,
- $x_1 = 15$.

Substitute these values into the general solution to form a system of equations that can be solved for C_1 and C_2 .

19.4.3 Iterative Solutions

In many practical situations, second-order difference equations are difficult to solve analytically, particularly if they are non-linear or if the forcing term is complex. In such cases, we may resort to iterative methods to find numerical solutions.

Iterative Methods

Iterative methods involve solving the difference equation step by step, using an initial condition and applying the recurrence relation to compute the next values in the sequence. This approach is particularly useful for systems where an analytical solution is not feasible.

The basic steps in an **iterative solution** are:

1. Set the initial conditions:

Specify the initial values x_0 and x_1 (or other appropriate initial conditions depending on the equation).

2. Apply the recurrence relation:

Use the difference equation to calculate the next values, $x_2, x_3, \dots$, based on the recurrence relation.

3. Repeat until the desired number of steps is reached:

Continue applying the recurrence relation to find the values of x_t for as many periods as needed.

Example 3 : Iterative Solution

Consider the equation:

$$x_{t+2} = 0.5x_{t+1} + 0.3x_t + 10$$

With initial conditions:

- $x_0=5$,
- $x_1=8$.

We will use the iterative method to compute x_2 , x_3 , and x_4 .

1. Step 1: Calculate x_2 :

$$x_2 = 0.5x_1 + 0.3x_0 + 10 = 0.5 \times 8 + 0.3 \times 5 + 10 = 4 + 1.5 + 10 = 15.5$$

2. Step 2: Calculate x_3 :

$$x_3 = 0.5x_2 + 0.3x_1 + 10 = 0.5 \times 15.5 + 0.3 \times 8 + 10 = 7.75 + 2.4 + 10 = 20.15$$

3. Step 3: Calculate x_4 :

$$\begin{aligned} x_4 &= 0.5x_3 + 0.3x_2 + 10 = 0.5 \times 20.15 + 0.3 \times 15.5 + 10 \\ &= 10.075 + 4.65 + 10 = 24.725 \end{aligned}$$

Advantages of Iterative Solutions

Iterative solutions are particularly useful when:

- The equation is complex, and an analytical solution is difficult to obtain.
- The system involves non-linear dynamics, where exact solutions are not feasible.
- Numerical methods are required for large-scale simulations or forecasting in applied economics.

In this section, we explored three primary methods for solving second-order difference equations:

1. **Homogeneous Equations:** For equations with no external forcing term, we use characteristic equations to determine the solution, which could involve real, repeated, or complex roots.
2. **Non-Homogeneous Equations:** For equations with external forcing terms, we find a general solution by combining the complementary solution of the homogeneous equation with a particular solution to account for the external forces.
3. **Iterative Solutions:** When analytical methods are not feasible, iterative methods are used to numerically solve difference equations step by step, which is particularly useful in applied fields.

These methods provide the foundation for understanding and solving a wide range of dynamic systems in economics and beyond.

Check your Progress - I

Based on your understanding of the previous section, answer these questions in your own words:

1. What is a second-order difference equation, and how does it differ from a first-order difference equation?

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-
2. What are the two main types of second-order difference equations, and how are they classified?

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3. Explain the role of the characteristic equation in solving homogeneous second-order difference equations.

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4. What is the difference between a homogeneous and a non-homogeneous second-order difference equation?

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-
-
5. How does the discriminant of the characteristic equation determine the form of the solution in homogeneous second-order difference equations?

6. What are the common methods for solving non-homogeneous second-order difference equations, and when is the method of undetermined coefficients used?

7. In iterative solutions for second-order difference equations, why are initial conditions essential for solving the system?

8. Solve the following homogeneous second-order difference equation:

$$x_{t+2} = 2x_{t+1} + 3x_t$$

Given the initial conditions $x_0 = 1$ and $x_1 = 4$, find the general solution.

9. Solve the non-homogeneous second-order difference equation:

$$x_{t+2} = 0.4x_{t+1} + 0.6x_t + 5$$

Given the initial conditions $x_0 = 2$ and $x_1 = 3$, determine the first three values of x .

10. Using the iterative method, compute the first three terms of the sequence for the following equation:

$$x_{t+2} = 0.5x_{t+1} + 0.3x_t + 8$$

Given the initial conditions $x_0 = 3$ and $x_1 = 6$.

19.5 INTUITIVE INTERPRETATION AND SOLUTION BEHAVIOR

Second-order difference equations are mathematical tools that model systems where the future state depends on not just the current but also the past two states. These equations are widely used in economic modeling, engineering, physics, and population studies to capture the dynamic behavior of systems over discrete time intervals. In this section, we will explore the **intuitive interpretation** of second-order difference equations, specifically focusing on the role of initial conditions, real-world applications, and the solution behavior of these systems, including convergence, oscillation, and divergence.

Second-order difference equations can describe a variety of real-world phenomena. To understand them thoroughly, we need to explore how initial conditions influence the system's evolution, what real-world processes these equations model, and how the solutions behave over time. By doing so, we gain a clearer understanding of their significance and applicability to complex systems.

19.5.1 Understanding the Role of Initial Conditions

In second-order difference equations, the **initial conditions** are crucial because they help to determine the specific path or trajectory that the system will follow over time. The initial conditions provide the starting values for the sequence that the equation generates. These conditions typically include the value of the variable at two initial time points, say x_0 and x_1 , which represent the state of the system at the beginning of the observed time period.

Why Initial Conditions Matter

When solving a second-order difference equation, we derive a **general solution** that encompasses a wide range of possible behaviors for the system. The general solution involves unknown constants that need to be determined. These constants are what link the abstract mathematical solution to the specific system being modeled. This is where initial conditions come into play.

For example, in a homogeneous second-order difference equation like:

$$x_{t+2} = a_1 x_{t+1} + a_2 x_t$$

The general solution is of the form:

$$x_t = C_1 r_1^t + C_2 r_2^t$$

Where r_1 and r_2 are the roots of the characteristic equation and C_1 and C_2 are constants. These constants, C_1 and C_2 , depend entirely on the initial conditions, specifically the values of x_0 and x_1 . For instance, if you know that $x_0 = 1$ and $x_1 = 2$, you can use these values to determine C_1 and C_2 , which in turn gives you a unique solution for the system's evolution over time.

Effect of Different Initial Conditions

The specific values of x_0 and x_1 influence the trajectory of the solution in meaningful ways. For instance:

- **Stability:** Small differences in initial conditions can lead to vastly different outcomes, especially in non-linear systems. In linear systems, the influence of initial conditions is more predictable, and the system's behavior tends to be more stable and smooth.
- **Oscillation:** If the system exhibits oscillatory behavior (e.g., in cases with complex roots), different initial conditions can alter the frequency or amplitude of the oscillations.
- **Growth or Decay:** In systems that involve exponential growth or decay (e.g., due to positive or negative real roots), initial conditions will determine whether the system grows, decays, or stabilizes over time.

In non-homogeneous equations, the initial conditions still play a role, but the external forcing term $f(t)$ also affects the trajectory. The system's response to the external input, combined with the initial conditions, defines how the system evolves over time.

19.5.2 Real-World Intuition: What Do These Equations Model?

Second-order difference equations are widely used in various fields, including economics, engineering, biology, and physics. They are particularly valuable when modeling systems that have feedback loops or delayed responses, where the current state depends not only on the immediate past but also on earlier states.

Common Real-World Applications

1. Economics:

In economics, second-order difference equations are often used to model the evolution of variables such as investment, output, or capital over time. For example:

- **Capital Accumulation:** The level of capital in an economy at time $t+2$ might depend not only on the current level of capital at time $t+1$ but also

on the level of capital at time t . The relationship might include an external forcing term to represent policy interventions or technological advancements.

2. Population Dynamics:

Second-order difference equations are used to model populations with delays in growth, such as species with reproductive lags. A population at time $t+2$ might depend on both the population at time $t+1$ and at time t . These models often incorporate external factors like immigration or environmental changes.

3. Engineering and Physics:

In control systems and signal processing, second-order difference equations are used to model systems that exhibit damped oscillations or vibrations. For instance, in mechanical systems with springs and dampers, the displacement of the system at time $t+2$ might depend on its displacement at $t+1$ and t , with external forces acting on the system.

Example: Economic Investment Model

Consider a simple investment model where the future level of investment depends on both the current and past levels of investment, with the equation:

$$I_{t+2} = 0.7I_{t+1} + 0.3I_t + 50$$

Here:

- It represents the level of investment at time t ,
- The coefficients 0.7 and 0.3 represent how the investment from the past two periods influences the current investment,
- The term 50 represents an external subsidy or policy change that injects additional investment into the economy.

In this case, the second-order difference equation models how investment evolves over time, with the current level of investment affected by both past decisions and an external input. This equation captures the intertemporal dependencies that are common in economic systems, where present decisions are influenced not only by the immediate past but also by longer-term trends.

19.5.3 Solution Behavior: Convergence, Oscillation, or Divergence?

The behavior of a second-order difference equation solution depends on the **roots** of the characteristic equation. These roots determine whether the system converges, oscillates, or diverges over time. Understanding this behavior is crucial for predicting the long-term dynamics of the system.

Types of Solution Behavior

1. **Convergence:**

If the roots of the characteristic equation are both less than 1 in absolute value, the solution will converge to a fixed value (i.e., the system stabilizes). This means that the system eventually settles into a steady state.

2. **Oscillation:**

If the roots of the characteristic equation are complex conjugates (i.e., they have both real and imaginary parts), the solution will exhibit oscillatory behavior. The system will alternate between increasing and decreasing values over time, but the magnitude of the oscillations can either grow, decay, or remain constant depending on the real part of the roots.

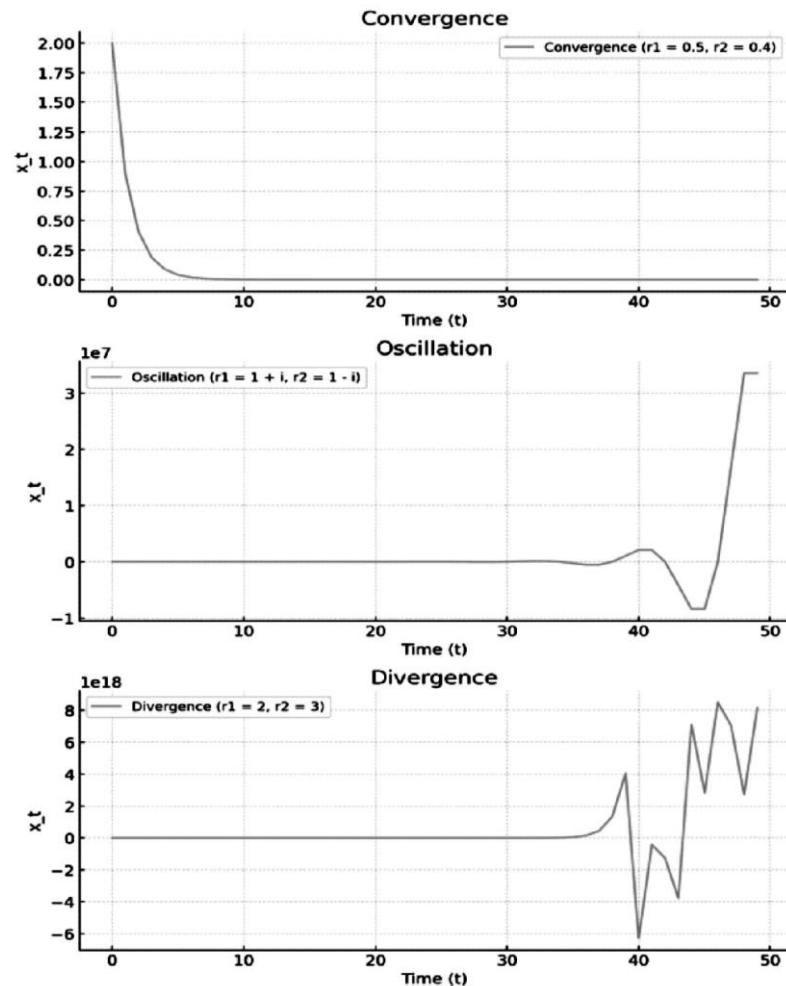
3. **Divergence:**

If the roots of the characteristic equation are greater than 1 in absolute value, the solution will diverge over time. This means that the system's values grow without bound.

Here are the plots representing the different behaviors of solutions to second-order difference equations:

1. **Convergence:** The solution gradually approaches zero over time ($r_1 = 0.5$, $r_2 = 0.4$). As the roots are less than 1 in absolute value, the system's values decay to zero.
2. **Oscillation:** The solution exhibits oscillatory behavior ($r_1 = 1 + i$, $r_2 = 1 - i$). With complex conjugate roots, the system oscillates between positive and negative values.

3. **Divergence:** The solution grows without bound ($r_1 = 2, r_2 = 3$). Since the roots are greater than 1, the system experiences exponential growth over time.



In this section, we explored the intuitive interpretation of second-order difference equations, focusing on the role of initial conditions, the real-world applications they model, and the solution behaviors such as convergence, oscillation, and divergence.

- Initial conditions play a key role in determining the trajectory of the system. They directly influence the constants in the general solution and, consequently, the behavior of the system over time.

- Real-world intuition reveals that these equations model systems with memory or delayed responses, such as economic investment cycles, population growth, and mechanical vibrations.
- The solution behavior of second-order difference equations can vary significantly depending on the nature of the roots of the characteristic equation, leading to stable equilibrium (convergence), periodic behavior (oscillation), or unbounded growth (divergence).

Understanding these concepts is crucial for analyzing dynamic systems and predicting their long-term behavior in real-world applications.

Check your Progress - II

Based on your understanding of the previous section, answer these questions in your own words:

1. What is the role of initial conditions in determining the solution of a second-order difference equation? How do different initial conditions affect the system's behavior over time?

2. In real-world applications, what types of systems are typically modeled using second-order difference equations? Provide two examples.

3. Explain the difference between convergence, oscillation, and divergence in the context of the solution behavior of second-order difference equations.

4. How does the nature of the roots of the characteristic equation influence the long-term behavior of the system in a homogeneous second-order difference equation?

5. What does it mean for a solution to exhibit oscillatory behavior? How can complex conjugate roots of the characteristic equation lead to oscillations in the solution?

6. Given the second-order difference equation:

$$x_{t+2} = 1.2x_{t+1} - 0.6x_t$$

Solve for the general solution and determine the solution's behavior if $x_0 = 2$ and $x_1 = 4$.

19.6 KEYWORDS

1. **Initial Conditions:** The starting values of a system in a difference equation, typically given as x_0 and x_1 for a second-order difference equation. Initial conditions help determine the specific solution for the system's evolution over time. The trajectory of the system over time depends on these starting values.
2. **Convergence:** The behavior of a system where the values of the variable approach a stable equilibrium point as time progresses. In second-order difference equations, convergence occurs when the absolute values of the roots of the characteristic equation are less than 1, leading the system to stabilize at a fixed point.
3. **Oscillation:** A cyclical or repetitive pattern in a system where the values of the variable fluctuate between positive and negative values or within a certain range over time. Oscillatory behavior in difference equations often arises when the system has complex conjugate roots as solutions to the characteristic equation.
4. **Divergence:** A behavior in which the values of the system grow without bound, often due to large roots (greater than 1 in absolute value) in the characteristic equation. Divergence can lead to instability in the system, where even small deviations from initial conditions are amplified over time.
5. **Homogeneous Equation:** A type of second-order difference equation where the right-hand side is equal to zero. In a homogeneous equation, the system's evolution is determined solely by its previous states without any external influences or forcing terms.
6. **Non-Homogeneous Equation:** A type of second-order difference equation that includes an external forcing term. The forcing term represents an external influence, such as a policy intervention or environmental shock, that affects the system's behavior. Non-homogeneous equations are typically solved by combining the homogeneous solution with a particular solution for the external input.

7. **Real-World Systems:** Systems or processes modeled by difference equations that evolve over discrete time steps. Common examples include population growth, capital accumulation in economics, and mechanical systems like damped oscillators. Second-order difference equations are particularly useful when modeling systems that involve both short-term and long-term dependencies.
8. **Solution Behavior:** The long-term trend or pattern followed by the solution to a difference equation. Depending on the nature of the roots of the characteristic equation, solution behavior can involve convergence to a steady state, oscillations around an equilibrium point, or divergence leading to unbounded growth or decay.
9. **Stability:** The tendency of a system to return to equilibrium after small perturbations or deviations from its initial state. A stable system will return to a fixed point after disturbances, while an unstable system may grow without bound or oscillate indefinitely.
10. **Exponential Growth:** A type of behavior where the value of a system increases exponentially over time. In difference equations, this occurs when the roots of the characteristic equation are greater than 1, leading to unbounded growth.
11. **Neutral Stability:** A scenario where the system neither grows nor decays. For example, in first-order difference equations, if the coefficient $a=1$, the system exhibits neutral stability, meaning deviations from equilibrium persist over time without growing or shrinking.

19.7 LET'S SUM UP

1. **Second-Order Difference Equations:** These equations model systems where the future state depends on both the current and previous states. They are particularly useful in representing dynamic systems with delayed responses or interdependencies.
2. **Homogeneous vs Non-Homogeneous:** Homogeneous equations have no external forcing term (i.e., $f(t) = 0$), while non-homogeneous equations include an external input that affects the system.

3. **Solution Methods:** Solutions to second-order difference equations involve finding the roots of the characteristic equation. These roots determine the system's behavior (convergence, oscillation, or divergence) over time.
4. **Initial Conditions:** The initial conditions (x_0 and x_1) are crucial for determining the exact solution of a second-order difference equation. They influence the system's trajectory and can change the long-term behavior.
5. **Real-World Applications:** Second-order difference equations are used to model various real-world processes, including economic growth, capital accumulation, population dynamics, and mechanical systems with feedback.
6. **Convergence, Oscillation, and Divergence:** The system's long-term behavior depends on the roots of the characteristic equation. Convergence leads to stabilization, oscillation to cyclical patterns, and divergence to unbounded growth or decay.
7. **Iterative Solutions:** In cases where analytical solutions are difficult or impossible, iterative methods provide a numerical approach to solving second-order difference equations by computing values step by step.
8. **Stability and Sensitivity:** The stability of a system is influenced by its initial conditions. In some systems, small changes in initial conditions can lead to vastly different outcomes, making sensitivity analysis important in forecasting and decision-making.

19.8 EXAMINATION-ORIENTED QUESTIONS

1. **Define second-order difference equations. How do they differ from first-order difference equations?**
 - **Explanation:**
 - First, briefly explain what a difference equation is.
 - Then, explain that a **second-order difference equation** involves a variable at time $t+2$, which depends on both the values at time $t+1$ and time t . Contrast it with a **first-order equation**, where only one previous value is involved.

- Key points to include:
- Definition of difference equations
- Differences in time dependencies between first and second-order equations

2. Explain the role of initial conditions in solving second-order difference equations.

- **Explanation:**
 - Discuss how initial conditions, typically x_0 and x_1 , determine the specific solution from the general solution.
 - Mention that the constants C_1 and C_2 are found by substituting the initial conditions into the general solution.
 - Key points to include:
 - Importance of x_0 and x_1
 - Determining constants based on initial values

3. What is the characteristic equation, and how is it used to solve homogeneous second-order difference equations?

- **Explanation:**
 - Explain that the characteristic equation is derived by assuming a solution of the form $x_t = r^t$.
 - Discuss how the characteristic equation helps in determining the roots that influence the system's behavior (whether it converges, oscillates, or diverges).
 - Key points to include:
 - Definition of the characteristic equation
 - Steps in solving the characteristic equation
 - Types of roots (real, complex, repeated)

4. Differentiate between homogeneous and non-homogeneous second-order difference equations. Provide examples.

○ **Explanation:**

- Define both homogeneous and non-homogeneous equations.
- In homogeneous equations, the right-hand side equals zero, while in non-homogeneous equations, an external forcing term $f(t)$ is present.
- Provide examples, like a capital accumulation model for homogeneous and a model with government subsidy for non-homogeneous.
- Key points to include:
 - Definition of homogeneous and non-homogeneous equations
 - Differences in the presence of forcing terms
 - Examples to illustrate the difference

5. Solve the homogeneous second-order difference equation:

$$x_{t+2} = 1.2x_{t+1} + 0.5x_t$$

Given the initial conditions $x_0 = 3$, $x_1 = 6$, find the general solution.

○ **Explanation:**

- Write the equation in standard form.
- Form the characteristic equation: $r^2 - 1.2r - 0.5 = 0$.
- Solve the characteristic equation to find the roots r_1 and r_2 .
- Use the initial conditions to determine the constants in the general solution.
- Key points to include:
 - Solving the characteristic equation
 - Applying initial conditions to find constants

6. Solve the non-homogeneous second-order difference equation:

$$x_{t+2} = 0.5x_{t+1} + 0.3x_t + 10$$

Given the initial conditions $x_0 = 2$, $x_1 = 4$, compute the first three terms of x_t .

○ **Explanation:**

- Start by solving the homogeneous part: $x_{t+2} = 0.5x_{t+1} + 0.3x_t$.
- Find the particular solution for the non-homogeneous part where the forcing term is constant (guess a particular solution $x_p(t) = A$).
- Combine the homogeneous and particular solutions.
- Use the initial conditions to compute the constants.
- Key points to include:
 - Solving the homogeneous equation
 - Finding the particular solution for constant forcing term
 - Determining the constants using initial conditions

7. Using the iterative method, compute the first four terms of the sequence for the equation:

$$x_{t+2} = 0.4x_{t+1} + 0.6x_t + 8$$

Given the initial conditions $x_0 = 3$ and $x_1 = 5$, compute x_2 , x_3 , and x_4 .

○ **Explanation:**

- Use the iterative approach to solve for x_2 , x_3 , and x_4 .
- Starting with $x_0 = 3$ and $x_1 = 5$, apply the recurrence relation step by step to calculate the next values.
- For each step:
 - $x_2 = 0.4 \times 5 + 0.6 \times 3 + 8$

- Continue for x_3 and x_4 .
- Key points to include:
- Step-by-step computation using the recurrence relation
- Emphasis on iterative methods for numerical solutions

Explanation for Attempting Questions:

□ Theoretical Questions:

- For each theoretical question, define the terms first, followed by the key differences or explanations asked in the question.
- Always include examples where relevant, and be clear about how certain concepts apply to the overall solution of second-order difference equations.
- Keep your answers concise and to the point while covering the core concepts discussed in the relevant sections.

□ Numerical Questions:

- For solving homogeneous equations, first find the characteristic equation, solve for the roots, and then write the general solution.
- For non-homogeneous equations, follow the two-step process: solve the homogeneous part first, then find the particular solution for the forcing term, and combine them.
- Iterative solutions involve applying the recurrence relation step by step, which is key to getting the correct sequence values.

19.9 HINTS TO CHECK YOUR PROGRESS (CYP)

CYP 1 :

1. What is a second-order difference equation, and how does it differ from a first-order difference equation?

- **Section 19.3** explains that a second-order difference equation is one where the future state depends not only on the immediate past but also

on the second most recent state. In contrast, a first-order difference equation only depends on the immediate past state.

2. What are the two main types of second-order difference equations, and how are they classified?

- **Section 19.3.2** covers the two main types: homogeneous (no external forcing term) and non-homogeneous (includes an external forcing term, $f(t)$).
- For homogeneous equations, the right-hand side is zero, while non-homogeneous equations have an external input influencing the system.

3. Explain the role of the characteristic equation in solving homogeneous second-order difference equations.

- **Section 19.4.1** discusses that the characteristic equation is derived by assuming a solution of the form $x_t = r^t$. Solving this quadratic equation helps determine the **roots** (r_1, r_2) that dictate the solution form, whether exponential, oscillatory, or involving growth/decay.

4. What is the difference between a homogeneous and a non-homogeneous second-order difference equation?

- **Section 19.3.2:** A homogeneous equation has no external forcing term (i.e., $f(t) = 0$), while a non-homogeneous equation has an external term influencing the system's evolution. The presence of $f(t)$ distinguishes the non-homogeneous case.

5. How does the discriminant of the characteristic equation determine the form of the solution in homogeneous second-order difference equations?

- **Section 19.4.1** explains that the discriminant of the characteristic equation ($\Delta = a_1^2 - 4a_2$) determines whether the roots are real or complex. The solution's form changes based on whether the discriminant is positive (two real roots), zero (repeated real root), or negative (complex conjugates).

6. What are the common methods for solving non-homogeneous second-order difference equations, and when is the method of undetermined coefficients used?

- **Section 19.4.2** outlines two primary methods:
 - **Undetermined Coefficients:** Used when the forcing term $f(t)$ is a polynomial, exponential, or sinusoidal. You guess a particular solution form and solve for the coefficients.
 - **Variation of Parameters:** Used when the forcing term is more complicated, adjusting the constants in the homogeneous solution.

7. In iterative solutions for second-order difference equations, why are initial conditions essential for solving the system?

- **Section 19.4.3** emphasizes that initial conditions (x_0 and x_1) are required to determine the constants in the general solution and start the process of calculating future values of the sequence. Without initial conditions, the system's exact solution cannot be determined.

CYP 2:

1. Solve the following homogeneous second-order difference equation:

- **Section 19.4.1** explains that to solve a homogeneous equation, first form the characteristic equation by substituting $x_t = r^t$ into the equation. Then solve for the roots of the characteristic equation and use them to determine the general solution, which can take the form of $x_t = C_1 r_1^t + C_2 r_2^t$ (if distinct roots) or $x_t = (C_1 + C_2 t) r^t$ (if repeated roots).

2. Solve the non-homogeneous second-order difference equation:

- **Section 19.4.2** guides the solution for non-homogeneous equations:
 1. Solve the corresponding homogeneous equation to get the complementary solution.

2. Find a particular solution based on the form of $f(t)$. For instance, if $f(t)$ is a constant, guess a constant solution. If $f(t)$ is sinusoidal, guess a sinusoidal form, etc.
3. Combine the complementary and particular solutions to get the general solution.

3. Using the iterative method, compute the first three terms of the sequence for the following equation:

- **Section 19.4.3** describes how to use iterative methods to solve second-order difference equations numerically. Start with the initial conditions and compute the next terms using the recurrence relation:

$$x_{t+2} = a_1 x_{t+1} + a_2 x_t + f(t)$$

Apply this formula step by step to calculate each term in the sequence.

19.10 SUGGESTED READINGS

- “Mathematical Methods for Economics” by Michael Hoy, John Livernois, Chris McKenna, and Ray Rees (2001)
- “Dynamic Economics: Quantitative Methods and Applications” by Jerome S. Rothenberg (1995)
- “Difference Equations and Applications” by V. K. B. Kota and S. G. Vaidya (2007)
- “Discrete Mathematics with Applications” by Susanna S. Epp (2003)
- “Mathematical Models in Economics” by V. L. Makarov (2010)
- “Introduction to the Theory of Difference Equations” by Saber Elaydi (2013)

M.A. Economics	Lesson No. 20
C.No. : Eco-103	Semester - 1st
	Unit III

SIMULTANEOUS DIFFERENTIAL EQUATIONS AND PHASE DIAGRAMS

STRUCTURE

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20.12 Suggested Readings

20.0 OBJECTIVES

The main objectives of this lesson are to enable you to:

- 1. Understand the Concept of Simultaneous Differential Equations:** Gain a clear understanding of what simultaneous differential equations are and why they are important in modeling dynamic systems with multiple interdependent variables.
- 2. Learn the Types of Simultaneous Differential Equations:** Differentiate between linear and non-linear simultaneous differential equations, and understand the impact of time-dependent vs. time-independent systems.
- 3. Develop Methods for Solving Simultaneous Differential Equations:** Learn analytical techniques (such as matrix methods, eigenvalues) and numerical methods (e.g., Euler's method) for solving systems of differential equations.
- 4. Construct and Interpret Phase Diagrams:** Understand the process of creating phase diagrams for 2-variable systems and how they help visualize system dynamics over time.
- 5. Analyze the Stability of Systems:** Study linear and non-linear stability analysis methods to determine whether equilibrium points are stable or unstable, and how the system behaves around these points.
- 6. Apply Simultaneous Differential Equations to Real-World Systems:** Learn how to apply simultaneous differential equations to biological models.

(e.g., predator-prey), engineering systems (e.g., oscillators), and economic models (e.g., market equilibrium).

7. **Understand the Role of Phase Diagrams in System Behavior:** Learn how phase diagrams are used to study system trajectories, equilibrium states, and long-term behavior, including convergence, oscillation, and divergence.

20.1 LEARNING OUTCOMES

By the end of this lesson, you should be able to:

1. Define and explain the concept of simultaneous differential equations.
2. Solve linear and non-linear simultaneous differential equations using analytical and numerical methods.
3. Construct and interpret phase diagrams for two-variable systems.
4. Perform stability analysis using the Jacobian matrix and other linear/non-linear methods.
5. Apply simultaneous differential equations to model real-world systems in biology, engineering, and economics.
6. Analyze the long-term behavior of systems, including convergence, oscillation, and divergence using phase diagrams.

20.2 INTRODUCTION

Simultaneous differential equations are essential tools in modeling systems where multiple variables interact and evolve over time. These equations are particularly useful when analyzing dynamic systems in which the future behavior of one variable depends not only on its current state but also on the states of other variables. The interaction between these variables can create complex behavior that cannot be described by a single equation alone. For example, in ecosystems, the populations of different species may affect each other through predator-prey dynamics or competition for resources. Similarly, in economics, the prices of goods might depend on both supply and demand, and their interaction affects the overall economic equilibrium.

A simultaneous differential equation is a system of differential equations that

describe how several variables change in relation to each other over time. Each variable in the system typically depends on its own rate of change and on the rate of change of other variables in the system. These systems can represent both linear relationships, where changes in one variable are directly proportional to changes in another, and non-linear relationships, where the changes may be more complex, involving powers, products, or other non-linear interactions.

The main challenge with simultaneous differential equations lies in solving them. While a single differential equation can often be solved relatively easily, multiple interacting equations require a more systematic approach. Solving these equations helps us understand the dynamics of the system, such as how changes in one part of the system affect other parts. The solutions can provide insights into long-term behaviors such as equilibrium states, oscillations, or growth trends, depending on the nature of the equations.

An important tool for analyzing the behavior of systems of simultaneous differential equations is the phase diagram. A phase diagram is a graphical representation that shows the trajectories of the system's variables over time. By plotting the system's variables in a two-dimensional or three-dimensional space, we can visualize how they interact and evolve. Phase diagrams are particularly helpful in studying stability, as they can reveal whether a system will settle into a steady state (equilibrium), oscillate over time, or grow without bound. In a phase diagram, equilibrium points represent the values of the variables where the system's rates of change are zero, and the trajectories around these points indicate whether the system is stable or unstable.

This lesson will focus on understanding and solving simultaneous differential equations, exploring the methods used to solve these systems both analytically (using algebraic techniques and standard solution methods) and numerically (through computational methods like the Euler method and Runge-Kutta methods). We will also learn how to construct and interpret phase diagrams, which offer visual insight into the behavior of systems, helping to predict their long-term evolution.

Real-world applications of simultaneous differential equations are abundant across various fields, including biology, where they model population dynamics and ecological relationships; engineering, where they describe systems such as electrical circuits or mechanical oscillators; and economics, where they model interactions such as supply-demand dynamics, market equilibrium, and economic growth. Understanding how to

model and analyze these systems allows us to make predictions and gain insights into the behavior of complex, interconnected systems.

In summary, simultaneous differential equations provide a framework for analyzing systems with multiple interdependent variables. By solving these equations, constructing phase diagrams, and studying their behavior, we can gain a deeper understanding of how dynamic systems evolve and interact over time. This lesson will equip you with the necessary tools to model, solve, and interpret real-world systems involving multiple variables and their interactions.

20.3 TYPES OF SIMULTANEOUS DIFFERENTIAL EQUATIONS

Simultaneous differential equations are a set of equations that involve multiple dependent variables, each with a derivative that depends on the other variables in the system. These equations are essential in modeling dynamic systems where the interactions between variables influence the system's evolution over time. In many real-world scenarios, the variables are interconnected, and their behavior is governed by their mutual dependencies. For instance, in population dynamics, the growth of one species might depend on the population of another, or in engineering, the behavior of interconnected electrical circuits can be modeled using simultaneous differential equations.

In this section, we will explore the different **types of simultaneous differential equations**:

- 1. Linear Simultaneous Differential Equations**
- 2. Non-linear Simultaneous Differential Equations**
- 3. Time-dependent and Time-independent Systems**

Each of these types has unique characteristics and methods of solving. We will look at the general forms of these equations and present numerous examples to help demonstrate their practical applications and solution techniques.

20.3.1 Linear Simultaneous Differential Equations

A linear system of simultaneous differential equations refers to a system where each equation is linear with respect to the dependent variables and their derivatives. Linear systems are easier to solve and analyze compared to non-linear systems because

their behavior is generally predictable and the solutions can often be expressed in closed form.

General Form of Linear Simultaneous Differential Equations

The general form of a linear simultaneous differential equation for a two-variable system is:

$$\frac{dx}{dt} = a_1 x + b_1 y + f_1(t)$$

$$\frac{dy}{dt} = a_2 x + b_2 y + f_2(t)$$

Where:

- x and y are the dependent variables (e.g., two populations in a predator-prey model, or two chemical concentrations in a reaction),
- a_1, a_2, b_1, b_2 are constants representing the interactions between the variables,
- $f_1(t)$ and $f_2(t)$ are external forcing functions, which could be functions of time (for non-homogeneous systems).

For **homogeneous systems**, the right-hand side of the equations would be zero:

$$\frac{dx}{dt} = a_1 x + b_1 y$$

$$\frac{dy}{dt} = a_2 x + b_2 y$$

In these equations, the rate of change of x depends linearly on both x and y , and the rate of change of y depends on both x and y .

1. Example 1 : Simple Linear System

Consider the following linear system:

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = -x + 4y$$

Here, x and y could represent the population of two species in an ecosystem. The constants 2, 3, -1, 4 define how the populations affect each other. The system is linear because both equations are linear in x and y .

Solution:

- First, we write this system in matrix form:

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- To solve the system, we would typically solve the matrix equation for the eigenvalues and eigenvectors to find the general solution.

2. Example 2: Homogeneous Linear System with Zero Forcing Term

Consider the following system without external forcing:

$$\frac{dx}{dt} = 4x + 3y$$

$$\frac{dy}{dt} = -2x + 5y$$

This system can be solved by finding the eigenvalues and eigenvectors of the coefficient matrix:

$$\begin{pmatrix} 4 & 3 \\ -2 & 5 \end{pmatrix}$$

The eigenvalues give information about the stability of the system (whether solutions grow, decay, or oscillate).

20.3.2 Non-linear Simultaneous Differential Equations

A non-linear system of simultaneous differential equations involves equations where the dependent variables appear in non-linear terms, such as products, powers, or other non-linear functions. These systems are often much more difficult to solve than linear systems, and their behavior can be chaotic, making analytical solutions challenging. However, such systems are more realistic in many real-world applications, where interactions between variables often do not follow simple proportional relationships.

General Form of Non-linear Simultaneous Differential Equations

The general form of a non-linear system for two variables is:

$$\begin{aligned}\frac{dx}{dt} &= f_1(x, y, t) \\ \frac{dy}{dt} &= f_2(x, y, t)\end{aligned}$$

Where $f_1(x, y, t)$ and $f_2(x, y, t)$ are non-linear functions of x and y , which could involve products like xy , powers like x^2 , or more complex terms.

Numerical Examples of Non-linear Simultaneous Differential Equations

1. Example 1: Predator-Prey Model

A well-known example of a non-linear system is the Lotka-Volterra model, used to describe predator-prey interactions:

$$\begin{aligned}\frac{dx}{dt} &= \alpha x - \beta xy \\ \frac{dy}{dt} &= \delta xy - \gamma y\end{aligned}$$

Where:

- x is the prey population,
- y is the predator population,
- α , β , δ , and γ are positive constants representing birth rates, death rates, and interaction coefficients.

This system is **non-linear** because the terms xy (the interaction between predators and prey) make the system non-linear.

Solution:

- This system is typically solved using numerical methods like the **Euler method** or **Runge-Kutta method**. Analytical solutions are challenging, but the system can provide rich dynamics such as oscillations (where the predator and prey populations oscillate around equilibrium values).

2 Example 2: Coupled Oscillators

Consider two coupled oscillators with the following non-linear system:

$$\frac{dx}{dt} = y + \sin(x)$$

$$\frac{dy}{dt} = -x + \cos(y)$$

This system represents two interacting oscillators, where x and y could represent the displacement of each oscillator. The sine and cosine terms introduce non-linearity.

Solution:

- Numerical methods like Runge-Kutta are often used to solve such systems. The behavior of the system might include phenomena such as chaotic oscillations or entrainment depending on the interaction strength.

20.3.3 Time-dependent and Time-independent Systems

Simultaneous differential equations can either have time-dependent coefficients or time-independent coefficients. Understanding whether the system's parameters vary over time is crucial for determining the long-term behavior of the system.

Time-independent Systems

In a time-independent system, the coefficients of the differential equations do not change with time. These systems are simpler to analyze because their behavior is constant, and their solutions are often more predictable.

General Form of Time-independent Systems

The general form of a time-independent linear system is:

$$\frac{dx}{dt} = a_1 x + b_1 y$$

$$\frac{dy}{dt} = a_2 x + b_2 y$$

Where the coefficients a_1, a_2, b_1, b_2 are constants, and the system's behavior does not change over time.

Example: Time-independent System

Consider a simple time-independent system:

$$\frac{dx}{dt} = 3x + 2y$$

$$\frac{dy}{dt} = -x + 4y$$

Here, the system's coefficients are constant, making it a time-independent system. This can be solved using matrix methods, where the eigenvalues of the coefficient matrix give information about the stability and behavior of the system.

Time-dependent Systems

In time-dependent systems, the coefficients of the differential equations vary with time. These systems are more complex because the behavior of the system can change as time progresses. Time-dependent systems are used to model situations where external forces or inputs vary over time, such as changing market conditions or fluctuating environmental factors.

General Form of Time-dependent Systems

The general form of a time-dependent system is:

$$\frac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t)$$

$$\frac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t)$$

Where $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$ are functions of time, and $f_1(t)$, $f_2(t)$ represent external forcing functions.

Example: Time-dependent System

Consider the following time-dependent system:

$$\frac{dx}{dt} = tx + y$$

$$\frac{dy}{dt} = -x + ty$$

Here, the coefficients depend on time (t), making the system time-dependent. This system can be solved using numerical methods, where the variation of the coefficients with time complicates the solution but makes it more reflective of real-world dynamics.

Simultaneous differential equations are a powerful tool for modeling dynamic systems with interdependent variables. In this section, we have discussed the three main types of systems:

1. **Linear Simultaneous Differential Equations:** These systems are easier to solve and often involve constant coefficients.
2. **Non-linear Simultaneous Differential Equations:** These systems are more complex and can exhibit behaviors such as chaos or periodicity, making them more challenging to solve.
3. **Time-dependent and Time-independent Systems:** Time-dependent systems have coefficients that change over time, adding complexity to the analysis.

The choice of solving technique depends on the type of system, and numerical methods are often used for non-linear and time-dependent systems. The next step is to apply these concepts in real-world applications such as biology, engineering, and economics to better understand the dynamics of interacting systems.

Check your Progress – I

Based on your understanding of the previous section, answer these questions in your own words:

1. What is the difference between linear and non-linear simultaneous differential equations?

2. What is the general form of a linear simultaneous differential equation?

3. How do time-dependent systems differ from time-independent systems in simultaneous differential equations?

4. Why are non-linear simultaneous differential equations often more challenging to solve than linear ones?

5. What are the typical real-world applications where simultaneous differential equations are used?

6. Solve the following linear system of simultaneous differential equations:

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = -x + 4y$$

Given the initial conditions $x_0 = 1, y_0 = 2$.

7. Solve the following non-linear system:

$$\frac{dx}{dt} = x + y^2$$

$$\frac{dy}{dt} = x^2 + y$$

Given the initial conditions $x_0 = 1, y_0 = 2$.

20.4 METHODS OF SOLVING SIMULTANEOUS DIFFERENTIAL EQUATIONS

Simultaneous differential equations arise when multiple variables depend on each other and change over time. The complexity of solving these equations varies based on the type of system—whether it is linear, non-linear, time-dependent, or time-independent. In this section, we will explore the methods of solving simultaneous

differential equations, which are vital in modeling real-world systems in fields like biology, physics, economics, and engineering.

We will discuss three primary methods for solving simultaneous differential equations:

1. **Analytical Methods:** These involve exact solutions derived through algebraic manipulations, integrations, and other standard mathematical tools.
2. **Numerical Methods:** These are used when analytical solutions are either difficult or impossible to obtain. Numerical methods approximate the solution through computational techniques.
3. **Matrix and Eigenvalue Techniques:** These are especially useful for solving linear systems of simultaneous differential equations and allow for a systematic approach using matrix algebra.

20.4.1 Analytical Methods

Analytical methods involve solving simultaneous differential equations through exact techniques, typically resulting in a formula that describes the behavior of the system over time. These methods are most useful when the system of equations is linear and the coefficients do not change with time (time-independent systems). Analytical methods can also be applied to non-linear systems in specific cases where simplifications are possible.

General Form of Simultaneous Differential Equations for Analytical Methods

Consider a system of linear simultaneous differential equations:

$$\begin{aligned} \frac{dX}{dt} &= a_1 X + b_1 Y + f_1(t) \\ \frac{dY}{dt} &= a_2 X + b_2 Y + f_2(t) \end{aligned}$$

Where x and y are dependent variables, a₁,a₂,b₁,b₂ are constants (or time-dependent coefficients), and f₁(t) and f₂(t) are external forcing functions (time-dependent).

Steps in Solving Using Analytical Methods

1. For Homogeneous Systems (when $f_1(t) = f_2(t) = 0$):

- We start by writing the system in matrix form:

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- The matrix approach is efficient when we deal with multiple variables.

2. Find the Eigenvalues and Eigenvectors:

- To solve the system, we solve the characteristic equation:

$$\det \begin{pmatrix} a_1 - r & b_1 \\ a_2 & b_2 \end{pmatrix} = 0$$

- The solutions to this equation (the eigenvalues) tell us the growth/decay or oscillatory nature of the solution.

3. Solve the System Using the Eigenvectors:

- Once the eigenvalues are found, the general solution of the system is a combination of the eigenvectors corresponding to those eigenvalues.

Example for Analytical Methods

Consider the following homogeneous system of linear simultaneous differential equations:

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = -x + 4y$$

Step 1: Write the system in matrix form:

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Step 2: Find the eigenvalues by solving the characteristic equation:

$$\det \begin{pmatrix} 2 - r & 3 \\ -1 & 4 - r \end{pmatrix} = 0$$

$$\begin{vmatrix} 2-r & 3 \\ -1 & 4-r \end{vmatrix} = 0$$

$$(2-r)(4-r) - (-3) = 0$$

$$r^2 - 6r + 11 = 0$$

Solving for r gives the eigenvalues $r_1 = 3 + i$ and $r_2 = 3 - i$.

Step 3 : Solve the system using the eigenvectors:

For complex eigenvalues, the general solution involves **oscillatory** behavior and can be written as:

$$x(t) = C_1 e^{3t} \cos(t) + C_2 e^{3t} \sin(t)$$

$$y(t) = C_3 e^{3t} \cos(t) + C_4 e^{3t} \sin(t)$$

Where C_1, C_2, C_3, C_4 are constants determined by initial conditions.

20.4.2 Numerical Methods

In many cases, simultaneous differential equations cannot be solved analytically, especially for non-linear systems. For such cases, numerical methods provide an approximate solution by discretizing the differential equations and solving them step by step. These methods are particularly useful when the system is non-linear or time-dependent.

Common Numerical Methods for Solving Simultaneous Differential Equations

1. Euler's Method:

Euler's method approximates the solution by using the slope at each point to estimate the value of the function at the next time step. The method is simple but can be inaccurate for stiff equations.

The formula for Euler's method is:

$$x_{t+1} = x_t + h \cdot f(t, x_t)$$

Where h is the step size.

2. Runge-Kutta Method:

The Runge-Kutta method provides a more accurate solution by considering multiple intermediate points in each time step. The most commonly used version is the fourth-order Runge-Kutta method, which has an error of $O(h^4)$.

The formulas for the fourth-order Runge-Kutta method are:

$$\begin{aligned}
 k_1 &= hf(t, x_t) \\
 k_2 &= hf\left(t + \frac{h}{2}, x_t + \frac{k_1}{2}\right) \\
 k_3 &= hf\left(t + \frac{h}{2}, x_t + k_2\right) \\
 k_4 &= hf(t + h, x_t + k_3) \\
 x_{t+1} &= x_t + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
 \end{aligned}$$

Numerical Example for Euler's Method

Consider the following system of differential equations:

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = -x + 4y$$

Given the initial conditions $x(0) = 1$ and $y(0) = 2$, use Euler's method to estimate the values of x and y at $t = 1$, with a step size $h = 0.1$.

Step 1 : Initialize values

$$x_0 = 1, y_0 = 2, h = 0.1$$

Step 2 : Apply Euler's method

For $t_0 = 0$:

$$\begin{aligned}
 \frac{dx}{dt} &= 2(1) + 3(2) = 8 \quad \Rightarrow \quad x_1 = x_0 + h \cdot 8 = 1 + 0.1 \cdot 8 = 1.8
 \end{aligned}$$

$$\frac{dy}{dt} = -(1) + 4(2) = 7 \Rightarrow y_1 = y_0 + h \cdot 7 = 2 + 0.1 \cdot 7 = 2.7$$

Thus, $x_1=1.8$ and $y_1 = 2.7$.

Repeat this process to calculate the next time steps.

20.4.3 Matrix and Eigenvalue Techniques

Matrix methods and eigenvalue techniques are especially effective for solving linear simultaneous differential equations. They provide a systematic approach to solving multiple equations at once and are particularly useful in understanding the behavior of multi-variable systems.

Example Using Eigenvalue Method

Consider the following linear system:

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = -x + 4y$$

Write this system in matrix form:

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Now, find the eigenvalues and eigenvectors of the matrix:

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$$

The characteristic equation is:

$$| \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} - r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} | = 0$$

$$(2 - r)(4 - r) - (-3) = 0$$

$$r^2 - 6r + 11 = 0$$

The solutions are $r_1 = 3 + i$ and $r_2 = 3 - i$.

Now find the eigenvectors corresponding to these eigenvalues and use them to construct the general solution for $x(t)$ and $y(t)$.

In this section, we discussed three primary methods for solving simultaneous differential equations: analytical methods, numerical methods, and matrix/eigenvalue techniques. Each method has its advantages, and the choice of method depends on the complexity of the system and the desired accuracy. Analytical methods are best for linear systems with constant coefficients, while numerical methods are useful for more complex systems or when an exact solution is difficult to obtain. The matrix and eigenvalue techniques provide a systematic approach to solving linear systems and help understand the system's behavior, especially in multi-variable cases.

20.5 STABILITY ANALYSIS IN SIMULTANEOUS SYSTEMS

In many scientific and economic applications, systems with multiple interacting variables evolve over time according to a set of differential equations. These systems can often exhibit complex behaviors, such as oscillations, growth, or decay, depending on their structure and initial conditions. The stability of such systems is crucial for understanding their long-term behavior. This chapter explores the concept of stability analysis in simultaneous systems, focusing on both linear and nonlinear systems. We will delve into methods of analyzing stability, providing intuitive explanations and examples to help solidify the concepts. By the end of this section, you will understand how to determine whether a system will settle into an equilibrium, oscillate, or diverge over time.

Stability analysis in simultaneous differential equations helps determine the long-term behavior of a system. Specifically, it tells us whether the system's solutions approach a steady state (equilibrium), oscillate, or grow without bound. A key component of stability analysis is determining the nature of equilibrium points and understanding the behavior of solutions near those points.

Equilibrium points occur when the derivatives of all the variables in the system are zero. Stability is then analyzed by looking at the system's behavior in the vicinity of these equilibrium points. The two primary methods of stability analysis are linear stability analysis and nonlinear stability analysis. Linear stability analysis uses approximations around equilibrium points, while nonlinear stability analysis provides a deeper understanding of more complex, non-linear interactions.

20.5.1 Linear Stability Analysis

Linear stability analysis is a fundamental technique used to study the behavior of dynamical systems, particularly those governed by simultaneous differential equations. In this context, stability refers to whether the system will return to a stable state after being disturbed, or whether it will diverge, moving away from the equilibrium point. In linear stability analysis, we approximate a potentially complex nonlinear system by linearizing it around an equilibrium point and examining the behavior of small perturbations.

This technique simplifies the analysis of a system's dynamics and provides critical insights into the long-term behavior of solutions. Let's dive deeper into the concept of linear stability analysis, its method, and its applications, along with step-by-step numerical examples to illustrate the concept.

Understanding Linear Stability Analysis

A dynamical system is a system of differential equations that describe how variables change over time. For example, in economics, such systems might describe how supply and demand evolve, while in biology, they could model population dynamics between predators and prey. These systems can either be linear or nonlinear, depending on whether the relationships between variables are proportional (linear) or more complex (nonlinear).

A linear system is one in which the dependent variables and their derivatives appear linearly in the equations. This means that each variable is raised only to the first power, and there are no products of variables or higher powers.

In linear stability analysis, we focus on analyzing the system's equilibrium points. These equilibrium points are the points where the rate of change of all variables is zero. Once we have found these equilibrium points, we study their stability by linearizing the system around each equilibrium point and analyzing the resulting linear system.

The General Approach to Linear Stability Analysis

Here is a step-by-step guide to performing linear stability analysis on a system of simultaneous differential equations:

1. Identify the Equilibrium Points:

- The first step is to find the equilibrium points, which are the values of the variables where all the derivatives are zero. For a system of differential equations, this involves solving a system of algebraic equations obtained by setting the derivatives equal to zero.

2. Linearize the System:

- Next, we approximate the system of equations by linearizing it around the equilibrium points. This is typically done by computing the Jacobian matrix, which contains the first derivatives of the system with respect to each variable.

3. Determine the Eigenvalues:

- The Jacobian matrix helps us find the eigenvalues of the system. The eigenvalues reveal the stability of the equilibrium points. If the real parts of the eigenvalues are negative, the system will return to equilibrium (stable). If they are positive, the system will diverge away from equilibrium (unstable). If the eigenvalues are purely imaginary, the system may oscillate around the equilibrium (neutral stability).

4. Interpret the Stability:

- Based on the eigenvalues, we classify the equilibrium points as stable, unstable, or neutrally stable. This gives us a clear understanding of how the system behaves near the equilibrium.

Let's now apply this procedure with two numerical examples to understand how linear stability analysis works in practice.

Example 1 : A Simple Linear System

Consider a simple linear system of differential equations modeling two interacting populations, such as prey and predators in an ecosystem. The system is given by:

$$\frac{dx}{dt} = 3x - y$$

$$\frac{dy}{dt} = -x + 2y$$

Where:

- $x(t)$ represents the prey population,
- $y(t)$ represents the predator population.

Step 1 : Find the Equilibrium Points

To find the equilibrium points, we set the right-hand sides of the equations equal to zero:

$$3x - y = 0 \quad (1)$$

$$-x + 2y = 0 \quad (2)$$

From equation (1), we can express y as $y = 3x$. Substituting this into equation (2), we get:

$$-x + 6x = 0 \Rightarrow -x + 6x = 0$$

$$5x = 0$$

Thus, $x = 0$. Substituting $x = 0$ into equation (1), we get $y = 0$.

So, the equilibrium point is $(x^*, y^*) = (0, 0)$.

Step 2 : Linearize the System

To linearize the system around the equilibrium point, we first need to compute the Jacobian matrix. The Jacobian matrix consists of the partial derivatives of the system's right-hand sides with respect to x and y . For the system

$$\frac{dx}{dt} = 3x - y$$

$$\frac{dy}{dt} = -x + 2y$$

The Jacobian matrix is:

$$J = \begin{pmatrix} \frac{a}{ax}(3x - y) & \frac{a}{ay}(3x - y) \\ \frac{a}{ax}(-x + 2y) & \frac{a}{ay}(-x + 2y) \end{pmatrix} \mathbf{1}$$

Calculating the partial derivatives:

$$J = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$$

Step 3 : Find the Eigenvalues

The next step is to find the eigenvalues of the Jacobian matrix. The characteristic equation is:

$$\det(J - \lambda I) = 0$$

$$\begin{vmatrix} 3 - \lambda & -1 \\ -1 & 2 - \lambda \end{vmatrix} = 0$$

Expanding the determinant:

$$(3 - \lambda)(2 - \lambda) - (-1)(-1) = 0$$

$$6 - 5\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 5\lambda + 5 = 0$$

Solving this quadratic equation for λ using the quadratic formula:

$$\lambda_1 = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(5)}}{2(1)}$$

$$\lambda_1 = \frac{5 \pm \sqrt{25 - 20}}{2}$$

$$\lambda_1 = \frac{5 \pm \sqrt{5}}{2}$$

Thus, the eigenvalues are:

$$\lambda_1 = \frac{5 + \sqrt{5}}{2} \approx 3.618$$

$$\lambda_2 = \frac{5 - \sqrt{5}}{2} \approx 1.382$$

Step 4 : Interpret the Stability

Both eigenvalues are positive, which means that the equilibrium point (0, 0) is unstable. Small perturbations will cause the system to diverge away from this equilibrium point, indicating that the equilibrium is not stable.

20.5.2 Nonlinear Stability Analysis

While linear stability analysis is highly effective for understanding the behavior of systems that can be approximated by linear equations near equilibrium points, many real-world systems exhibit more complex, nonlinear behavior that cannot be captured accurately by linear models. In such cases, nonlinear stability analysis becomes essential. This technique is used to analyze the stability of equilibrium points in systems where the differential equations are nonlinear, meaning the variables may appear in products, powers, or other complex relationships.

Nonlinear stability analysis provides a deeper understanding of how systems behave near equilibrium points, even when those systems exhibit oscillations, chaos, or more intricate behavior. It is a crucial tool in studying systems in fields such as biology, economics, engineering, and ecology, where nonlinear interactions between variables often lead to complex dynamics that cannot be understood solely through linear approximations.

Key Concepts in Nonlinear Stability Analysis

1. Equilibrium Points in Nonlinear Systems:

- Just as in linear systems, equilibrium points in nonlinear systems occur where the derivatives of all the variables are zero. These points represent steady states where the system's dynamics "settle down."
- However, in nonlinear systems, the behavior around these equilibrium points can be more complicated, potentially leading to oscillations, bifurcations, or even chaotic behavior.

2. Linearizing the System:

- In some cases, nonlinear systems can be approximated by linear systems near the equilibrium points using linearization. This is done by finding the Jacobian matrix, which describes how the system behaves locally around an equilibrium point.
- While this step is similar to linear stability analysis, it does not always provide a full understanding of the system's global dynamics, particularly if the system exhibits nonlinearity on a larger scale.

3. Lyapunov's Direct Method:

- In situations where linearization doesn't give a complete picture, Lyapunov's direct method can be used. This method involves constructing a Lyapunov function—a scalar function that helps determine whether the system will return to equilibrium (i.e., whether the equilibrium is stable) or diverge away from it.

4. Phase Portraits and Trajectories:

- Another key tool in nonlinear stability analysis is the phase portrait, a graphical representation of the system's trajectories in the phase space (a space where each axis represents one of the system's variables).
- By analyzing the phase portrait, we can observe the behavior of solutions over time, including whether they settle into equilibrium, exhibit periodic oscillations, or show chaotic behavior.

Steps in Nonlinear Stability Analysis

To perform nonlinear stability analysis, follow these key steps:

1. Find the Equilibrium Points:

- The first step is to find the equilibrium points by setting the right-hand side of the differential equations to zero and solving the resulting algebraic system.

2. Linearize the System (if possible):

- Next, for each equilibrium point, we linearize the system by computing the Jacobian matrix. This matrix contains the partial derivatives of the system's equations with respect to the variables, evaluated at the equilibrium point.
- The eigenvalues of the Jacobian matrix give initial insights into the stability of the equilibrium point.

3. Nonlinear Analysis:

- If linearization alone doesn't provide enough information (for example, if the eigenvalues are purely imaginary or zero), a nonlinear stability analysis is required. This typically involves:
 - **Lyapunov's Direct Method:** Construct a Lyapunov function to examine the system's stability.
 - **Phase Portrait Analysis:** Visualize the system's trajectories in phase space.

4. Interpret the Results:

- Finally, based on the findings, classify the equilibrium points as stable, unstable, or neutrally stable. Nonlinear systems may exhibit various types of behaviors, including periodic orbits, limit cycles, and chaotic dynamics.

Example 1 : Predator-Prey Model (Lotka-Volterra Equations)

A well-known example of a nonlinear system is the **Lotka-Volterra model**, which describes the interaction between two species, one being prey and the other predators. The model is given by the following system of differential equations:

$$\frac{dx}{dt} = ax - \beta xy$$

$$\frac{dy}{dt} = \alpha xy - \gamma y$$

Where:

- $x(t)$ is the prey population,
- $y(t)$ is the predator population,
- $\alpha, \beta, \gamma, \delta$ are positive constants representing birth rates, death rates, and interaction coefficients.

This system is nonlinear because of the xy term, which represents the interaction between the two species. Let's go through the steps of nonlinear stability analysis for this system.

Step 1 : Find the Equilibrium Points

To find the equilibrium points, set the right-hand sides of the equations equal to zero:

$$\alpha x - \beta xy = 0. \quad (1)$$

$$\delta y - \gamma y = 0. \quad (2)$$

From equation (1), we can factor out x and get:

$$x(\alpha - \beta y) = 0$$

Thus, $x = 0$ or $y = \frac{\alpha}{\beta}$

From equation (2), we can factor out y and get:

$$y(\delta - \gamma) = 0$$

Thus, $y = 0$ or $x = \frac{\gamma}{\delta}$

We now have two equilibrium points:

1. $x = 0$ and $y = 0$ (trivial equilibrium)
2. $x = \frac{\gamma}{\delta}$ and $y = \frac{\alpha}{\beta}$ (non-trivial equilibrium)

Step 2 : Linearize the System

Now, we linearize the system by computing the Jacobian matrix. The Jacobian matrix for the system is:

$$J = \begin{bmatrix} \frac{\partial}{\partial x} (ax - \beta xy) & \frac{\partial}{\partial y} (ax - \beta xy) \\ \frac{\partial}{\partial x} (\alpha y - \beta y) & \frac{\partial}{\partial y} (\alpha y - \beta y) \end{bmatrix}$$

Calculating the partial derivatives:

$$J = \begin{bmatrix} a - \beta y & -\beta x \\ \alpha & \alpha - \beta \end{bmatrix}$$

Substituting the equilibrium values $x = \frac{\gamma}{\delta}$ and $y = \frac{\alpha}{\beta}$ into the Jacobian matrix:

$$J = \begin{bmatrix} a - \beta \cdot \frac{\gamma}{\delta} & -\beta \cdot \frac{\gamma}{\delta} \\ \alpha & \alpha - \beta \end{bmatrix}$$

Simplifying the matrix:

$$J = \begin{bmatrix} a - \frac{\beta \gamma}{\delta} & -\frac{\beta \gamma}{\delta} \\ \alpha & \alpha - \beta \end{bmatrix}$$

Step 3 : Find the Eigenvalues

The eigenvalues of the Jacobian matrix are the solutions to the characteristic equation:

$$\det(J - \lambda I) = 0$$

$$\begin{vmatrix} a - \frac{\beta \gamma}{\delta} - \lambda & -\frac{\beta \gamma}{\delta} \\ \alpha & \alpha - \beta - \lambda \end{vmatrix} = 0$$

Expanding the determinant:

$$\lambda^2 - \frac{f\lambda y_0 a}{8f} = 0$$

Solving for λ we get:

$$\lambda^2 + \frac{f\lambda y_0 a}{8f} = 0$$

Thus, the eigenvalues are purely imaginary, which suggests that the system will exhibit oscillatory behavior around the equilibrium point $\left(\frac{y_0}{\delta}, \frac{\alpha}{\beta}\right)$. This implies that the predator and prey populations will oscillate over time, and the equilibrium point is a limit cycle.

Example 2 : Nonlinear Damped Oscillator

Consider the following system of differential equations representing a nonlinear damped oscillator:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -x - \alpha y - f x^2$$

Where:

- $x(t)$ represents the displacement of the oscillator,
- $y(t)$ represents the velocity of the oscillator,
- α and β are constants.

This system is nonlinear because of the x^2 term in the second equation. Let's perform a nonlinear stability analysis on this system.

Step 1 : Find the Equilibrium Points

Set the right-hand sides equal to zero:

$$y = 0. \quad (1)$$

$$-x - \alpha y - \beta x^2 = 0. \quad (2)$$

From equation (1), we know $y = 0$. Substituting $y = 0$ into equation (2):

$$-x - 3x^2 = 0$$

$$x(1 + 3x) = 0$$

$$\text{Thus, } x = 0 \text{ or } x = -\frac{1}{3}.$$

So, the equilibrium points are:

$$1. \quad x = 0, y = 0,$$

$$2. \quad x = -\frac{1}{3}, y = 0.$$

Step 2 : Linearize the System

The Jacobian matrix is:

$$J = \begin{pmatrix} \frac{\partial}{\partial x}(-x - 3x^2) & \frac{\partial}{\partial y}(-x - 3x^2) \\ \frac{\partial}{\partial x}(-x - ay - 3x^2) & \frac{\partial}{\partial y}(-x - ay - 3x^2) \end{pmatrix}$$

The partial derivatives are:

$$J = \begin{pmatrix} 0 & 1 \\ -1 - 2\{3x & -a \end{pmatrix}$$

Substituting the equilibrium point $x = 0, y = 0$ into the Jacobian matrix:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & -a \end{pmatrix}$$

Step 3 : Find the Eigenvalues

The characteristic equation is:

$$\det(J - \lambda I) = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -a - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + a\lambda + 1 = 0$$

Expanding the determinant:

$$\lambda(\lambda + a) - (-1) = 0 \Rightarrow \lambda^2 + a\lambda + 1 = 0$$

Solving for λ using the quadratic formula:

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4}}{2}$$

If $\alpha^2 > 4$, the roots are real and negative, implying that the equilibrium at $(0, 0)$ is stable. If $\alpha^2 < 4$, the roots are complex, implying oscillatory behavior.

Nonlinear stability analysis is a powerful method for studying the stability of equilibrium points in systems that exhibit complex, nonlinear behavior. By finding the equilibrium points, linearizing the system around those points, and analyzing the eigenvalues of the Jacobian matrix, we can determine whether the system will return to equilibrium or diverge away. For systems where linear analysis isn't sufficient, Lyapunov's direct method or phase portrait analysis can provide additional insights into the system's behavior. The two examples we discussed—predator-prey dynamics and a nonlinear damped oscillator—demonstrate how nonlinear stability analysis helps us understand the long-term behavior of real-world systems.

20.6 PHASE DIAGRAMS AND THEIR IMPORTANCE

Phase diagrams are crucial tools in the study of dynamical systems, particularly when dealing with systems of simultaneous differential equations. They offer a visual representation of how the system behaves over time, helping us understand the long-term evolution of the system. Phase diagrams plot the system's state variables (e.g., population sizes, concentrations, economic variables) on the axes, and their trajectories over time show how these variables evolve. By examining phase diagrams, we can gain insights into the stability of equilibrium points, the presence of oscillations, limit cycles, and even chaotic behavior.

In this section, we will explore the fundamentals of phase diagrams, including how to construct and interpret them.

20.6.1 Introduction to Phase Diagrams

A phase diagram is a graphical representation of a system of differential equations that describes the evolution of the system's variables over time. In a phase diagram, each axis represents a different state variable, and the paths (or trajectories) of the system are plotted as the system evolves. The main purpose of a phase diagram is to

provide a visual tool that helps analyze the qualitative behavior of the system without requiring the explicit solution of the system's differential equations.

Phase diagrams are particularly useful for studying the behavior of systems near equilibrium points, and they help us categorize the system's dynamics into categories such as:

- **Stable equilibrium:** The system tends to return to equilibrium after small perturbations.
- **Unstable equilibrium:** The system moves away from equilibrium after perturbations.
- **Limit cycles:** Closed trajectories that represent periodic solutions.
- **Oscillatory behavior:** Systems where variables periodically increase and decrease.

The trajectories on a phase diagram indicate how the system's state changes over time and whether it converges to an equilibrium point, oscillates, or diverges.

Key Features of Phase Diagrams:

1. **Equilibrium Points:** These are points where the system does not change, meaning that the derivatives of all state variables are zero. They are the points where the system's behavior can be analyzed for stability.
2. **Trajectories:** These are the paths that the system follows as it evolves from an initial state. Each trajectory corresponds to a set of values for the state variables over time.
3. **Stability and Attractors:** Equilibrium points can be stable (where nearby trajectories converge to them) or unstable (where nearby trajectories move away from them).
4. **Limit Cycles:** Closed trajectories indicating periodic behavior. Systems exhibiting limit cycles oscillate indefinitely, and these are often seen in predator-prey models and mechanical systems.

20.6.2 Constructing Phase Diagrams

Constructing a phase diagram involves several steps. Let's go through the general process of constructing a phase diagram, using a simple two-variable system as an example.

Steps to Construct a Phase Diagram:

1. **Write the System of Differential Equations:** Start with a system of simultaneous differential equations that describe the dynamics of the system. These equations describe how the state variables change over time.
2. **Find the Equilibrium Points:** Set the right-hand sides of the equations to zero and solve for the equilibrium points. These are the points where the system doesn't change over time.
3. **Determine the Direction of the Trajectories:** Use the system's equations to determine the direction of motion (positive or negative) in different regions of the phase space.
4. **Sketch the Phase Portrait:** Plot the equilibrium points on the phase diagram and sketch the trajectories that emanate from them, using arrows to indicate the direction of movement.

Example : Predator-Prey System

Let's consider a simple predator-prey model:

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = \gamma xy - \delta y$$

Where:

- $x(t)$ is the prey population,
- $y(t)$ is the predator population,
- α , β , γ , and δ are positive constants.

Step 1 : Find the Equilibrium Points

To find the equilibrium points, set the right-hand sides of the equations equal to zero:

$$\alpha x - \beta xy = 0$$

$$\delta xy - \gamma y = 0$$

From the first equation:

$$x(\alpha - \beta y) = 0$$

$$\text{So, } x = 0 \text{ or } y = \frac{\alpha}{\beta}$$

From the second equation:

$$y(\delta x - \gamma) = 0$$

$$\text{So, } y = 0 \text{ or } x = \frac{\gamma}{\delta}$$

Thus, the equilibrium points are:

- (0, 0) (trivial equilibrium)
- $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$ (non-trivial equilibrium)

Step 2 : Construct the Phase Diagram

To construct the phase diagram, follow these steps:

- 1. Plot the equilibrium points** on a two-dimensional plane, where the x-axis represents the prey population and the y-axis represents the predator population.
- 2. Draw the trajectories** between and around the equilibrium points. The direction of the trajectories can be determined by analyzing the signs of the equations in different regions of the phase space.
- 3. Use arrows to indicate the direction** of the trajectories. This will show how the system evolves over time.

Let's understand through the phase diagram for the predator-prey model.

Predator-Prey Model : In this system, the equilibrium point (0,0) is unstable, while the non-trivial equilibrium point at $\frac{Y}{8} = \frac{a}{P}$, $(8y, Pa)$ is stable. The system exhibits periodic oscillations between predator and prey populations.

20.6.3 Interpreting Phase Diagrams

Interpreting phase diagrams involves analyzing the equilibrium points, trajectories, and the overall behavior of the system. Phase diagrams provide insight into the stability of equilibrium points and reveal important features such as oscillations, limit cycles, and stability.

Key Points to Look for in Phase Diagrams:

1. Equilibrium Points:

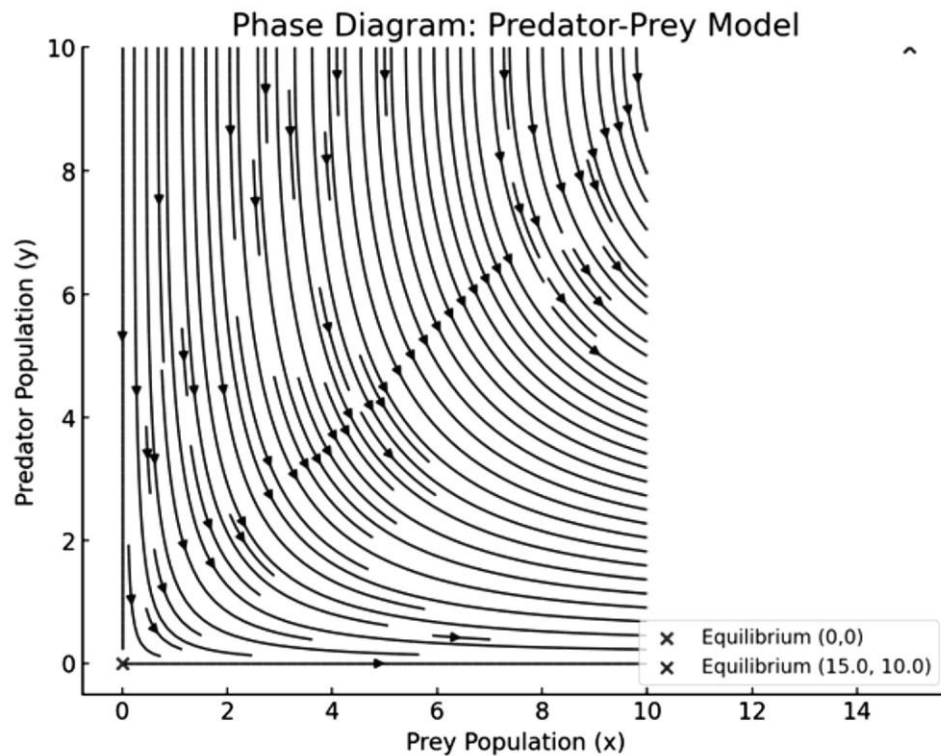
- A stable equilibrium is where trajectories tend to converge toward a point, indicating that the system will settle into that state.
- An unstable equilibrium is where trajectories diverge from a point, showing that the system will not return to that state if disturbed.

2. Limit Cycles:

- If trajectories form closed loops, it indicates periodic behavior, often seen in systems like predator-prey models or mechanical oscillators. These systems will oscillate between certain bounds.

3. Trajectories:

- The direction of trajectories (shown with arrows) indicates how the system evolves over time. Trajectories that move toward the equilibrium point suggest that the equilibrium is stable, while those moving away suggest instability.

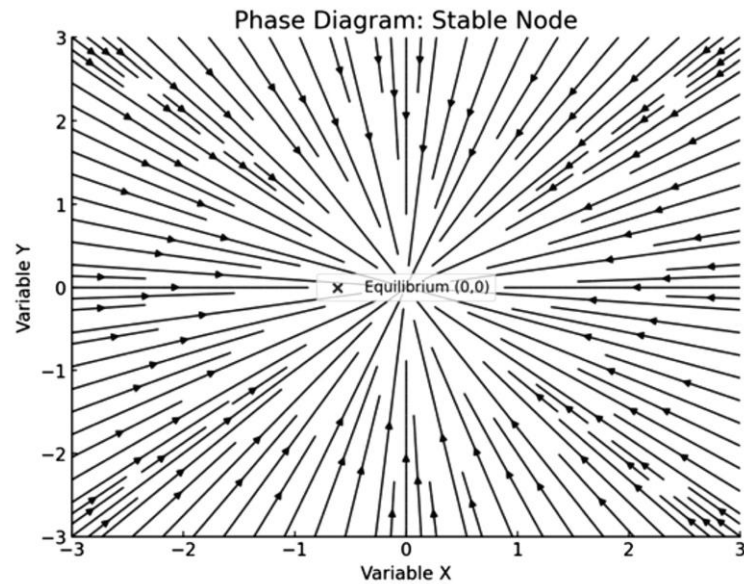


4. **Stability:**

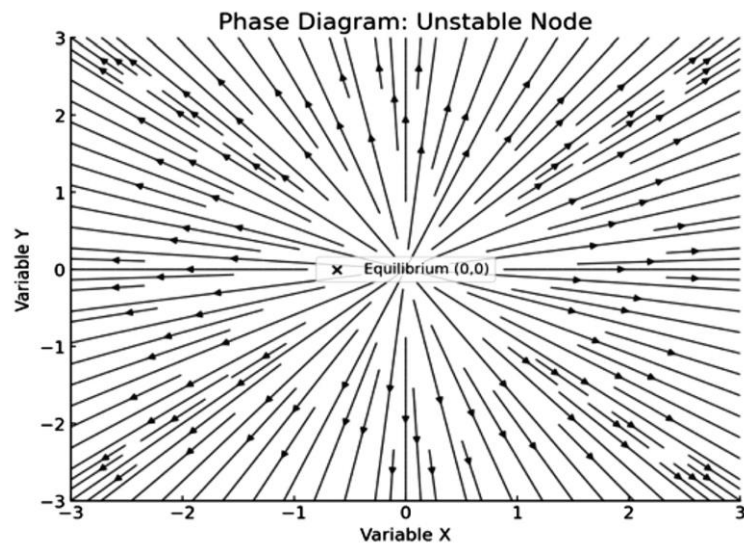
- In a phase diagram, stable equilibrium points are surrounded by trajectories that point towards the equilibrium. Unstable equilibrium points are surrounded by trajectories that point away.

Interpretation Example for the Predator-Prey Model:

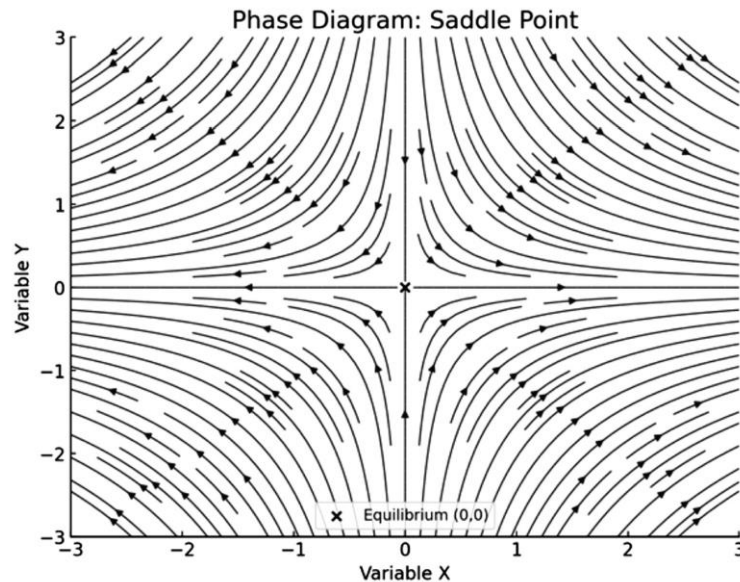
- The equilibrium point (0, 0) is unstable, as trajectories move away from it.
- The equilibrium point $\frac{\gamma}{\delta}, \frac{\alpha}{\beta}$ is stable, as trajectories converge towards it, indicating that the system will eventually settle into this equilibrium point.
- If the system exhibits oscillatory behavior, the trajectories will trace out a limit cycle, where the predator and prey populations continually oscillate.



Stable Node: In this diagram, all trajectories move toward the equilibrium point at (0,0). This behavior indicates a stable system, where any disturbance causes the system to return to equilibrium over time.



Unstable Node: Here, all trajectories move away from the equilibrium point (0,0), showing that the system is unstable. Small perturbations will cause the system to diverge from the equilibrium point.



Saddle Point: The equilibrium point $(0,0)$ behaves as a saddle point. Trajectories move toward the equilibrium along one axis (in this case, the x-axis) and away from it along the other axis (the y-axis). This behavior indicates instability in one direction and stability in another.

Check your Progress – II

Based on your understanding of the previous section, answer these questions in your own words:

1. What is the main difference between linear and nonlinear simultaneous differential equations?

2. What is the purpose of stability analysis in simultaneous systems?

3. What is the role of the Jacobian matrix in linear stability analysis?

4. How do eigenvalues of the Jacobian matrix help in determining the stability of an equilibrium point?

5. What are the characteristics of a stable equilibrium point in a phase diagram?

6. How does a phase diagram help in understanding the long-term behavior of a system?

7. What is the significance of a limit cycle in a phase diagram?

8. Solve the following linear system of simultaneous differential equations:

$$\frac{dx}{dt} = 2x - y$$
$$\frac{dy}{dt} = -x + 3y$$

with initial conditions $x(0) = 1$, $y(0) = 2$.

9. Given the nonlinear predator-prey system:

$$\frac{dx}{dt} = 2x - 3xy$$
$$\frac{dy}{dt} = -x + 4y$$

Find the equilibrium points of the system.

10. For the following linear system, determine the stability of the equilibrium point:

$$\begin{aligned}\frac{dx}{dt} &= -x + 2y \\ \frac{dy}{dt} &= 3x - y\end{aligned}$$

Analyze the stability using linear stability analysis.

20.8 KEYWORDS

Here are the key terms from the chapter summarized :

1. **Simultaneous Differential Equations:** A system of differential equations in which multiple dependent variables change with respect to one another over time. The interactions between variables are described by these equations, which are used to model dynamic systems in various fields, such as biology, economics, and engineering.
2. **Linear System:** A system of differential equations where the variables and their derivatives appear in linear terms, i.e., there are no products or higher powers of the variables. Linear systems are generally easier to solve and analyze.

3. **Nonlinear System:** A system where the differential equations contain nonlinear terms, such as products or powers of the variables. These systems often exhibit complex behaviors like chaos, periodicity, and bifurcations, and are more challenging to solve compared to linear systems.
4. **Equilibrium Point:** A point where the system's state variables do not change, meaning the derivatives are zero. The system remains constant at this point unless disturbed. Equilibrium points can be stable or unstable, depending on the system's dynamics around them.
5. **Stability Analysis:** A method used to determine the long-term behavior of a system near its equilibrium points. Stability analysis can identify whether an equilibrium is stable (system returns to equilibrium), unstable (system moves away from equilibrium), or neutrally stable (system oscillates around equilibrium).
6. **Phase Diagram:** A graphical representation of a dynamical system, where the state variables are plotted against each other. It shows the trajectories of the system over time, providing insights into the system's behavior such as stability, oscillations, and limit cycles.
7. **Eigenvalues and Eigenvectors:** In linear stability analysis, eigenvalues are values that indicate the growth or decay rate of the system's solutions. Eigenvectors are the directions in which the system's state changes. The sign of the eigenvalues determines the stability of equilibrium points (negative eigenvalues indicate stability, positive indicate instability).

20.9 LET'S SUM UP

1. **Simultaneous Differential Equations:** These are systems where multiple variables change with respect to each other over time. They are used to model complex systems in various fields, such as biology, economics, and engineering, where interactions between variables are essential for understanding the dynamics.
2. **Linear vs. Nonlinear Systems:** Linear systems are simpler to solve as they involve equations where variables appear to the first power, while

nonlinear systems are more complex and involve higher powers or products of variables. Nonlinear systems often exhibit more intricate behaviors like chaos or periodic oscillations.

3. **Methods for Solving Systems:** There are two primary methods for solving simultaneous differential equations:
 - **Analytical methods** provide exact solutions, usually for linear systems with constant coefficients.
 - **Numerical methods** are used for more complex systems (especially nonlinear ones), where exact solutions are difficult or impossible to obtain.
4. **Stability Analysis:** Stability analysis helps determine how a system behaves near its equilibrium points. Linear stability analysis is used for systems that can be approximated by linear equations near equilibrium, while nonlinear stability analysis is needed for more complex systems.
5. **Phase Diagrams:** These are visual tools that help represent the behavior of a system over time. By plotting variables on a graph and showing how they evolve, phase diagrams can help us identify equilibrium points, stability, and whether the system will oscillate or settle into steady states.
6. **Eigenvalues and Eigenvectors:** In linear stability analysis, eigenvalues help determine the nature of equilibrium points. If eigenvalues are negative, the equilibrium is stable; if positive, the equilibrium is unstable. Eigenvectors provide the directions along which the system's state evolves.
7. **Applications in Real Life:** Simultaneous differential equations are crucial for modeling and analyzing real-world systems. They can be applied to population dynamics, market equilibrium in economics, oscillations in engineering, and many other dynamic systems where multiple variables interact.
8. **Key Takeaways:** Understanding simultaneous differential equations and their stability helps us predict the long-term behavior of complex systems. Whether it's oscillations in predator-prey models or growth trends in economics, these mathematical tools provide a framework for analyzing and understanding the world around us.

20.10 EXAMINATION-ORIENTED QUESTIONS

1. What are Simultaneous Differential Equations and why are they important in modeling dynamic systems?

- Discuss the role of simultaneous differential equations in modeling systems where multiple variables interact and evolve over time. Highlight applications in biology, economics, and engineering.

2. Differentiate between Linear and Non-Linear Simultaneous Differential Equations.

- Define linear and nonlinear systems, emphasizing the difference in terms of the relationships between variables and the ease of solving the systems.

3. What is the significance of the Jacobian matrix in Linear Stability Analysis?

- Explain how the Jacobian matrix is used to approximate the behavior of a system near equilibrium points. Discuss how its eigenvalues help classify equilibrium points as stable, unstable, or neutrally stable.

4. Describe the difference between Time-Dependent and Time-Independent Systems in the context of Simultaneous Differential Equations.

- Explain the role of time-dependent coefficients in changing the system's behavior over time, compared to time-independent systems where coefficients remain constant.

5. How can Phase Diagrams help in visualizing the dynamics of a system?

- Discuss how phase diagrams provide a graphical representation of the system's trajectories over time, showing the system's equilibrium points, stability, and the nature of its long-term behavior (oscillations, growth, decay, etc.).

6. What is a Limit Cycle in a Phase Diagram, and what does it represent?

- **Explanation:** Define limit cycles and explain their importance in phase diagrams. Discuss how they represent periodic behavior in systems, such as predator-prey models.

7. Solve the following Linear System of Simultaneous Differential Equations:

$$\frac{dx}{dt} = 3x - 2y$$

$$\frac{dy}{dt} = -x + 4y$$

Given the initial conditions $x(0) = 2$, $y(0) = 3$, solve the system and interpret the equilibrium points.

- Use matrix methods or substitution to solve the system. Find the equilibrium points by setting the right-hand sides to zero and solve for x and y .

8. Given the nonlinear Predator-Prey model:

$$\frac{dx}{dt} = x(1 - x - 2y)$$

$$\frac{dy}{dt} = y(1 - 0.5x - y)$$

Find the equilibrium points of the system.

- Set both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$, solve for x and y , and identify the equilibrium points.

9. Using the Linear Stability Analysis, determine the stability of the equilibrium point for the system:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = 3x - y$$

Find the eigenvalues of the Jacobian matrix and discuss the stability of the equilibrium point $(0,0)$.

- Form the Jacobian matrix, calculate its eigenvalues, and analyze their signs to determine the stability of the equilibrium point.

20.11 HINTS TO CHECK YOUR PROGRESS (CYP)

Check Your Progress – I

1. What is the difference between linear and non-linear simultaneous differential equations?

- **Hint:** Refer to Section 20.3.1 and 20.3.2 where linear and non-linear systems are discussed. Pay attention to how the terms involving variables are different in each case.

2. What is the general form of a linear simultaneous differential equation?

- **Hint:** Look at Section 20.3.1 where the general form of linear simultaneous differential equations is provided. Focus on the structure of the equations.

3. How do time-dependent systems differ from time-independent systems in simultaneous differential equations?

- **Hint:** Refer to Section 20.3.3 where both time-dependent and time-independent systems are defined and the key difference is highlighted.

4. Why are non-linear simultaneous differential equations often more challenging to solve than linear ones?

- **Hint:** Check Section 20.3.2 for examples and explanations about the complexities of solving non-linear systems compared to linear ones.

5. What are the typical real-world applications where simultaneous differential equations are used?

- **Hint:** Review Section 20.0 (Introduction) where real-world applications like predator-prey dynamics, market equilibrium, and others are mentioned.

6. Solve the following linear system of simultaneous differential equations:

- **Hint:** See Section 20.4 for methods of solving linear systems, including the use of matrix equations and eigenvalue methods.

7. Solve the following non-linear system:

- **Hint:** Look at Section 20.4.2 for numerical methods such as the Euler method or Runge-Kutta method, which can be applied to non-linear systems like this.

Check Your Progress – II

1. What is the purpose of stability analysis in simultaneous systems?

- **Hint:** Refer to Section 20.5 where the concept of stability analysis is introduced. Focus on its role in understanding whether a system will settle into an equilibrium, oscillate, or grow without bound.

2. What is the role of the Jacobian matrix in linear stability analysis?

- **Hint:** Check Section 20.5.1 for a detailed explanation of the Jacobian matrix and its role in determining the stability of equilibrium points in linear systems.

3. How do eigenvalues of the Jacobian matrix help in determining the stability of an equilibrium point?

- **Hint:** Section 20.5.1 discusses the computation and interpretation of eigenvalues. Understand how the signs of eigenvalues (positive, negative, or imaginary) relate to the stability of equilibrium points.

4. What are the characteristics of a stable equilibrium point in a phase diagram?

- **Hint:** Look at Section 20.6.3 where phase diagrams are explained. Focus on how stable equilibrium points are represented and their effect on system trajectories.

5. How does a phase diagram help in understanding the long-term behavior of a system?

- **Hint:** Section 20.6.3 provides insights into how phase diagrams allow for visualizing the system's behavior over time, such as oscillations, convergence, and divergence.

6. What is the significance of a limit cycle in a phase diagram?

- **Hint:** Refer to Section 20.6.3 where limit cycles are discussed as periodic solutions in phase diagrams that indicate oscillatory behavior.

7. Given the nonlinear predator-prey system, find the equilibrium points of the system:

- **Hint:** Refer to Section 20.3.2 for the method of finding equilibrium points in non-linear systems by setting the right-hand sides of the equations to zero and solving the resulting algebraic system.

8. For the following linear system, determine the stability of the equilibrium point:

- **Hint:** Use the method outlined in Section 20.5.1. Find the Jacobian matrix, calculate the eigenvalues, and determine the stability based on the signs of the eigenvalues.

20.12 SUGGESTED READINGS

- “Introduction to Ordinary Differential Equations” by Earl A. Coddington (1989)
- “Differential Equations and Dynamical Systems” by Lawrence Perko (2001)
- “Nonlinear Dynamics and Chaos” by Steven H. Strogatz (1994)
- “An Introduction to Mathematical Modelling” by Edward A. Bender (2000)
- “Mathematical Methods in the Physical Sciences” by Mary L. Boas (2006)
- “Dynamical Systems and Chaos” by Robert L. Devaney (2003)
- “Applied Mathematics for the Managerial, Life, and Social Sciences” by Soo T. Tan (2009)

M.A. Economics	Lesson No. 21
C.No. : Eco-103	Semester - 1st
	Unit III

APPLICATIONS OF DIFFERENCE AND DIFFERENTIAL EQUATIONS IN ECONOMICS

STRUCTURE

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21.5 Keywords

21.6 Let's Sum Up

21.7 Examination-Oriented Questions

21.8 Hints to Check Your Progress (CYP)

21.9 Suggested Readings

21.0 OBJECTIVES

The main objectives of this lesson are to enable you to:

- ☐ Understand the role of difference and differential equations in economic modeling.
- ☐ Learn the differences between discrete-time and continuous-time economic models.
- ☐ Explore applications of difference equations in investment and capital accumulation models.
- ☐ Analyze business cycle models using difference equations.
- ☐ Study the application of differential equations in continuous-time economic models.
- ☐ Learn about the use of differential equations for modeling capital accumulation and economic growth.
- ☐ Understand the impact of monetary and fiscal policies through dynamic models.

21.1 LEARNING OUTCOMES

By the end of this lesson, you should be able to:

1. Develop the ability to apply difference and differential equations to model economic systems.
2. Analyze dynamic economic models in both discrete and continuous time.

3. Understand how economic growth models are constructed and analyzed using differential equations.
4. Solve investment and capital accumulation models using difference equations.
5. Evaluate business cycles using discrete-time and continuous-time models.
6. Apply differential equations to model monetary policy and fiscal policy effects in economics.
7. Understand and apply optimal control theory in economic decision-making.

21.2 INTRODUCTION

Differential and difference equations play a crucial role in modeling and analyzing dynamic economic systems where multiple variables evolve over time and interact with one another. These equations serve as foundational tools for economists, policymakers, and analysts who seek to understand and predict complex behaviors in areas such as economic growth, market equilibrium, fiscal policy, and population dynamics. These models allow for the study of the interplay between variables like capital, labor, population, and demand, providing valuable insights into how these factors evolve and influence each other.

In economics, difference equations are primarily used to model systems in discrete time, where changes occur at specific intervals, such as yearly, quarterly, or monthly. These models are particularly useful in situations where data is collected at discrete time points. For example, they can be used to model investment decisions, economic growth over successive periods, and even business cycles. One key characteristic of difference equations is that they model how the state of the system evolves from one period to the next, based on the current state and the interactions between variables.

On the other hand, differential equations are used for continuous-time models, where changes in variables happen continuously over time. These equations are employed in models that describe processes occurring in real-time, like capital accumulation or economic growth, where the system's state changes smoothly over time rather than at discrete intervals. Continuous-time models are particularly valuable for studying long-term trends, such as growth patterns, inflation, or interest rates. In these models, the rate

of change of one variable is directly related to other variables in the system, allowing for the modeling of dynamic economic processes in a more fluid and continuous manner.

The application of difference equations in economics is particularly important in modeling discrete-time systems. For instance, in investment and capital accumulation models, businesses or economies can be modeled over successive time periods to predict future levels of investment or capital stock based on current values. Similarly, business cycle models utilize difference equations to model economic fluctuations such as periods of expansion and recession, helping to understand how economic variables evolve through cycles.

Meanwhile, differential equations are widely used in continuous-time economic models to capture the dynamics of processes such as capital accumulation, where the rate of capital investment influences the overall economic growth of a country. These models are useful for studying optimal growth paths, particularly when looking at the Solow Growth Model, which helps to explain long-term economic growth based on capital accumulation and technological progress. Moreover, monetary and fiscal policy analysis uses differential equations to understand the impact of changes in government spending, taxation, or money supply on overall economic performance in real-time.

Overall, both difference and differential equations are indispensable tools in economic modeling, allowing economists to analyze and forecast the impact of various economic policies and real-world changes. These models provide deeper insights into how economies function, evolve, and react to internal and external influences. By studying these mathematical frameworks, economists can offer valuable predictions about market behaviors, growth patterns, and policy outcomes, helping to inform better decision-making and planning at both the macroeconomic and microeconomic levels.

21.3 APPLICATIONS OF DIFFERENCE EQUATIONS IN ECONOMICS

Difference equations are a fundamental mathematical tool used to model dynamic systems where changes occur at discrete intervals over time. In economics, difference equations are especially useful for modeling scenarios where economic variables evolve in a discrete time framework. Unlike continuous models, where changes are represented as instantaneous, discrete models represent changes at specific time points, often in yearly, quarterly, or monthly intervals.

In this section, we will explore the applications of difference equations in various areas of economics, highlighting how they are used to model economic systems such as investment accumulation, business cycles, market equilibrium, and population dynamics. Each subsection will provide an in-depth explanation along with examples to illustrate the relevance of difference equations in these economic models.

21.3.1 Discrete-Time Economic Models

Discrete-time economic models are systems where the state of the economy changes at specific intervals, such as annual, quarterly, or monthly periods. These models are typically used when data is available in time-series form at regular intervals, and they are particularly useful for analyzing economic behaviors that evolve step by step over time.

Example: Simple Economic Growth Model

A basic example of a discrete-time model is the Solow Growth Model, which can be adapted to a discrete-time framework. The equation for output (Y) in terms of capital (K) and labor (L) in a simple version of this model is given by:

$$Y_t = A K_t^\alpha L_t^{1-\alpha}$$

Where:

- Y_t is output at time t ,
- A is the level of technology,
- K_t is capital at time t ,
- L_t is labor at time t ,
- α is the output elasticity of capital.

To make this model discrete, we focus on the evolution of capital over time. Suppose capital accumulation follows the equation:

$$K_{t+1} = K_t + I_t - \delta K_t$$

Where:

- I_t is investment at time t ,
- δ is the depreciation rate of capital.

This discrete equation shows how the capital stock evolves over time based on investment, depreciation, and previous capital stock.

Application: Forecasting Economic Growth

In practice, economists use discrete-time models to forecast economic growth by solving these difference equations based on historical data. For instance, using past data on output, capital, and investment, economists can project future capital accumulation, helping to predict future economic performance. Such models are widely used in forecasting GDP growth, particularly when only annual or quarterly data is available.

21.3.2 Investment and Capital Accumulation

Investment and capital accumulation are central to economic growth models. In discrete time, the evolution of capital in an economy is typically modeled using difference equations that describe how investment contributes to the capital stock while accounting for depreciation.

Example : Investment and Capital Accumulation

Consider an economy where the capital accumulation process is governed by the following difference equation:

$$K_{t+1} = (1 - \delta)K_t + I_t$$

Where:

- K_{t+1} is the capital stock at time $t+1$,
- K_t is the capital stock at time t ,
- δ is the depreciation rate of capital,
- I_t is the investment at time t .

The equation indicates that the capital stock at the next period depends on the current stock of capital (K_t), the depreciation of capital (δK_t), and the investment (I_t) made during the current period.

Application : Analyzing Economic Development

This model is crucial for understanding the process of **capital deepening**, where an increase in investment results in a larger capital stock and potentially higher output in the future. Economists use this model to analyze how changes in investment (e.g., from increased savings or foreign investment) influence capital accumulation and long-term economic growth.

For example, if the government increases investment by offering tax incentives to businesses, this will increase. It and thus lead to higher capital stock in the next period, boosting productivity and output in the future.

21.3.3 Business Cycle Models Using Difference Equations

The business cycle refers to the fluctuations in economic activity, such as periods of expansion followed by recessions. Economists often use discrete-time difference equations to model these cycles by capturing the dynamics of economic variables like output, employment, and inflation.

Example: Simple Business Cycle Model

Consider a basic model of output dynamics that captures the cyclical behavior of an economy. A simple difference equation for output (Y_t) could be written as:

$$Y_{t+1} = \alpha Y_t + \beta I_t$$

Where:

- Y_t is output at time t ,
- I_t is investment at time t ,
- α and β are parameters that determine how past output and investment affect future output.

In this model, output at time $t+1$ depends on both past output (Y_t) and current investment (I_t), capturing the cyclical behavior of the economy. If I_t (which could represent business investment or government spending) increases, it stimulates future output growth, contributing to an economic expansion.

Application : Modeling Recessions and Expansions

This model can be used to simulate different business cycle scenarios. For instance, if investment decreases during a recession, output may decrease in the subsequent period. Conversely, an increase in investment (such as government spending or a rise in business confidence) can lead to economic expansion.

Economists use such models to predict the phases of the business cycle and to design policies aimed at mitigating recessions or sustaining expansions.

Example : Keynesian Cross Model

The Keynesian Cross is a well-known model used to explain business cycles and economic fluctuations. In this model, output (Y) is determined by aggregate demand, which consists of consumption, investment, government spending, and net exports. A difference equation for output in this model can be written as:

$$Y_{t+1} = cY_t + I_t + G$$

Where:

- Y_t is output at time t ,
- c is the marginal propensity to consume,
- I_t is investment at time t ,
- G is government spending (assumed constant).

This model shows how output depends on past output, investment, and government spending. If the government increases G , output will rise in the next period. The model captures the multiplier effect, where an initial increase in spending leads to a larger increase in overall economic activity.

Application : Fiscal Policy Impact

This model is used to assess the impact of fiscal policies on the economy. By adjusting GG (government spending), policymakers can influence aggregate demand and smooth out fluctuations in the business cycle. For example, during a recession, an increase in government spending can boost demand and help stabilize output.

21.3.5 Dynamic Systems in Market Equilibrium

Market equilibrium occurs when the supply and demand in an economy are balanced, with no tendency for change. In a dynamic system, market equilibrium is achieved when the economic variables (such as price, quantity, and production) adjust over time.

Example : Supply and Demand Model

Consider a simple model of supply and demand in a competitive market. The equilibrium price (P_t) adjusts over time as supply and demand change. A basic difference equation for price might be written as:

$$P_{t+1} = P_t + \gamma(D_t - S_t)$$

Where:

- P_t is the price at time t ,
- D_t is demand at time t ,
- S_t is supply at time t ,
- γ is a parameter that determines the rate of price adjustment.

This equation shows that the price at time $t+1$ is influenced by the difference between demand and supply at time t . If demand exceeds supply ($D_t > S_t$), prices will increase, and if supply exceeds demand ($S_t > D_t$), prices will fall. The system will move toward equilibrium over time.

Application : Analyzing Market Adjustments

This model can be used to analyze how markets adjust over time in response to changes in demand or supply. For example, if there is a shock to demand (such as

an increase in consumer preference for a product), this will lead to a higher price in the next period, helping to restore equilibrium.

21.3.6 Population Dynamics and Economics

Population dynamics models help economists understand the effects of population changes on economic variables such as labor supply, consumption, and resource allocation. These models often use difference equations to capture the growth or decline of populations over discrete time periods.

Example : Simple Population Growth Model

A basic population growth model can be written as:

$$P_{t+1} = P_t + rP_t \left(1 - \frac{P_t}{K}\right)$$

Where:

- P_t is the population at time t ,
- r is the growth rate of the population,
- K is the carrying capacity (the maximum population size the environment can support).

This model, called the logistic growth model, describes how a population grows over time, considering both the natural growth rate and the limiting factor of the environment (carrying capacity). As the population approaches the carrying capacity, growth slows down.

Application : Labor Supply and Economic Growth

In economics, population growth impacts the labor force, which in turn affects economic growth. A growing population means more workers are available, potentially increasing output. However, if the population exceeds the carrying capacity of the economy (due to resource constraints), growth may slow, and additional factors like technological innovation or migration may be needed to sustain economic development.

Difference equations are powerful tools for modeling and analyzing dynamic economic systems in discrete time. They help economists understand how key variables

evolve over time, whether it's investment, capital accumulation, market equilibrium, or population dynamics. By using these equations, economists can predict future economic outcomes, evaluate the impact of policies, and identify potential challenges and opportunities within an economy. These models are essential for both theoretical analysis and decision-making in real-world economic scenarios.

Check Your Progress – I

Based on your understanding of the previous section, answer these questions in your own words:

1. What are difference equations, and how do they differ from differential equations in economic modeling?

2. How are discrete-time models used to represent economic systems, and what are some examples of economic phenomena that can be modeled using difference equations?

3. In the context of investment and capital accumulation, how can difference equations help us understand the dynamic behavior of capital stock over time?

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4. How do business cycle models utilizing difference equations help in understanding economic fluctuations such as recessions and expansions?

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5. What role do difference equations play in analyzing dynamic systems in market equilibrium, particularly in supply and demand models?

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6. Explain the importance of population dynamics in economic modeling. How do difference equations help in studying the effects of population changes on economic variables like labor supply?

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7. What is the relationship between discrete-time models of economic growth and capital accumulation? How do difference equations describe this relationship?

8. How do changes in investment affect the evolution of capital accumulation in an economy according to difference equations? Provide an example of such a model.

9. In the context of business cycle models, how do difference equations account for fluctuations in key economic indicators such as output and inflation?

10. What are the key assumptions of population dynamics models in economics, and how do difference equations help in modeling the impact of population growth on economic performance?

11. How does the application of difference equations in market equilibrium help explain how markets adjust over time? What factors might influence this adjustment process?

12. What are the applications of discrete-time models for forecasting economic growth, and how do difference equations play a role in projecting future economic conditions?

21.4 APPLICATIONS OF DIFFERENTIAL EQUATIONS IN ECONOMICS

Differential equations are fundamental to economic modeling, particularly when studying systems that evolve continuously over time. Unlike discrete models, where changes are observed at specific intervals, continuous-time models describe the behavior of economic systems that change smoothly and continuously, such as capital accumulation, market dynamics, and economic growth. These models are used to represent the dynamics of variables like GDP, inflation, interest rates, and consumption, and they play a crucial role in analyzing and forecasting economic trends.

In this section, we explore the applications of differential equations in economics, highlighting key models that use continuous-time approaches. We will also provide examples to illustrate the relevance of these models in understanding real-world economic systems.

21.4.1 Continuous-Time Economic Models

In continuous-time economic models, variables evolve smoothly over time. These models are particularly useful for studying long-term trends and understanding dynamic relationships between economic variables. The key advantage of continuous-time models is that they allow for the analysis of systems where changes are assumed to occur instantaneously, providing more nuanced insights into economic dynamics.

Example : Solow Growth Model

A classic example of a continuous-time economic model is the Solow Growth Model, which describes the long-term economic growth of a country based on capital accumulation, labor growth, and technological progress. The model is expressed by the following differential equation:

$$\frac{dK}{dt} = sY - \delta K$$

Where:

- K is the capital stock at time t,
- Y is the output at time t,
- s is the savings rate,
- δ is the depreciation rate of capital.

The Solow model assumes that output is produced using capital and labor, and a portion of output is saved and invested back into the economy, increasing the capital stock. The differential equation reflects how the capital stock changes over time based on savings and depreciation. This model helps economists understand how savings, investment, and technological progress contribute to economic growth over time.

Application : Economic Growth Forecasting

Using the Solow model, economists can predict long-term economic growth by analyzing how capital accumulation and technological progress will affect GDP over time. For example, a country with higher savings rates can increase its capital stock

more rapidly, leading to higher output in the long run. However, the model also highlights diminishing returns to capital, meaning that as the capital stock grows, additional increases in capital lead to smaller increases in output.

21.4.2 Capital Accumulation and Optimal Growth Models

Capital accumulation models focus on how an economy builds up its capital stock over time. This process is crucial for understanding economic growth, as capital is essential for increasing productivity and output. The optimal growth model seeks to determine the best path for capital accumulation to maximize output or utility over time, taking into account factors such as investment, depreciation, and technological progress.

Example : Ramsey-Cass-Koopmans Model

The Ramsey-Cass-Koopmans (RCK) model is a widely used optimal growth model that examines how an economy chooses its consumption and investment over time to maximize overall welfare. The model is represented by the following system of differential equations:

$$\frac{dC}{dt} = \alpha Y - \rho C - \delta K$$

Where:

- C is consumption at time t,
- Y is output at time t,
- α is the output elasticity of capital,
- ρ is the time preference rate (how much people value current consumption over future consumption),
- δ is the depreciation rate of capital.

The equation shows how consumption evolves over time, based on output, the rate of return on capital, and the depreciation of capital. The goal of this model is to determine the optimal path of consumption and investment that maximizes the present value of future utility.

Application : Optimal Consumption and Investment Decisions

In the real world, governments and policymakers use models like the RCK to determine optimal saving and consumption rates to achieve sustainable economic growth. For instance, during times of economic expansion, it might be optimal for the economy to save and invest more in capital, whereas during recessions, the government might reduce taxes and encourage higher consumption to stimulate the economy.

21.4.3 Monetary and Fiscal Policy Analysis in Continuous Time

Monetary and fiscal policies have profound impacts on the economy, and continuous-time models help economists understand how these policies influence key economic variables like output, inflation, and employment. These models typically involve differential equations that describe the effects of government spending, taxation, and monetary supply changes on economic performance over time.

Example : IS-LM Model in Continuous Time

The IS-LM model (Investment-Savings, Liquidity-Money) is a cornerstone of macroeconomic analysis and is often expressed in continuous time. The IS curve represents equilibrium in the goods market, while the LM curve represents equilibrium in the money market. In continuous time, the model can be written as:

$$\frac{dY}{dt} = a(I(Y, i) - G)$$

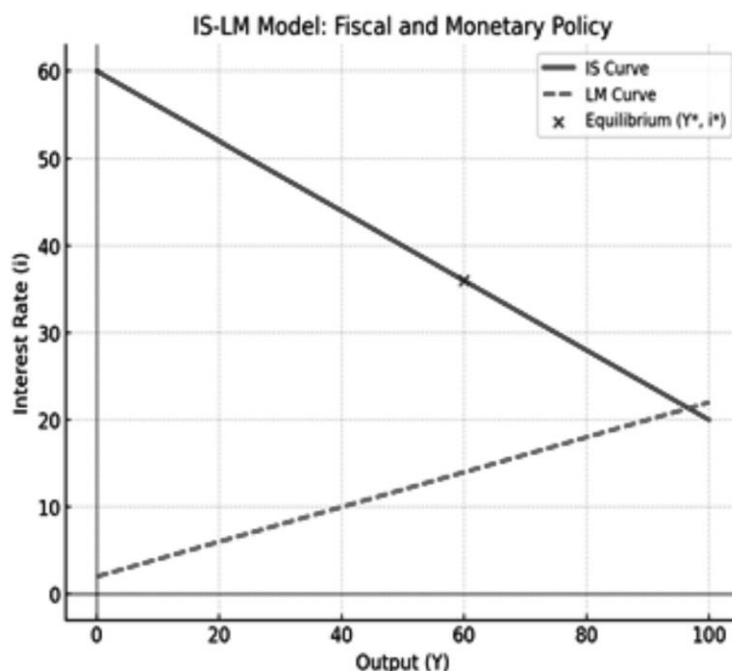
Where:

- Y is output at time t ,
- i is the interest rate,
- $I(Y, i)$ is investment as a function of output and the interest rate,
- G is government spending.

IS Curve (Investment-Savings): The IS curve represents equilibrium in the goods market, where investment (I) equals savings (S). The curve has a negative slope because as the output (Y) increases, the interest rate (i) decreases to restore equilibrium in the goods market. This is based on the idea that higher output encourages more investment, but as investment increases, interest rates fall, balancing the market.

LM Curve (Liquidity Preference-Money Supply): The LM curve represents equilibrium in the money market, where the demand for money equals the supply of money. The curve has a positive slope because as output (Y) increases, the demand for money increases, pushing up the interest rate (i) for a given money supply.

Equilibrium Point: The intersection of the IS and LM curves represents the **equilibrium** in both the goods and money markets, where both output (Y) and the interest rate (i) stabilize. At this point, the economy is in a state of balance, with no pressures for change.



Impact of Policies:

- Fiscal Policy (Shift in IS curve):** An increase in government spending or investment shifts the IS curve to the **right**, increasing output and raising the interest rate.
- Monetary Policy (Shift in LM curve):** An increase in the money supply shifts the LM curve downward (to the right), decreasing the interest rate and increasing output.

The model captures how output changes in response to changes in investment, government spending, and the interest rate. By analyzing how these variables evolve over time, economists can predict the impact of fiscal policies (such as changes in government spending or taxes) and monetary policies (such as changes in the money supply or interest rates) on economic performance.

Application : Policy Design and Economic Stabilization

In practice, central banks and governments use continuous-time models like the IS-LM model to design policies aimed at stabilizing the economy. For example, during a recession, a government might increase spending (G) to shift the IS curve to the right, thereby increasing output. Simultaneously, the central bank might reduce interest rates to stimulate investment, shifting the LM curve down and increasing output.

21.4.4 Resource Allocation and Optimal Control Theory

Optimal control theory is a branch of mathematics that deals with finding the best possible control policies for a dynamic system. In economics, it is used to model situations where resources (such as capital, labor, or natural resources) need to be allocated optimally over time to achieve specific objectives, such as profit maximization or welfare optimization.

Example : Optimal Resource Allocation

A simple example of optimal resource allocation is the problem of managing a renewable resource, such as a fishery or a forest, to maximize long-term economic benefits. The dynamics of the resource stock can be modeled by a differential equation, such as:

$$\frac{dR}{dt} = rR \left(1 - \frac{R}{K} \right) - C$$

Where:

- R is the resource stock at time t ,
- r is the growth rate of the resource,
- K is the carrying capacity of the environment,
- C is the rate of consumption or extraction.

The objective is to choose the optimal rate of consumption (C) that maximizes the economic benefits of resource extraction over time while ensuring the sustainability of the resource.

Application : Environmental Economics

Optimal control models are often used in environmental economics to manage natural resources sustainably. For example, policymakers use such models to determine the best harvest rates for fisheries or the optimal timber extraction rate from forests. By solving these models, they can ensure that the resource is not overexploited while still maximizing economic returns.

21.4.5 Market Dynamics and Equilibrium Adjustments

In economics, market dynamics refer to the forces that influence supply and demand in a market, causing prices and quantities of goods and services to adjust over time. These dynamics can be influenced by various factors, such as changes in consumer preferences, technological advances, government policies, and external economic shocks. Equilibrium adjustments refer to how the market moves toward a new equilibrium state when these factors change, and the system is out of balance.

Market Equilibrium in Economics

Market equilibrium occurs when the quantity demanded of a good or service equals the quantity supplied, and there is no incentive for the price to change. In continuous-time models, this adjustment process is typically modeled using differential equations, which describe how variables such as price (P) and quantity (Q) adjust over time to reach equilibrium.

Modeling Market Dynamics with Differential Equations

In continuous time, market dynamics can be modeled by using the following differential equations for price and quantity. A simple market model could be described as:

$$\frac{dP}{dt} = a(D(P) - S(P))$$

Where:

- P is the price of the good,
- $D(P)$ is the demand for the good, which is a function of price,
- $S(P)$ is the supply of the good, which is also a function of price,
- α is a constant that represents the speed of price adjustment.

In this equation, the price P adjusts over time in response to the difference between demand $D(P)$ and supply $S(P)$. If demand exceeds supply $D(P) > S(P)$, prices will rise, and if supply exceeds demand $S(P) > D(P)$, prices will fall. This model captures how markets adjust to reach a new equilibrium after a shock or change in market conditions.

Adjustment Mechanism:

When the market is not in equilibrium, prices and quantities move towards the equilibrium position over time. This adjustment process can happen through the following steps:

- **Excess Demand:** If demand is greater than supply, there is upward pressure on prices, which encourages producers to increase supply and consumers to reduce their demand.
- **Excess Supply:** If supply exceeds demand, there is downward pressure on prices, which reduces production and encourages consumers to buy more.

The rate of adjustment depends on the responsiveness of demand and supply to changes in price, represented by the slopes of the demand and supply curves. The speed of adjustment is influenced by the constant α , which captures how quickly prices and quantities adjust in the market.

Practical Example:

Consider a market for a commodity, such as oil. If a new technological breakthrough increases the supply of oil, the supply curve shifts to the right, leading to a temporary excess supply. In response, the price of oil falls over time as suppliers

compete to sell their excess stock. The adjustment process continues until the market reaches a new equilibrium price and quantity, where supply matches demand.

21.4.6 Growth Models and Sustainability

Economic growth models describe how an economy's output (GDP) increases over time. Sustainable growth refers to the idea of achieving long-term economic growth while maintaining environmental health, social equity, and resource availability. Growth models help economists understand how economies expand, the factors that drive this expansion, and how growth can be achieved without depleting resources or causing environmental damage.

Key Features of Growth Models

There are different types of growth models used to explain how economies grow, with the Solow Growth Model being one of the most widely discussed in economics. Growth models typically focus on factors like capital accumulation, technological progress, and labor force growth as the main drivers of economic growth.

The Solow Growth Model:

The **Solow Growth Model** is a standard neoclassical model that explains how economic output grows over time based on three key factors:

1. **Capital Accumulation:** As more capital (e.g., machines, infrastructure) is accumulated, it enhances productivity, leading to higher output.
2. **Labor Growth:** An increasing labor force boosts the capacity to produce goods and services.
3. **Technological Progress:** Technological improvements increase the productivity of both labor and capital, driving growth.

The continuous-time version of the Solow model can be expressed using the following differential equation:

$$\frac{dK}{dt} = sY - \delta K$$

Where:

- K is the capital stock,
- Y is output,
- s is the savings rate,
- δ is the depreciation rate of capital.

This equation shows how capital accumulates over time, where a portion of output is saved and invested back into the economy to increase capital. Depreciation (δ) reduces the capital stock, while investment (sY) increases it. The economy eventually reaches a steady state, where capital accumulation balances depreciation.

Sustainability in Growth Models

Sustainable growth models extend traditional growth models by incorporating environmental factors, such as resource constraints and pollution, into the analysis. Sustainable growth seeks to achieve long-term economic growth while ensuring that natural resources are not depleted and the environment is preserved.

One approach to modeling sustainable growth is through the Environmental Kuznets Curve (EKC), which suggests that economic growth initially leads to environmental degradation, but after a certain level of income, growth becomes less harmful to the environment. As income rises, there is greater demand for cleaner technologies and more resources are available for environmental protection.

Example: Growth with Environmental Constraints

Consider a growth model that includes an environmental factor, such as pollution, as a negative externality. The differential equation for sustainable growth might be:

$$\frac{dY}{dt} = aK - f3E$$

Where:

- Y is output,
- K is capital,

- E is the environmental factor (e.g., pollution),
- α is the output elasticity of capital,
- β represents the impact of pollution on growth.

In this model, economic output grows with the accumulation of capital, but pollution (E) reduces the potential for growth over time. Sustainable growth is achieved when the economy manages to balance capital accumulation with environmental preservation, such that growth does not come at the cost of environmental degradation.

Practical Example : Renewable Resource Management

Consider an economy that relies on a renewable resource, such as fish in a fishery. A model for the sustainable use of this resource might look like:

$$\frac{dR}{dt} = rR \left(1 - \frac{R}{K}\right) - C$$

Where:

- R is the stock of the renewable resource,
- r is the growth rate of the resource,
- K is the carrying capacity of the environment,
- C is the rate of consumption or extraction.

This equation describes the population growth of a resource (e.g., fish) over time, accounting for natural growth and consumption. The goal is to determine the optimal extraction rate (C) that maximizes economic returns without depleting the resource.

Example : Sustainable Growth Model

A sustainable growth model might involve the following differential equation, which describes the growth of an economy's output while accounting for environmental constraints:

$$\frac{dY}{dt} = \alpha K^\alpha \beta L^{1-\alpha} - \gamma Y$$

Where:

- Y is output,
- K is capital,
- L is labor,
- E is the environmental factor (e.g., pollution or resource depletion),
- α , β , and γ are parameters that describe the efficiency of capital, labor, and the environmental factor.

The model shows that economic growth depends on capital, labor, and environmental resources. Sustainable growth is achieved by balancing the need for capital and labor with the need to protect the environment.

Application : Environmental Sustainability

This type of model is crucial for understanding how economies can grow while minimizing environmental damage. For example, policymakers can use these models to assess the impact of industrial activities on the environment and design policies that encourage green technologies or sustainable practices, thus ensuring long-term economic stability and ecological health.

Differential equations are indispensable tools in economics for modeling continuous-time processes. They allow economists to study the dynamic evolution of economic variables, from capital accumulation and market equilibrium to population dynamics and sustainability. These models provide valuable insights into the behavior of economic systems and help policymakers make informed decisions to promote sustainable growth, economic stability, and efficient resource allocation. Through examples such as the Solow growth model, optimal growth models, and market dynamics, we see how differential equations are applied to real-world economic challenges, guiding decisions that affect the long-term health of economies.

Check your Progress – II

Based on your understanding of the previous section, answer these questions in your own words:

1. What is the difference between continuous-time models and discrete-time models in economics? How are differential equations used in continuous-time models?

2. How does the Solow Growth Model use differential equations to describe the relationship between capital, labor, and technological progress?

3. Explain the Ramsey-Cass-Koopmans model of optimal growth. What role do differential equations play in determining the optimal path of consumption and investment over time?

4. How do differential equations help in analyzing the effects of monetary and fiscal policies on economic growth? Provide an example of a continuous-time model for fiscal policy analysis.

5. What is the importance of resource allocation models in economics, and how do optimal control theory and differential equations contribute to efficient resource management?

6. Explain how the dynamics of market equilibrium are modeled using differential equations. How do supply and demand changes affect equilibrium prices over time?

7. In the context of environmental economics, how do differential equations model sustainable economic growth? Discuss the role of environmental constraints in growth models.

8. How are growth models in economics, such as the Solow model or endogenous growth models, used to predict long-term economic performance? Explain the significance of differential equations in these models.

9. Describe how optimal control theory is used to allocate resources efficiently over time. How do differential equations represent the optimal path in resource management?

10. What is the role of capital accumulation in economic growth models, and how do differential equations describe the accumulation of capital over time?

11. How do differential equations model the adjustments in prices and outputs in response to economic shocks or policy interventions? Provide an example of a model for price adjustments in a market.

12. What are the applications of differential equations in understanding economic sustainability? How do these models guide decisions in areas like environmental protection, energy use, and natural resource management?

21.5 KEYWORDS

1. **Differential Equation:** A mathematical equation involving derivatives, which represents how a variable changes with respect to another variable (usually time). In economics, it is used to model continuous-time systems, such as economic growth and capital accumulation.
2. **Capital Accumulation:** The process by which an economy builds its capital stock over time through investments. Differential equations are used to model how capital accumulates, depreciates, and contributes to economic growth.
3. **Optimal Growth Models:** Economic models that describe the best path for capital accumulation, consumption, and investment to maximize welfare over time. These models often use differential equations to determine the optimal trajectory of economic variables.

4. **Monetary Policy:** Actions by a central bank or monetary authority to manage the money supply and interest rates. Differential equations help model how changes in monetary policy influence output, inflation, and economic growth over time.
5. **Fiscal Policy:** Government policies related to taxation and spending that influence the economy. Differential equations are used to understand the dynamic effects of fiscal policy on variables like national income, employment, and inflation.
6. **Resource Allocation:** The process of distributing resources such as capital, labor, and natural resources across different sectors of the economy. Optimal control theory and differential equations are used to determine the most efficient allocation over time.
7. **Market Equilibrium:** The condition where the supply of goods and services equals demand at a given price level. Differential equations are used to model how prices and quantities adjust over time to reach equilibrium in competitive markets.
8. **Sustainability:** Economic growth that does not deplete natural resources or harm the environment in the long term. Differential equations model sustainable economic growth by accounting for the limits of resources and environmental factors that constrain economic activities.

21.6 LET'S SUM UP

1. **Role of Differential Equations in Economics:** Differential equations are fundamental in economics, particularly in continuous-time models. They allow economists to model how economic variables evolve over time, providing insights into the dynamics of growth, investment, policy effects, and market adjustments.
2. **Applications in Economic Growth:** Models like the Solow Growth Model and Ramsey-Cass-Koopmans model use differential equations to describe capital accumulation, optimal growth paths, and the effects of technological progress and labor force changes on long-term economic growth.

3. **Monetary and Fiscal Policy Analysis:** Continuous-time models help in analyzing the dynamic impact of monetary and fiscal policies. Differential equations provide a framework for understanding how changes in interest rates, government spending, and taxation influence economic activity over time.
4. **Optimal Control Theory in Resource Allocation:** Differential equations, combined with optimal control theory, are used to model how resources like capital and labor should be allocated over time to maximize output or welfare, while considering constraints like depreciation and environmental factors.
5. **Market Dynamics and Equilibrium:** In markets, differential equations are used to describe how prices and quantities adjust in response to changes in supply, demand, and external factors. These models help predict the path to market equilibrium and the effect of shocks.
6. **Environmental Economics and Sustainability:** Differential equations also play a key role in modeling sustainable growth by accounting for resource limitations and environmental impacts. These models guide decision-making on how to achieve growth without depleting resources or harming ecosystems.
7. **Importance of Continuous-Time Models:** Continuous-time models, represented by differential equations, offer more flexibility than discrete-time models. They are essential for understanding real-time dynamics of economic systems, especially when analyzing continuous processes like inflation, output growth, or capital accumulation.
8. **Future Directions in Economic Modeling:** As computational tools improve, the use of complex differential equation models in economics will continue to expand. These models will enable economists to better simulate and predict the behavior of economies, providing more accurate policy recommendations and growth strategies.

21.9 EXAMINATION-ORIENTED QUESTIONS

1. **What are the main differences between discrete-time and continuous-time models in economics?**

- Discrete-time models analyze changes in economic variables at specific intervals (e.g., yearly, quarterly).
- Continuous-time models describe the smooth, continuous evolution of variables over time (e.g., capital accumulation or economic growth).
- Discrete-time models are easier to compute but may ignore real-time dynamics.
- Continuous-time models provide more accurate reflections of ongoing processes like inflation, interest rates, or growth.
- Examples: Solow Growth Model (continuous) vs. simple investment models (discrete).

2. **How does the Solow Growth Model use differential equations to model long-term economic growth?**

- The Solow model uses differential equations to describe how capital (K) accumulates over time.
- The key equation is $\frac{dK}{dt} = sY - \delta K$, where s is the savings rate, and δ is the depreciation rate.
- The equation shows that capital accumulation depends on savings (investment) and depreciation.
- The model helps predict the steady-state level of capital and output, where capital accumulation equals depreciation.
- This continuous equation gives insights into long-term growth and the impact of savings and technological progress.

3. **What is the role of monetary and fiscal policies in economic models? How do differential equations help in analyzing their effects?**

- Monetary policy involves controlling the money supply and interest rates to stabilize the economy, while fiscal policy focuses on government spending and taxation.

- Differential equations model how these policies affect variables like output, inflation, and interest rates over time.
- In the IS-LM model, fiscal policy shifts the IS curve, and monetary policy shifts the LM curve.
- Differential equations in these models show how adjustments in policy influence economic equilibrium in both the goods and money markets.
- Policymakers use these models to predict how changes in tax rates or interest rates will affect economic variables in the short and long term.

4. What is the significance of optimal control theory in resource allocation, and how does it utilize differential equations?

- Optimal control theory helps determine the best way to allocate limited resources over time to maximize a certain objective (e.g., profit, welfare).
- Differential equations are used to model the dynamics of resource allocation (e.g., how capital, labor, or land evolves over time).
- The objective is to maximize a function, such as total output or utility, subject to resource constraints.
- Example: In managing renewable resources (e.g., fisheries or forests), optimal control theory can determine the optimal extraction rate while ensuring sustainability.
- These models are essential for policymakers aiming to achieve the best outcomes while balancing competing interests like economic growth and environmental sustainability.

5. How do business cycle models using difference equations help in understanding economic fluctuations?

- Business cycle models describe fluctuations in economic activity, such as expansions and recessions, and are often modeled using difference equations.
- Difference equations model how output (Y_t) changes over time based on investment, government spending, and other variables.

- These models show how an initial shock (e.g., a decrease in investment) can lead to lower output, which may persist until the economy adjusts back to equilibrium.
- The models are used to forecast the timing and severity of business cycles and to design appropriate policy interventions.

6. Explain how differential equations model the dynamics of market equilibrium. How do supply and demand affect market prices over time?

- Market equilibrium occurs when supply equals demand at a given price. Differential equations describe how prices adjust over time to restore equilibrium when disrupted.
- The equation $\frac{dP}{dt} = \alpha(D(P) - S(P))$ shows how price (P) changes in response to the difference between demand (D(P)) and supply (S(P)).
- If demand exceeds supply, prices rise; if supply exceeds demand, prices fall. The market adjusts until $D(P) = S(P)$, restoring equilibrium.
- Differential equations model how quickly the price adjusts, depending on the responsiveness of demand and supply.
- These models are crucial for understanding how external factors (e.g., technological changes, policy changes) affect market prices and quantities over time.

7. In the context of population dynamics, how do difference equations help model the effects of population growth on economic variables like labor supply?

- Population dynamics models use difference equations to describe how populations evolve over time, considering birth rates, death rates, and migration.
- Population growth influences economic variables like labor supply, which in turn affects output and productivity.

- Example: A growing population increases the labor force, leading to higher output, but at some point, diminishing returns to labor may set in.
- These models help policymakers understand how population growth impacts economic performance and inform decisions regarding resource allocation and economic planning.

8. Given the following equation for capital accumulation in an economy:

$$K_{t+1} = 0.9K_t + 50$$

If $K_0=200$, calculate the capital stock after 5 periods.

- **Explanation:** This is a recursive difference equation. Start by calculating K_1 , then use K_1 to calculate K_2 , and so on up to K_5 .
- **Steps:**
 - $K_1=0.9 \times 200 + 50$
 - $K_2=0.9 \times K_1 + 50$
 - Continue until K_5 .
- This shows how capital accumulates over time based on the given savings and depreciation rate.

9. Solve the following difference equation for output growth:

$$Y_{t+1} = 1.2Y_t + 300$$

Given that $Y_0=1000$, find the output at $t = 3$.

- **Explanation:** This recursive difference equation models the growth of output, with a constant increase of 300 units in each period.
- **Steps:**
 - $Y_1=1.2 \times 1000 + 300$
 - $Y_2=1.2 \times Y_1 + 300$
 - $Y_3=1.2 \times Y_2 + 300$

- Calculate each step to find Y_3 , demonstrating how output grows over time.

10. In a continuous-time Solow Growth Model, the capital accumulation equation is given by:

$$\frac{dK}{dt} = 0.2Y - 0.05K$$

If $Y = 500$ and $K = 100$, calculate the rate of change of capital $\left(\frac{dK}{dt}\right)$.

- Explanation: Substitute the given values into the equation to find the rate of change of capital. This tells you how fast the capital stock is growing or shrinking at the given point in time.
- Steps:
 - $\frac{dK}{dt} = 0.2 \times 500 - 0.05 \times 100$
 - Simplify to find $\frac{dK}{dt}$.
- This shows how capital accumulates or depreciates based on output and depreciation rates.

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INTRODUCTION TO LINEAR PROGRAMMING

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22.0 Objectives

The main objectives of this lesson are to enable you to:

- Discuss the fundamental concepts and terminology of Linear Programming (LP).
- Learn how to formulate linear programming problems from real-world scenarios.
- Master the graphical method for solving linear programming problems with two variables.
- Explore the Simplex Method for solving LP problems involving multiple variables.
- Explain the concept of duality in linear programming and its significance.
- Describe the importance of sensitivity analysis in LP and its application in decision-making.

- Discuss the limitations and assumptions of linear programming models in real-world applications.

22.1 Learning Outcomes

By the end of this lesson, you should be able to:

1. Define key terms such as decision variables, objective function, constraints, and feasible region in the context of LP.
2. Formulate linear programming problems from practical economic or business situations.
3. Solve LP problems using the graphical method and interpret the results.
4. Use the Simplex Method to solve LP problems with more than two variables.
5. Discuss the duality principle in linear programming and how it offers insights into economic problems.
6. Perform sensitivity analysis to assess how changes in parameters affect the optimal solution.
7. Critically evaluate the assumptions and limitations of linear programming models when applied to complex, real-world problems.

22.2 Introduction

Linear Programming (LP) is a powerful mathematical technique used to optimize a linear objective function, subject to a set of linear constraints. The primary goal of linear programming is to determine the best possible outcome in a mathematical model, often by maximizing or minimizing a particular quantity, such as profit, cost, or time, given certain restrictions on resources. It is a crucial tool in fields like economics, business, operations research, and industrial engineering, where optimal resource allocation is necessary for effective decision-making.

At its core, linear programming involves finding the values of decision variables that optimize an objective function, which is a linear equation. These decision variables represent quantities to be determined, such as the amount of a product to be manufactured

or the number of units of a service to be provided. The objective function could represent various economic goals, such as maximizing profit or minimizing costs. The problem is subject to constraints, which are linear inequalities or equations that limit the values of decision variables, such as resource limitations, budgetary constraints, or capacity restrictions. The solution to the linear programming problem lies in the feasible region, which is the set of all possible values that satisfy the constraints. The optimal solution is the point within this region where the objective function is either maximized or minimized, depending on the problem's goal.

Linear programming has become a vital tool in modern economics and business because it provides a structured method for making decisions in situations involving limited resources. In economics, LP is widely used for resource allocation, where scarce resources such as labor, capital, and raw materials need to be allocated efficiently to achieve the best possible economic outcomes. For instance, a company may use linear programming to determine the most profitable combination of products to manufacture, considering production capacity, labor, and material costs. Similarly, it can be applied to optimize supply chains, determine the most cost-effective transportation routes, and manage inventories.

One of the most common methods used to solve linear programming problems is the graphical method, particularly for problems involving two variables. This method involves plotting the constraints on a graph, identifying the feasible region, and then evaluating the objective function at the vertices of this region to find the optimal solution. While this method is useful for small problems, it is limited to problems with only two variables. For larger problems, the Simplex Method is often used. This method is a more sophisticated algorithm that iterates through possible solutions to find the optimal one, making it applicable to problems with many variables.

Linear programming also has important theoretical extensions, such as duality, which provides a different perspective on the optimization problem, and sensitivity analysis, which studies how changes in the parameters of the problem (such as coefficients of the objective function or the constraints) affect the optimal solution. These concepts enhance the applicability of LP to real-world economic problems, allowing for deeper insights into the impact of changes in assumptions or conditions.

Overall, linear programming is an indispensable tool for decision-making in economics and business. It allows managers, economists, and policymakers to make informed decisions that maximize efficiency and profitability while adhering to constraints, ensuring optimal use of resources.

22.3 Key Concepts and Terminology

Linear Programming (LP) is a mathematical technique used to solve optimization problems where the goal is to maximize or minimize a linear objective function subject to a set of linear constraints. Linear programming provides a method to optimize outcomes such as profit, cost, or efficiency, while adhering to certain restrictions or limitations. The terminology and concepts of LP are critical for understanding how to formulate problems, apply solution methods, and interpret results. This section will cover in-depth the key concepts and terminology that lay the foundation for learning and applying linear programming.

1. Decision Variables

In any linear programming problem, the decision variables represent the quantities that need to be determined in order to achieve the objective. These variables are the unknowns in the model, and solving for them provides the solution to the optimization problem. In a practical sense, decision variables are typically the quantities of products to produce, resources to allocate, or investments to make.

□ Examples:

- In a **production problem**, the decision variables could be x_1 and x_2 , representing the number of units of Product 1 and Product 2, respectively.
- In a **transportation problem**, the decision variables could be the amount of goods to be transported from each source to each destination.

Decision variables are often subject to non-negativity constraints, which means that they cannot take negative values. This reflects real-world limitations, such as not being able to produce or transport a negative quantity of a product.

□ **Non-Negativity Constraints:**

$$x_1 \geq 0, x_2 \geq 0$$

These constraints are important because negative values for decision variables, such as negative production or negative amounts of resources, are infeasible.

2 Objective Function

The objective function in Linear Programming is the mathematical expression that defines the goal of the optimization problem. The function is a linear equation representing the quantity that is to be maximized or minimized, such as profit, cost, time, or efficiency. The objective function contains the decision variables, and the goal is to find the values of these variables that either maximize or minimize the function.

- **Maximization Problem:** In a maximization problem, the objective is to maximize profit or revenue. The objective function is written as:

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

where $c_1, c_2, \dots, c_n$ are the profit coefficients associated with each decision variable $x_1, x_2, \dots, x_n$, respectively.

- **Minimization Problem:** In a minimization problem, the objective is to minimize costs or time. The objective function in a minimization problem can be written as:

$$\text{Minimize } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

where $c_1, c_2, \dots, c_n$ are the cost coefficients for each decision variable.

□ **Real-world Example:**

- In a manufacturing problem, the objective might be to maximize profit. The objective function could represent the profit from selling each product, and we want to find the optimal number of units to produce.
- In a logistics problem, the objective might be to minimize transportation costs. The objective function would represent the total transportation cost, and we seek to determine the most cost-effective route for distributing goods.

3 Constraints

Constraints in Linear Programming are conditions or restrictions that limit the values that the decision variables can take. These constraints represent limitations in the system, such as available resources, production capacity, or time. Constraints are typically expressed as linear inequalities or equations, which define the feasible region.

- **Resource Constraints:** A common type of constraint is a resource constraint, such as limited labor hours, materials, or budget. For example, if a factory has only 100 labor hours available, the constraint might be:

$$2x_1 + 3x_2 \leq 100$$

where $2x_1$ and $3x_2$ represent the labor hours required to produce one unit of Product 1 and Product 2, respectively. This inequality ensures that the total labor hours do not exceed the available amount.

- **Budget Constraints:** Budget constraints represent the amount of money available for different activities. For example:

$$x_1 + 8x_2 = 500$$

where $5x_1$ and $8x_2$ represent the costs of producing one unit of Product 1 and Product 2, respectively.

- **Capacity Constraints:** In production or manufacturing problems, capacity constraints might represent limitations in machine time, warehouse space, or raw material availability.
- **Non-Negativity Constraints:** As mentioned earlier, the decision variables are often required to be non-negative because negative values do not make sense in most real-world problems (e.g., negative production quantities).

$$x_1 \geq 0, x_2 \geq 0$$

4 Feasible Region

The **feasible region** is the set of all possible values for the decision variables that satisfy all the constraints of the linear programming problem. In other words, it is the region

where all constraints intersect and are simultaneously satisfied. The feasible region represents the set of all potential solutions to the problem, although not all of these solutions are necessarily optimal.

- The feasible region is typically a polygon (in two dimensions) or a polyhedron (in higher dimensions), and it is bounded by the constraints.
- Any solution within the feasible region is considered feasible, but the optimal solution lies at one of the vertices (or corner points) of the feasible region.
- **Graphical Representation:** When solving LP problems with two variables, the feasible region is represented graphically as the area where all the constraint lines or curves intersect. The optimal solution is usually found at one of the vertices of this region.
- **Example:** In a two-variable problem, constraints such as $x_1 + x_2 \leq 10$ and $x_1 \geq 0, x_2 \geq 0$ define a feasible region that is a triangle, and the optimal solution would be located at one of the corner points of this triangle.

5. Optimal Solution

The optimal solution is the set of values for the decision variables that maximizes or minimizes the objective function, while satisfying all the constraints. The optimal solution lies at a point in the feasible region, often at a vertex (corner point), where the objective function reaches its highest or lowest possible value.

- **Maximization:** In problems where the goal is to maximize a quantity (e.g., profit), the optimal solution occurs at the vertex of the feasible region where the objective function reaches its highest value.
- **Minimization:** In problems where the goal is to minimize a quantity (e.g., cost), the optimal solution occurs at the vertex where the objective function achieves its lowest value.

The corner-point theorem of linear programming states that the optimal solution will always lie at one of the vertices (or corner points) of the feasible region, assuming the problem has a finite and bounded feasible region. This is an essential property of LP because it allows for efficient solution methods, such as evaluating the objective function at the vertices to find the optimal solution.

6 Linear Programming Models

Linear programming models involve the representation of real-world problems using linear relationships. These models include the objective function, decision variables, and constraints, all of which are linear.

- **Product Mix Problem:** The product mix problem is one of the most common applications of LP. It involves deciding how much of each product to produce to maximize profit while respecting constraints on resources, labor, and materials.
- **Transportation Problem:** This problem involves determining the most cost-effective way to transport goods from multiple sources (such as factories) to multiple destinations (such as warehouses) while satisfying supply and demand constraints.
- **Diet Problem:** The diet problem involves selecting the quantities of various food items to minimize cost while meeting nutritional requirements. Constraints ensure that the total diet meets specific nutritional needs, and the objective is to minimize cost.
- **Portfolio Optimization:** LP is also used in finance to determine the optimal mix of assets in an investment portfolio. The goal is to maximize returns or minimize risk while adhering to constraints such as budget, risk tolerance, and investment proportions.

7. Graphical Method

The graphical method is a technique used for solving linear programming problems with two decision variables. It involves plotting the constraints on a graph, determining the feasible region, and evaluating the objective function at the vertices of the feasible region to find the optimal solution.

- **Plotting Constraints:** Each constraint is plotted as a straight line, and the feasible region is the area where all constraints are satisfied.
- **Objective Function Evaluation:** Once the feasible region is identified, the objective function is evaluated at the vertices (corner points) of the feasible region. The point that gives the maximum or minimum value is the optimal solution.

While the graphical method is effective for problems with two variables, it is not suitable for problems with more than two variables. For larger problems, methods like the **Simplex Method** are used.

Linear Programming is a highly effective tool for optimization, with a variety of applications in economics, business, production, and logistics. Understanding the key concepts and terminology — from decision variables and objective functions to constraints, feasible regions, and optimal solutions — is fundamental for applying LP techniques. This knowledge provides the basis for solving complex real-world problems, guiding decisions in areas such as resource allocation, production scheduling, and financial management. By mastering these core concepts, one can develop proficiency in formulating, solving, and interpreting linear programming models to make informed and optimized decisions.

22.4 Graphical Method for Solving Linear Programming Problems

The graphical method is one of the simplest techniques for solving Linear Programming (LP) problems with two variables. It provides a visual way to understand and solve optimization problems by finding the feasible region and determining the optimal solution using graphical representations. Though it is a powerful tool for small-scale problems with just two decision variables, the graphical method is not suitable for problems with more than two variables due to the limitations of visualizing multi-dimensional spaces. In this section, we will explore the introduction to the graphical method, the steps to solve LP problems using this method, and the limitations that come with it.

22.4.1 Introduction to Graphical Method

Linear Programming problems typically involve maximizing or minimizing a linear objective function subject to a set of linear constraints. The graphical method is a technique used to solve these problems when there are only two decision variables.

In the graphical method:

- Each constraint in the problem is represented by a linear equation, which divides the plane into two regions: one that satisfies the constraint (feasible region) and one that does not.
- The objective function is also represented as a line (or plane in higher dimensions),

and the goal is to find the point within the feasible region that maximizes or minimizes the objective function.

The graphical method works by:

1. Plotting the constraints on a graph, typically in two dimensions.
2. Determining the feasible region where all the constraints are satisfied.
3. Identifying the vertices of the feasible region, where the optimal solution is likely to lie.
4. Evaluating the objective function at these vertices to find the optimal solution.

The graphical method is very intuitive and gives a clear visual representation of the problem, allowing us to observe how changes in constraints or the objective function affect the optimal solution.

22.4.2 Steps to Solve Using Graphical Method

The graphical method involves several systematic steps to solve LP problems. These steps help us transform a word problem into a visual model and determine the optimal solution. The steps to solve a linear programming problem using the graphical method are as follows:

Step 1: Write the LP Problem in Standard Form

Before solving the problem, it is crucial to represent it in standard linear programming form, which includes:

- **Objective function:** Define what is to be maximized or minimized.
- **Constraints:** Express each constraint as a linear inequality or equation.
- **Non-negativity conditions:** Ensure that decision variables cannot take negative values.

For example, consider the following simple LP problem:

$$\text{Maximize } Z = 3x_1 + 2x_2$$

Subject to:

$$x_1 + x_2 \leq 4$$

$$2x_1 + x_2 \leq 5$$

$$x_1 \geq 0, x_2 \geq 0$$

Here, the objective is to maximize the function $Z = 3x_1 + 2x_2$, subject to the given constraints and non-negativity conditions.

Step 2: Plot the Constraints on a Graph

For each constraint, we will plot it as a straight line on the graph. The inequality signs in the constraints determine which side of the line is feasible.

- To plot the line, convert each inequality into an equation by replacing the inequality sign with an equal sign.
 - For $x_1 + x_2 \leq 4$, plot the line $x_1 + x_2 = 4$.
 - For $2x_1 + x_2 \leq 5$, plot the line $2x_1 + x_2 = 5$.
 - For $x_1 \geq 0$ and $x_2 \geq 0$, the constraints indicate that the solution is confined to the first quadrant (positive values of x_1 and x_2).
- **Graphing Method:**
 - To graph $x_1 + x_2 = 4$, find the intercepts. Set $x_1 = 0$, then $x_2 = 4$. Set $x_2 = 0$, then $x_1 = 4$. Plot these points and draw the line.
 - Similarly, graph $2x_1 + x_2 = 5$ by finding the intercepts. Set $x_1 = 0$, then $x_2 = 5$. Set $x_2 = 0$, then $x_1 = 2.5$.

Step 3: Identify the Feasible Region

The feasible region is the area on the graph where all the constraints are satisfied simultaneously. This is the intersection of all the half-planes created by the constraint lines. In other words, it is the region where:

- The solution lies below or on the line $x_1 + x_2 = 4$.
- The solution lies below or on the line $2x_1 + x_2 = 5$.

- The solution lies within the first quadrant (since $x_1 \geq 0$ and $x_2 \geq 0$).

The feasible region will be the area bounded by these lines. This region can be polygonal, with vertices corresponding to potential optimal solutions. For linear problems, the optimal solution will always be at one of the vertices of the feasible region.

Step 4: Determine the Vertices of the Feasible Region

The vertices (corner points) of the feasible region are crucial, as the optimal solution of an LP problem will always lie at one of these vertices. To identify the vertices, you can either:

- Find the intersection points of the constraint lines by solving their equations algebraically.
- Use the graphical method to locate the points where the constraint lines intersect visually.

For the above example:

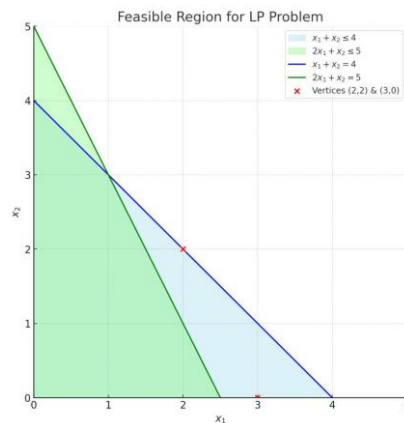
- The intersection of $x_1 + x_2 = 4$ and $2x_1 + x_2 = 5$ gives one vertex.
- The intersection of $x_1 + x_2 = 4$ with the x_1 -axis (i.e., where $x_2 = 0$) gives another vertex.
- The intersection of $2x_1 + x_2 = 5$ with the x_2 -axis (i.e., where $x_1 = 0$) gives the third vertex.

Step 5: Evaluate the Objective Function at Each Vertex

After determining the vertices of the feasible region, evaluate the objective function at each vertex. The optimal solution corresponds to the vertex where the objective function achieves its maximum (for maximization problems) or minimum (for minimization problems).

- **Example:** For the objective function $Z = 3x_1 + 2x_2$, calculate Z at each vertex:
 - At the first vertex, calculate $Z = 3x_1 + 2x_2$.
 - At the second vertex, calculate $Z = 3x_1 + 2x_2$.
 - At the third vertex, calculate $Z = 3x_1 + 2x_2$.

The vertex that results in the highest value of Z will be the optimal solution.



Here is the **graph** representing the feasible region for the linear programming problem, which is based on the following constraints:

1. $x_1 + x_2 \leq 4$
2. $2x_1 + x_2 \leq 5$
3. $x_1 \geq 0$ and $x_2 \geq 0$

The graph highlights the following:

- **Feasible region:** The area where the constraints overlap, shaded with different colors.
- **Constraint lines:** The blue and green lines represent the constraints $x_1 + x_2 = 4$ and $2x_1 + x_2 = 5$, respectively.
- **Vertices:** The red points mark the intersection points of the constraint lines, which represent potential solutions to the LP problem. The optimal solution lies at one of these vertices.

22.4.3 Limitations of the Graphical Method

While the graphical method is an intuitive and effective tool for solving small LP problems

with two variables, it has several limitations, especially when the number of decision variables exceeds two. These limitations make the graphical method impractical for large-scale or high-dimensional problems. Below are some of the key limitations:

1. Limited to Two Variables

The graphical method is only applicable to LP problems with two decision variables. When there are more than two decision variables, it becomes impossible to visualize the feasible region and optimal solution graphically, as we cannot represent higher-dimensional spaces on a two-dimensional plane. For LP problems with more than two variables, methods like the Simplex Method or Interior Point Method are more appropriate.

- **Example:** If a problem involves three or more variables (e.g., x_1, x_2, x_3), the graphical method cannot be used to plot the feasible region and determine the optimal solution. The graphical method only works when we can graph the constraints and visually observe the feasible region.

2. Inefficiency for Large Problems

Even with two variables, the graphical method may not be the most efficient solution technique for larger, more complex problems. For instance, finding the intersection points of multiple constraints can be time-consuming and error-prone. As the number of constraints increases, the graphical approach becomes more difficult to apply effectively, especially when the feasible region is irregular or complicated.

- **Example:** For problems with many constraints, even if there are only two variables, plotting all the constraints on a graph can result in a cluttered diagram that is difficult to interpret and find the feasible region.

3. No Guidance for Non-Linear Constraints

The graphical method can only be used for linear programming problems, which involve linear constraints. It cannot handle non-linear constraints or non-linear objective functions. Many real-world optimization problems involve non-linear relationships, and these cannot be solved using the graphical method.

- **Example:** If a constraint is of the form $x^2 + y^2 \leq 4$ (which represents a circular boundary), the graphical method cannot handle this kind of non-linearity. The

graphical method is limited to problems with linear equations and inequalities only.

4. Difficulty in Finding Exact Values

In practice, solving for exact intersection points and evaluating the objective function at vertices can be challenging with the graphical method, especially when the coordinates of the vertices are not integers. This can lead to approximation errors, particularly in problems where high precision is required.

- **Example:** If the intersection of two constraint lines occurs at $x_1 = 2.5$ and $x_2 = 3.7$, finding the exact value of the objective function at this point can be cumbersome and prone to calculation errors.

5. Inability to Handle Large Data Sets

In real-world applications, decision-making often involves large data sets with many variables and constraints. The graphical method is impractical for such scenarios, as the visual representation becomes increasingly complicated. Larger problems require computational methods that can handle high-dimensional data efficiently.

- **Example:** A logistics company trying to optimize transportation routes for 100 cities would require a solution with many variables (such as the number of trucks assigned to each route, and the time taken for each trip). Solving this problem with the graphical method would be unmanageable.

The graphical method is a powerful tool for solving Linear Programming problems with two variables, offering an intuitive and visual approach to optimization. However, its practical limitations—such as being confined to two variables, inefficiency for large problems, inability to handle non-linear constraints, and difficulties in achieving high precision—make it unsuitable for larger or more complex problems. While useful for understanding basic LP concepts, more advanced methods like the Simplex Method or Interior Point Methods are needed for real-world problems involving more variables or non-linearities. Despite its limitations, the graphical method provides a solid foundation for understanding LP and its applications in various optimization scenarios.

Example 1: Maximizing Profit

Problem Statement:

A company manufactures two products, P1 and P2. The profit from each unit of Product 1 is \$3 and from Product 2 is \$2. The company has two resources with the following constraints:

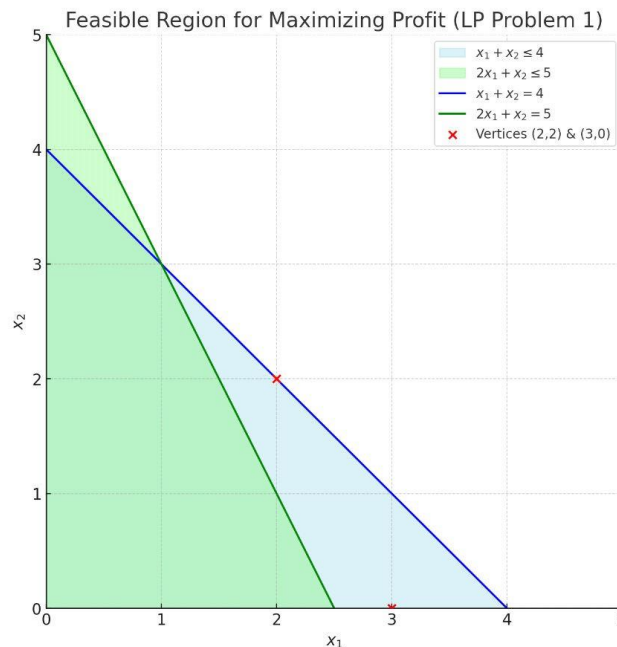
- It takes 1 unit of P1 and 1 unit of P2 to produce a product, and there are at most 4 units of this resource available.
- The second resource requires 2 units of P1 and 1 unit of P2, and there are at most 5 units of this resource available.
- Both products must be produced in non-negative quantities.

Maximize the profit $Z = 3x_1 + 2x_2$, subject to the constraints:

$$x_1 + x_2 \leq 4 \text{ (Resource 1)}$$

$$2x_1 + x_2 \leq 5 \text{ (Resource 2)}$$

$$x_1 \geq 0, x_2 \geq 0$$



Solution Steps:

1. **Graph the constraints:** Plot the constraints on the graph to determine the feasible region.
2. **Identify the vertices of the feasible region:** Mark the intersection points of the constraints.
3. **Evaluate the objective function at the vertices:** Calculate the profit at each vertex and determine the maximum profit.

22.5 Formulating Linear Programming Problems

Linear Programming (LP) models are used to represent real-world decision-making situations that involve optimizing a certain objective subject to a set of constraints. The process of **formulating** a Linear Programming problem involves converting a real-world scenario into a mathematical model consisting of decision variables, an objective function, and constraints. Understanding how to formulate LP problems is crucial for applying LP techniques effectively in various fields, such as economics, business, and operations research.

22.5.1 Linear Programming Models

A Linear Programming Model is a mathematical representation of a decision problem that involves:

- **Decision Variables:** These are the unknowns that need to be determined. They represent the quantities to be optimized (e.g., the number of products to produce or the amount of resources to allocate).
- **Objective Function:** This is the function to be maximized or minimized. It represents the goal of the LP problem, such as maximizing profit or minimizing cost.
- **Constraints:** These are the limitations or restrictions on the decision variables. Constraints represent the available resources, capacities, or other limitations that must be adhered to.

The general form of an LP problem is:

Objective Function:

Maximize or Minimize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

where $c_1, c_2, \dots, c_n$ are the coefficients representing the contribution of each decision variable $x_1, x_2, \dots, x_n$ to the objective function.

Subject to the constraints:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

□ □

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

Where:

- $x_1, x_2, \dots, x_n$ are the decision variables.
- a_{ij} are the coefficients of the decision variables in the constraints.
- $b_1, b_2, \dots, b_m$ are the right-hand side values of the constraints, representing the limits on resources.
- The objective function can either be a maximization or minimization, depending on the goal of the problem.
- The constraints are linear inequalities that define the feasible region.

22.5.2 Steps to Formulate an LP Problem

Formulating a Linear Programming problem involves several key steps. These steps transform a real-world decision problem into a mathematical representation that can be solved using LP methods.

Step 1: Define the Decision Variables

The first step in formulating an LP problem is to define the decision variables, which represent the quantities you want to optimize.

- **Example:** In a manufacturing problem where a company produces two products, P1 and P2, the decision variables could be:
 - x_1 : the number of units of Product 1 to produce.
 - x_2 : the number of units of Product 2 to produce.

These decision variables represent the unknown quantities that will be determined through the optimization process.

Step 2: Write the Objective Function

Next, determine the objective function, which represents the goal of the problem. The objective function could be a profit function, cost function, or any other quantity that needs to be maximized or minimized. The objective function is written in terms of the decision variables.

- **Example:** If the profit per unit of Product 1 is \$3 and the profit per unit of Product 2 is \$5, the objective function to maximize profit would be:

$$\text{Maximize } Z = 3x_1 + 5x_2$$

This objective function seeks to maximize the total profit from producing x_1 units of Product 1 and x_2 units of Product 2.

Step 3: Write the Constraints

The next step is to define the constraints. Constraints represent limitations on resources, such as available labor, raw materials, or production capacity. Constraints are written as linear inequalities or equations involving the decision variables.

- **Example:** In a manufacturing problem, suppose that:
 - Each unit of Product 1 requires 2 hours of labor and each unit of Product 2 requires 1 hour of labor.
 - There are 100 hours of labor available.

The constraint for labor availability would be:

$$2x_1 + x_2 \leq 100$$

This constraint limits the total number of labor hours to 100.

Additionally, there could be constraints on raw materials, production capacity, or other resources. These constraints are written in similar form.

- **Example:** If each unit of Product 1 requires 3 units of raw material and each unit of Product 2 requires 2 units of raw material, and there are 180 units of raw material available, the constraint would be:

$$3x_1 + 2x_2 \leq 180$$

Also, constraints are typically required to ensure non-negative decision variables:

$$x_1 \geq 0, x_2 \geq 0$$

Step 4: Identify the Feasible Region

The feasible region is the set of all points that satisfy all the constraints simultaneously. This region is bounded by the constraint lines (or planes in higher dimensions) and is typically represented graphically. In two-dimensional problems, the feasible region is the area on the graph where all constraints overlap.

Example: The constraints $2x_1 + x_2 \leq 100$ and $3x_1 + 2x_2 \leq 180$

define a region on the graph where the feasible solutions must lie.

Step 5: Solve the LP Problem

Once the decision variables, objective function, and constraints are defined, the LP problem can be solved. For simple problems with two decision variables, the graphical method can be used to identify the feasible region and find the optimal solution. For more complex problems with more than two variables, methods like the Simplex Method or Interior Point Methods are used to find the optimal solution.

- **Example:** To maximize profit $Z = 3x_1 + 5x_2$ subject to the constraints:

$$x_1 + x_2 \leq 100$$

$$3x_1 + 2x_2 \leq 180$$

$$x_1 = 0, x_2 \geq 0$$

The feasible region is identified, and the objective function is evaluated at the vertices of this region to determine the maximum profit.

Example:

Let's consider a practical example of product mix optimization.

Problem Statement:

A factory manufactures two products, Product 1 and Product 2. The factory has 100 labor hours and 180 units of raw material available. The time required to make each product and the raw material usage is as follows:

- Product 1: 2 hours of labor, 3 units of raw material
- Product 2: 1 hour of labor, 2 units of raw material

The profit per unit of Product 1 is \$3, and the profit per unit of Product 2 is \$5. The objective is to maximize the total profit.

Step 1: Define the decision variables.

- Let x_1 be the number of units of Product 1 produced.
- Let x_2 be the number of units of Product 2 produced.

Step 2: Write the objective function.

Maximize $Z = 3x_1 + 5x_2$

Step 3: Write the constraints.

- Labor constraint: $2x_1 + x_2 \leq 100$
- Raw material constraint: $3x_1 + 2x_2 \leq 180$
- Non-negativity constraint: $x_1 \geq 0, x_2 \geq 0$

Step 4: Identify the feasible region by plotting the constraints.

- Plot the inequalities on a graph.
- The feasible region is where all constraints are satisfied.

Step 5: Solve the problem using graphical methods or Simplex Method.

Formulating a Linear Programming problem involves defining decision variables, writing the objective function, expressing the constraints, and then solving the problem either graphically (for two variables) or using computational methods (for larger problems). By following the steps outlined, complex decision-making problems can be translated into solvable LP models, helping businesses and organizations optimize resource use and maximize profits.

Check your Progress – I

Based on your understanding of the previous section, answer these questions in your own words:

1. What is a decision variable in Linear Programming?

2. What is the objective function in Linear Programming?

3. How do you graphically represent constraints in Linear Programming?

4. What does the feasible region represent in a Linear Programming problem?

5. What are the main steps involved in solving a Linear Programming problem using the graphical method?

6. What is the main limitation of the graphical method in Linear Programming?

7. What does the term “slack variable” mean in the context of Linear Programming?

8. What is the purpose of formulating a Linear Programming problem?

9. What is the objective of using Linear Programming models in real-world problems?

10. How do you formulate the constraints in a Linear Programming problem?

22.6 Simplex Method

The Simplex Method is a powerful and widely used algorithm for solving Linear Programming (LP) problems, especially when the problem has more than two decision variables. Unlike the graphical method, which is limited to problems with two variables,

the Simplex Method can handle LP problems with any number of variables and constraints. It is particularly effective for large-scale optimization problems where graphical solutions are impractical.

22.6.1 Introduction to the Simplex Method

The Simplex Method, developed by George Dantzig in 1947, is an iterative process used to find the optimal solution to a Linear Programming problem. It is designed to solve linear optimization problems of the following form:

Objective Function:

Maximize or Minimize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

Subject to:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

⋮

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

where:

- $x_1, x_2, \dots, x_n$ are the decision variables,
- $c_1, c_2, \dots, c_n$ are the coefficients in the objective function,
- a_{ij} are the coefficients in the constraints,
- $b_1, b_2, \dots, b_m$ are the right-hand side values for each constraint.

The Simplex Method is based on the idea that the optimal solution to an LP problem occurs at one of the vertices (corner points) of the feasible region defined by the constraints. It moves from one vertex to another along the edges of the feasible region, improving the value of the objective function at each step, until it reaches the optimal solution.

The method works by iteratively selecting a basic feasible solution (a vertex of the feasible region) and moving along the edges of the feasible region in the direction that

improves the objective function. The algorithm stops when it reaches the optimal solution, which is located at the vertex where no further improvement in the objective function is possible.

Key Terms:

- **Basic Feasible Solution:** A solution that satisfies all the constraints, where some of the variables (called basic variables) are set to positive values, and the rest (called non-basic variables) are set to zero.
- **Pivoting:** The process of moving from one basic feasible solution to another by changing the values of the basic and non-basic variables.
- **Optimal Solution:** The solution where the objective function achieves its maximum (in a maximization problem) or minimum (in a minimization problem), and no further improvement is possible.

22.6.2 Steps Involved in the Simplex Method

The Simplex Method proceeds through several steps to find the optimal solution. These steps can be broken down as follows:

Step 1: Convert the LP problem into standard form

Before applying the Simplex Method, the LP problem must be in **standard form**:

1. The objective function must be a maximization problem.
2. All constraints must be inequalities of the form $\leq$. If any constraint is in the form $\geq$, it is multiplied by -1.
3. Each constraint should have a non-negative right-hand side.

Additionally, slack variables (extra variables added to turn inequalities into equalities) are introduced. These slack variables represent unused resources and must be non-negative.

Step 2: Set up the initial simplex tableau

The initial tableau contains the coefficients of the objective function and the constraints, including the slack variables. The tableau is organized as follows:

- The first row corresponds to the objective function.
- The other rows represent the constraints and their corresponding slack variables.

Each row of the tableau represents one equation in the system, and the last column represents the right-hand side (the b_i values from the constraints).

Step 3: Check for optimality

In each iteration, the objective function coefficients in the tableau are checked:

- If all the coefficients in the objective function row (excluding the right-hand side) are non-negative, the current solution is optimal.
- If any coefficient is negative, the solution is not optimal, and we need to improve it by pivoting.

Step 4: Choose the entering variable

To improve the current solution, we must identify the entering variable. This is the non-basic variable that will replace one of the current basic variables. The entering variable corresponds to the most negative coefficient in the objective function row. This is because the most negative coefficient indicates the direction in which the objective function will improve the most.

Step 5: Choose the leaving variable

Once the entering variable is chosen, we need to identify the leaving variable, which will be replaced by the entering variable. This is done by finding the pivot row in the tableau. The pivot row corresponds to the constraint that will become tight (i.e., the constraint that limits the increase in the entering variable). The leaving variable is the basic variable associated with this row.

The pivot is determined by the ratio of the right-hand side values to the corresponding coefficients in the column of the entering variable. The smallest non-negative ratio determines the pivot row.

Step 6: Pivot and update the tableau

Once the entering and leaving variables are determined, the tableau is updated using the pivoting process:

- The pivot element (the element at the intersection of the pivot column and pivot row) is divided to make it 1.
- The other elements in the pivot column are updated by row operations to make them 0.
- The objective function row is also updated using the same row operations.

After this step, the tableau is updated, and the process repeats from Step 3.

Step 7: Repeat until optimality is reached

The process continues until all the coefficients in the objective function row are non-negative, indicating that the optimal solution has been reached. At that point, the values of the decision variables are found in the tableau.

22.6.3 Advantages of the Simplex Method

The Simplex Method has several advantages that make it widely used for solving Linear Programming problems:

1. Efficient for Large-Scale Problems

The Simplex Method is highly efficient, even for large LP problems with many decision variables and constraints. It has been shown to perform well in practice, often solving problems in a number of iterations that is much smaller than the worst-case theoretical bound.

2. Guaranteed to Find an Optimal Solution

The Simplex Method always finds an optimal solution if one exists, provided that the problem is feasible and the LP model is formulated correctly. It can handle both bounded and unbounded feasible regions, ensuring the optimal solution is found.

3. Flexibility

The Simplex Method can handle various types of Linear Programming problems, including problems with different forms of constraints (e.g., less than or equal to, greater than or equal to, and equality constraints). Additionally, it works for problems with multiple decision variables and constraints.

4. Handling Degeneracy

In certain cases, the Simplex Method may encounter degeneracy, where a basic feasible solution corresponds to multiple vertices of the feasible region. However, the Simplex Method can still handle degeneracy effectively, ensuring that the algorithm continues toward the optimal solution.

5. Practical Use in Real-World Problems

The Simplex Method is the most commonly used algorithm for solving Linear Programming problems in real-world applications. It is used in industries such as manufacturing, logistics, transportation, and financial portfolio optimization, among many others.

6. Simple to Implement

The Simplex Method is relatively easy to implement, and there are many software packages available that use the Simplex Method to solve LP problems. These include popular solvers like CPLEX, Gurobi, and Microsoft Excel Solver, which make the method accessible to a wide range of users.

The Simplex Method is an essential tool for solving Linear Programming problems with multiple decision variables and constraints. While the method operates iteratively by selecting entering and leaving variables, its efficiency and ability to handle large, complex problems make it an indispensable algorithm for optimization. By following the steps outlined in the process — from setting up the initial tableau to pivoting and updating the tableau — the Simplex Method provides a structured way to find the optimal solution to real-world decision-making problems.

22.7 Duality in Linear Programming

In Linear Programming (LP), duality is an important concept that provides an alternative way of looking at an optimization problem. Every Linear Programming problem, known as the primal problem, has an associated dual problem that offers additional insights into the problem's structure and provides a way to solve the problem from a different perspective. The theory of duality is a central part of LP theory and plays a critical role in both solving and interpreting LP problems.

22.7.1 Concept of Duality

The concept of duality in Linear Programming refers to the relationship between two LP problems: the primal problem and the dual problem. Every linear programming problem (primal problem) can be associated with a dual problem that provides a different perspective on the original problem.

- **Primal Problem:** The original optimization problem, either a maximization or minimization problem, is called the primal problem. It involves decision variables, an objective function, and constraints.
- **Dual Problem:** The dual problem is derived from the primal problem, and it offers an alternative formulation that is related to the primal problem. The dual problem expresses the optimization problem in terms of new variables (the dual variables) and a different objective function.

The dual problem provides insights into the economic interpretation of the original problem. For instance, the dual variables represent the shadow prices or marginal values of the constraints in the primal problem. These shadow prices show how much the objective function value (e.g., profit or cost) would change if the constraint values were relaxed or tightened.

22.7.2 Primal and Dual Problems

Primal Problem Formulation

The general form of a primal LP problem is as follows:

Maximize:

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Subject to:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

□ □

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

where:

- $x_1, x_2, \dots, x_n$ are the decision variables,
- a_{ij} are the coefficients in the constraints,
- $b_1, b_2, \dots, b_m$ are the right-hand side values,
- $c_1, c_2, \dots, c_n$ are the coefficients in the objective function.

The goal is to maximize the objective function Z , subject to the constraints that represent limitations on resources or other factors.

Dual Problem Formulation

The dual problem associated with the primal problem is formulated as follows:

Minimize:

$$W = b_1 y_1 + b_2 y_2 + \dots + b_m y_m$$

Subject to:

$$a_{11} y_1 + a_{12} y_2 + \dots + a_{1m} y_m \geq c_1$$

$$a_{21} y_1 + a_{22} y_2 + \dots + a_{2m} y_m \geq c_2$$

□

$$a_{n1} y_1 + a_{n2} y_2 + \dots + a_{nm} y_m \geq c_n$$

where:

- $y_1, y_2, \dots, y_m$ are the dual variables associated with each constraint in the primal problem,
- The objective is to minimize W , subject to constraints that are related to the primal objective coefficients.

The dual variables $y_1, y_2, \dots, y_m$ can be interpreted as the shadow prices or the value of relaxing or tightening each constraint in the primal problem.

Economic Interpretation of Duality

- In the primal problem, the decision variables $x_1, x_2, \dots, x_n$ represent the quantities to be determined (e.g., the number of products to produce).
- In the dual problem, the dual variables $y_1, y_2, \dots, y_m$ represent the value or marginal worth of relaxing the constraints in the primal problem. These dual variables provide important economic insights into how much the objective function would improve if the constraints were relaxed.

22.7.3 Duality Theorems

The Duality Theorem is a key result in Linear Programming that establishes the relationship between the primal and dual problems. The Duality Theorem guarantees that:

1. If the primal problem has an optimal solution, then the dual problem also has an optimal solution.
2. The optimal values of the objective functions in the primal and dual problems are equal, known as the duality gap. In other words, the optimal objective value of the primal problem is equal to the optimal objective value of the dual problem.

Primal and Dual Problems' Relationship

- The primal problem maximizes the objective function, while the dual problem minimizes the objective function.
- The optimal solution of the primal problem provides the best feasible solution to the primal constraints, and the optimal solution of the dual problem provides the best set of values for the dual variables (shadow prices) associated with the constraints.

Weak Duality Theorem

The weak duality theorem states that the value of the objective function for any feasible solution to the primal problem is always less than or equal to the value of the objective function for any feasible solution to the dual problem. This implies that the optimal value of the primal problem will never exceed the optimal value of the dual problem.

Strong Duality Theorem

The strong duality theorem asserts that if the primal problem has an optimal solution, then the dual problem has an optimal solution as well, and the optimal values of the objective functions in the primal and dual problems are equal. This is a critical property that ensures the dual problem is a valid and useful way of solving the primal problem.

Complementary Slackness Conditions

The complementary slackness conditions provide a direct relationship between the primal and dual variables. These conditions tell us that for each pair of corresponding primal and dual variables:

1. If a primal constraint is strictly satisfied (i.e., the inequality is not tight), then the corresponding dual variable must be zero.
2. If a dual constraint is strictly satisfied, then the corresponding primal variable must be zero.

These complementary slackness conditions provide a way to check whether a pair of solutions (one for the primal problem and one for the dual problem) is optimal.

The duality in Linear Programming offers a powerful method for analyzing optimization problems from a different perspective. By formulating both the primal and dual problems, we can gain valuable economic insights into the problem's structure and its constraints. The Duality Theorem guarantees that the optimal solutions for both problems will be related, providing an efficient way to solve LP problems. Furthermore, duality offers practical applications in sensitivity analysis, where the dual variables (shadow prices) indicate how much the objective function would improve if constraints were relaxed or tightened. Understanding the concept of duality is fundamental for solving real-world LP problems and gaining deeper insights into optimization and resource allocation.

22.8 Sensitivity Analysis

Sensitivity Analysis in Linear Programming (LP) is a method used to determine how changes in the parameters of an LP problem (such as coefficients in the objective function or changes in the right-hand side of the constraints) affect the optimal solution.

In other words, sensitivity analysis helps to understand how sensitive the optimal solution is to variations in input data. This is especially important in real-world applications, where data might change due to various factors, such as fluctuations in market prices, changes in available resources, or adjustments in production capacities.

22.8.1 Introduction to Sensitivity Analysis

Sensitivity analysis is a vital aspect of Linear Programming because, in practical scenarios, the parameters of an LP model are often uncertain or subject to change. Therefore, understanding the impact of these changes on the optimal solution is crucial for decision-making.

In the context of LP, sensitivity analysis typically involves studying the effects of:

- 1. Changes in the objective function coefficients:** These coefficients represent the contribution of each decision variable to the objective function (such as cost, profit, etc.). If these coefficients change, it could alter the optimal solution.
- 2. Changes in the right-hand side values of the constraints:** The right-hand side values of the constraints represent the limits on resources (e.g., labor hours, materials, budget). Modifying these values could change the feasible region and, consequently, the optimal solution.

Types of Sensitivity Analysis

- **Objective Function Sensitivity:** This type of sensitivity analysis investigates how changes in the coefficients of the objective function affect the optimal solution. It helps in understanding how robust the solution is to fluctuations in the objective coefficients.
- **Constraint Sensitivity:** This examines how changes in the right-hand side of the constraints (e.g., resource availability) influence the feasible region and the optimal solution. It helps in understanding how much a resource constraint can be relaxed or tightened without altering the optimal solution.

Graphical Interpretation of Sensitivity

In two-variable LP problems, the impact of changes can often be visualized graphically by observing shifts in the constraint lines and the feasible region. The solution may move

to a different vertex of the feasible region depending on how the objective function and constraints change.

For more complex problems (with three or more variables), sensitivity analysis involves examining the changes algebraically or using computational methods, as graphical interpretation is no longer feasible.

22.8.2 Importance of Sensitivity Analysis in LP

Sensitivity analysis plays a significant role in both theoretical and practical aspects of Linear Programming. Below are several reasons why sensitivity analysis is crucial:

1. Handling Uncertainty in Real-World Applications

Real-world problems often involve uncertainty in parameters such as costs, resource availability, and other constraints. Sensitivity analysis allows decision-makers to understand the extent to which changes in these parameters affect the optimal solution. This helps in making robust decisions that account for potential variations.

For example, in a manufacturing setting, the cost of raw materials or the availability of labor may change over time. Sensitivity analysis helps in assessing how these changes impact the optimal production plan and profit. If the optimal solution remains stable despite changes in resource availability or costs, the solution is considered more robust.

2. Identifying the Critical Constraints

Sensitivity analysis helps identify which constraints are binding (tight or active) and which are non-binding (loose or inactive). Binding constraints are those that hold equality in the optimal solution, while non-binding constraints do not affect the solution. Understanding which constraints are critical allows decision-makers to prioritize resources, adjust operations, and ensure that the system remains feasible.

For example, if a certain resource constraint is binding in the optimal solution, increasing the availability of that resource could lead to a higher profit or lower cost. On the other hand, if a constraint is non-binding, relaxing or tightening it would not affect the optimal solution.

3. Guiding Decision-Making Under Variability

In situations where input data may change due to external factors, sensitivity analysis provides guidance on how to adjust decisions. For instance, if there is a possibility of an increase in demand or a decrease in material costs, sensitivity analysis can help assess how such changes would impact the overall strategy. It enables decision-makers to prepare for a range of scenarios and avoid making decisions based solely on fixed parameters.

For example, in logistics planning, if the transportation cost per unit changes due to fluctuating fuel prices, sensitivity analysis allows the company to understand the effect of such changes on the overall transportation strategy. This helps in planning for both best-case and worst-case scenarios.

4. Determining the Flexibility of the Optimal Solution

Sensitivity analysis can indicate how much **flexibility** exists in the solution before a change in parameters renders the solution no longer optimal. For example, it can help in determining how much the price of a product can be reduced while still maintaining profitability or how much a resource can be increased before a new optimal solution is needed.

If the objective function coefficient for a product is reduced, sensitivity analysis shows how far this coefficient can drop before it no longer makes sense to produce that product. This is valuable for pricing decisions and cost reductions.

5. Exploring Trade-offs

In many LP problems, there are trade-offs between objectives or resources. Sensitivity analysis allows us to quantify these trade-offs. For instance, if relaxing one constraint leads to a higher objective function value, sensitivity analysis can help determine whether the improvement justifies the additional resources required to achieve it.

- **Example:** A company may need to decide whether it's worth relaxing the labor constraint to produce more units of a product. Sensitivity analysis helps evaluate the profit increase versus the additional labor cost incurred.

6. Providing Insights for Future Planning

Sensitivity analysis allows companies and organizations to plan for future scenarios and prepare strategies accordingly. It helps in anticipating future changes in market conditions, resource availability, or demand fluctuations. For instance, an analysis of resource constraints can guide long-term investment decisions, such as whether to expand capacity or improve resource utilization.

For example, if a company is unsure whether to invest in additional machinery, sensitivity analysis can show how much the company's profits would improve if the machinery were added and how much demand would have to increase for the investment to become profitable.

7. Assisting in Resource Allocation and Cost Management

By understanding how different constraints affect the optimal solution, sensitivity analysis helps in effective resource allocation and cost management. It enables companies to identify where cost savings can be achieved, whether by reducing resource usage or improving efficiency. Similarly, it can help prioritize resource allocation by showing which constraints are most important for achieving optimal outcomes.

- **Example:** In project management, sensitivity analysis can reveal which resources (such as labor or equipment) are most critical for project success. This helps project managers allocate resources efficiently and avoid wasteful expenditures.

Sensitivity Analysis is a powerful tool in Linear Programming that provides critical insights into how changes in parameters affect the optimal solution. It is widely used in real-world applications where data is uncertain and fluctuates over time. Sensitivity analysis not only helps in making robust decisions but also in identifying critical constraints, exploring trade-offs, and optimizing resource allocation. By analyzing how the solution responds to changes in parameters, decision-makers can ensure that the chosen strategy is adaptable and effective in the face of uncertainty. Whether it's in manufacturing, logistics, finance, or project management, sensitivity analysis adds value by providing a deeper understanding of the model's behavior under different scenarios.

22.9 Applications, Limitations, and Assumptions of Linear Programming

Linear Programming (LP) is a versatile optimization technique used to solve decision-making problems in various fields, such as economics, business, engineering, and logistics. By modeling real-world problems with linear objective functions and linear constraints, LP provides valuable solutions for optimizing resource allocation, cost minimization, profit maximization, and other objectives.

In this section, we will explore the applications, limitations, and assumptions of Linear Programming.

22.9.1 Applications of Linear Programming

Linear Programming is widely applied across different sectors and industries where optimization of resources is required. Here are some common applications:

1. Manufacturing and Production Planning

LP is used in production planning to determine the optimal quantity of products to produce given limited resources such as labor, raw materials, and machine time. In such problems, the objective is typically to maximize profit or minimize production costs, subject to constraints on resources.

- **Example:** A company manufactures two products, Product A and Product B. The company has limited raw materials, labor, and machine hours available. By applying LP, the company can determine the optimal mix of products to produce in order to maximize profit while staying within the resource limits.

2. Resource Allocation

In organizations where resources are limited, LP helps in resource allocation to achieve optimal utilization. This can apply to labor, raw materials, capital, land, or time.

- **Example:** A business needs to allocate a certain amount of budget to various departments (marketing, production, research & development) while maximizing the company's overall profit. LP can help allocate the budget optimally to each department

3. Transportation and Logistics

LP is extensively used in transportation problems to minimize transportation costs, where goods need to be moved from several suppliers to multiple consumers, subject to capacity and demand constraints.

- **Example:** A company needs to transport goods from multiple warehouses to retail stores. LP can help minimize the transportation cost by determining the optimal distribution plan.

4. Diet Problems

In the diet problem, LP is used to determine the least-cost combination of foods that meets the required nutritional needs. This application is widely used in the food industry, healthcare, and even in sports nutrition.

- **Example:** A hospital wants to provide a meal plan to patients while minimizing food costs, ensuring that nutritional requirements are met. LP can optimize the food selection to achieve the goal.

5. Financial Portfolio Optimization

LP is used in portfolio optimization to allocate investments in various assets (stocks, bonds, real estate, etc.) in such a way that maximizes the return while minimizing the risk, subject to certain constraints such as budget or risk tolerance.

- **Example:** An investor wants to allocate funds to different investment options to achieve the best return while minimizing the financial risk. By solving the LP problem, the investor can determine the optimal asset distribution.

6. Supply Chain Management

LP helps in optimizing the supply chain by minimizing costs or maximizing profits across various stages, including procurement, manufacturing, inventory management, and distribution.

- **Example:** A company wants to optimize its supply chain to minimize the total cost, including transportation, storage, and inventory holding costs. LP can determine the most cost-effective procurement and distribution strategy.

7. Marketing and Advertising

In marketing and advertising, LP can be applied to budget allocation to determine the optimal spending on different advertising channels (TV, radio, online ads, etc.) while maximizing the reach or sales.

- **Example:** A company has a fixed advertising budget and wants to allocate it optimally across different advertising media to maximize customer reach. LP can help identify the best allocation of the budget.

22.9.2 Limitations of Linear Programming

While Linear Programming is a powerful optimization tool, it has certain limitations:

1. Linearity Assumption

LP assumes that both the objective function and constraints are linear. This means that the relationships between decision variables are assumed to be proportional (i.e., the effect of increasing a variable by one unit is the same regardless of the starting point). In reality, many problems involve non-linear relationships that LP cannot handle.

- **Example:** In production problems, the relationship between the number of units produced and the cost of production may not always be linear. If the cost increases exponentially as production volume increases, LP would not provide accurate solutions.

2. Homogeneity of Constraints

LP assumes that all constraints are constant and deterministic, meaning they do not change over time. In practice, resource availability, costs, or demand may fluctuate, making the LP model less flexible when dealing with real-world uncertainty.

- **Example:** If a company's raw material supply fluctuates, it may not be appropriate to use LP without accounting for such uncertainty. Other methods, like stochastic optimization, would be more suitable in such cases.

3. Integer Solutions

LP typically provides continuous solutions, meaning it assumes that decision variables can take any real value. However, in many cases, decision variables must be integers

(e.g., the number of items to produce, the number of workers). In such cases, Integer Linear Programming (ILP) or Mixed Integer Linear Programming (MILP) methods should be used, which are computationally more complex than LP.

- **Example:** A transportation problem where the number of trucks required must be an integer, LP will not directly give integer solutions, and ILP or MILP would be required.

4. Limited to Static Problems

LP is typically applied to **static** problems where the objective function and constraints do not change over time. However, many real-world problems are dynamic, where the parameters can change over time (e.g., changing prices, fluctuating demand). LP does not account for time dynamics, making it less useful for dynamic problems.

- **Example:** In inventory management, demand might vary over time, which requires dynamic models like Dynamic Programming rather than static LP.

5. Sensitivity to Data Inaccuracy

The solutions obtained from LP models can be sensitive to the **accuracy of input data**. Small changes in input data (such as coefficients in the objective function or constraints) can lead to large changes in the optimal solution. Therefore, LP models may not always provide stable or reliable solutions when the data is uncertain or subject to estimation errors.

- **Example:** A small change in the cost coefficient in a manufacturing problem might significantly alter the optimal production mix, especially in problems with many variables.

22.9.3 Assumptions of Linear Programming

To develop and apply an LP model effectively, several assumptions are made. These assumptions form the foundation of LP and should be understood and considered when using this method in real-world applications.

1. Linearity of the Objective Function and Constraints

One of the main assumptions of LP is that both the objective function and the constraints are linear. This implies that the relationship between decision variables is proportional.

- **Example:** The total cost in a production problem is assumed to be directly proportional to the number of units produced.

2. Certainty in Data

LP assumes that all the data used in the model (e.g., coefficients of the objective function, resource availability) is certain and known with precision. There is no room for uncertainty or random fluctuations in the parameters.

- **Example:** The amount of available labor, raw materials, or budget is assumed to be fixed and accurately known.

3. Proportionality of Resources

LP assumes that resources are divisible and proportional. This means that if a certain amount of resources is used to produce a certain amount of output, then half the resources would produce half the output.

- **Example:** If 10 hours of labor are required to produce 100 units of a product, LP assumes that 5 hours of labor will produce exactly 50 units.

4. Constant Technology

LP assumes that the technology (i.e., the production processes) is constant, meaning that the amount of resources required to produce a unit of output does not change over time or with production scale.

- **Example:** In a production setting, LP assumes that the ratio of labor and materials required to produce a unit of output remains constant throughout the decision period.

5. No Interdependencies between Decision Variables

LP assumes that the decision variables do not **interact** or affect each other in non-linear ways. Each decision variable is treated independently, and changes in one variable do not influence others, except as specified by the constraints.

- **Example:** In a marketing budget allocation problem, LP assumes that spending on one media type (e.g., TV ads) does not affect the effectiveness or cost of another media type (e.g., online ads).

Linear Programming has a wide range of applications in various industries such as manufacturing, transportation, finance, and resource allocation. It provides valuable solutions for optimizing resources, minimizing costs, and maximizing profits. However, the method has several limitations, such as its assumption of linearity, the need for deterministic data, and its inability to handle integer solutions directly. Despite these limitations, LP is a powerful tool for solving optimization problems and is widely used in practice. Understanding the assumptions, applications, and limitations of LP is crucial for correctly applying the method and interpreting the results in real-world decision-making.

Check your Progress – II

Based on your understanding of the previous section, answer these questions in your own words:

1. What is the Simplex Method used for in Linear Programming?

2. How does the Simplex Method find the optimal solution to an LP problem?

3. What are the key steps involved in applying the Simplex Method?

4. What is the concept of dual in Linear Programming?

5. How are primal and dual problems related in Linear Programming?

6. What is the Duality Theorem in Linear Programming?

7. Why is sensitivity analysis important in Linear Programming?

8. How does sensitivity analysis help in decision-making under uncertainty?

9. What factors are analyzed in sensitivity analysis for an LP problem?

10. What are the main applications of Linear Programming?

11. What are the major limitations of Linear Programming?

12. What assumptions are made when using Linear Programming for optimization?

-
-
-
13. What are the advantages of using the Simplex Method over other LP solving methods?
-
-
-
-

22.10 Keywords

1. **Decision Variables:** These are the unknowns in a Linear Programming problem that represent the quantities to be determined. They are the values that will be optimized (either maximized or minimized) to achieve the objective of the problem.
2. **Objective Function:** A linear function that defines the goal of the optimization problem. It is what is to be either maximized (e.g., profit) or minimized (e.g., cost) subject to constraints.
3. **Constraints:** These are the restrictions or limitations in a Linear Programming problem. Constraints are usually expressed as linear inequalities or equations, and they define the feasible region within which the decision variables must lie.
4. **Feasible Region:** The set of all possible solutions that satisfy the given constraints. The feasible region is bounded by the constraint lines or planes and represents all the potential solutions to the problem.
5. **Slack Variables:** Extra variables added to an inequality constraint to transform it into an equality. Slack variables represent unused resources or capacities in a system and are typically introduced in the standard form of Linear Programming problems.

22.11 Let's Sum Up

1. **Linear Programming (LP) as an Optimization Tool:** LP is an essential tool used in various fields like economics, business, and operations research to solve optimization problems. It helps in maximizing or minimizing a specific objective (e.g., profit, cost) subject to constraints, providing an optimal solution to real-world resource allocation problems.
2. **Key Concepts in LP:** The core elements of LP include **decision variables**, the **objective function**, and **constraints**. The decision variables represent quantities to be optimized, the objective function defines the goal of the problem (either to maximize or minimize), and constraints represent the limitations or available resources.
3. **Methods for Solving LP Problems:** There are multiple ways to solve LP problems. The **graphical method** is effective for problems with two variables but becomes impractical for problems involving more than two variables. The **Simplex method** is more suitable for large-scale LP problems and helps in finding the optimal solution efficiently by moving through feasible solutions.
4. **Duality and Sensitivity Analysis:** The **duality** theory in LP provides an alternative perspective, helping in understanding the economic implications of constraints and objective functions. **Sensitivity analysis** helps determine how changes in the objective function coefficients or constraints affect the optimal solution, offering insights for decision-making under uncertainty.
5. **Applications of LP:** LP is widely applied in various industries such as manufacturing, transportation, logistics, resource allocation, portfolio optimization, and supply chain management. It helps in making decisions that optimize resources, reduce costs, and maximize profits.
6. **Limitations of LP:** LP assumes linear relationships and certainty in data, making it unsuitable for problems involving non-linearity or uncertain parameters. It is also limited to problems with continuous decision variables and may not be directly applicable when integer solutions are needed.

- 7. Assumptions of LP:** The assumptions behind LP include linearity of the objective function and constraints, certainty in input data, proportionality of resources, and independence of decision variables. These assumptions may not always hold true in real-world problems, limiting the generality of LP models.
- 8. Practical Use and Effectiveness:** Despite its limitations, LP remains a powerful tool for optimization in real-world applications. Understanding its concepts, methods, applications, and limitations helps in applying LP effectively to solve complex problems in resource allocation and decision-making.

This summary highlights the importance of Linear Programming as an optimization technique, its wide range of applications, and the need to understand its assumptions, limitations, and the various methods available to solve LP problems.

22.12 Examination-Oriented Questions

1. What is the Simplex Method? Explain its steps with an example.
 - Start by defining the Simplex Method: Mention that the Simplex Method is an iterative algorithm used to solve Linear Programming problems, particularly when there are more than two variables.
 - List the steps involved: Briefly explain the core steps of the Simplex Method (i.e., setting up the initial tableau, choosing the entering and leaving variables, pivoting, and iterating until the optimal solution is found).
 - Example: Provide a simple example with constraints and an objective function, and walk through the steps using the Simplex method to find the optimal solution.
 - Conclude with the outcome: Emphasize the result, indicating how the method helps in reaching the optimal solution and its advantages over other methods.
2. What are the assumptions and limitations of Linear Programming?
 - List the assumptions: Discuss the basic assumptions in LP, such as linearity, certainty in data, proportionality of resources, and independence of decision variables.

- Highlight the limitations: Address key limitations like the linearity assumption, the inability to handle non-linear problems, and the need for continuous decision variables.
- Provide real-world examples: Mention scenarios where LP might not work, such as in problems involving uncertain data or non-linear relationships.
- Conclude with the application scope: Mention where LP is effective and where alternative methods like Integer Programming or non-linear optimization might be used.

3. Define the concept of Duality in Linear Programming. How is the dual problem related to the primal problem?

- Introduce duality: Start by explaining the concept of duality in LP, where every primal problem has a corresponding dual problem.
- Primal and Dual relationships: Discuss how the dual problem is formulated based on the primal problem, and explain the relationship between their objective functions and constraints.
- Primal and dual solutions: Clarify that the optimal solution of the primal problem is linked to the optimal solution of the dual problem, and the objective function values are equal.
- Economic interpretation: Provide an example to explain how dual variables represent shadow prices or the marginal value of resources in the primal problem.

4. Explain the process of sensitivity analysis in Linear Programming. Why is it important?

- Define sensitivity analysis: Explain that sensitivity analysis examines how changes in the objective function coefficients or the right-hand side of the constraints affect the optimal solution.
- Types of sensitivity analysis: Mention that it includes changes in the objective function coefficients and changes in resource availability (right-hand side of constraints).

- Practical applications: Provide examples where sensitivity analysis helps, such as understanding the impact of price fluctuations in a manufacturing or transportation problem.
- Importance: Conclude by explaining that sensitivity analysis helps in making decisions under uncertainty and ensuring that the optimal solution remains robust when the problem parameters change.

5. What are the steps involved in formulating a Linear Programming problem from a real-world scenario? Provide an example.

- Explain the steps of formulation: Discuss how to formulate a real-world problem into an LP problem, starting with defining decision variables, followed by writing the objective function, constraints, and non-negativity conditions.
- Real-world example: Provide a practical example, such as a production problem, where the goal is to maximize profit subject to resource constraints.
- Identify the components: Clearly identify decision variables (e.g., number of products), the objective (e.g., maximize profit), and constraints (e.g., labor hours, material limits).
- Conclude with the model: Show how the objective function and constraints are put together to create a mathematical model that can be solved using LP methods.

6. Explain the graphical method for solving a Linear Programming problem with two variables. Provide an example.

- Start by introducing the graphical method: Mention that the graphical method is used to solve Linear Programming problems with two decision variables, where the constraints and objective function can be visually represented on a graph.
- Steps to solve using the graphical method: Explain the key steps—plot the constraints as straight lines, identify the feasible region, and evaluate the objective function at the vertices of the feasible region to find the optimal solution.
- Provide a simple example: Present an LP problem with two variables, such as

maximizing profit subject to constraints on resources (e.g., labor, material).
Show how to plot the constraints and find the feasible region.

- Conclusion: Conclude by showing how the objective function is evaluated at the vertices, leading to the optimal solution.

7. What are the limitations of the graphical method for solving Linear Programming problems?

- Discuss the limitations of the graphical method: Start by acknowledging that while the graphical method is useful for small problems with two decision variables, it has several limitations when applied to larger problems.
- Limitations in terms of dimensionality: Explain that the graphical method is limited to problems with only two decision variables, as higher-dimensional problems cannot be effectively represented on a 2D graph.
- Complexity in identifying feasible regions: Mention that for problems with numerous constraints, identifying the feasible region graphically becomes increasingly difficult and impractical.

Inaccuracy for large-scale problems: Highlight that as the number of variables and constraints increases, the graphical method becomes cumbersome, and solving such problems is better suited to methods like the Simplex method.

22.13 Hints to Check Your Progress (CYP)

CYP 1:

1. **Objective Function and Constraints:** Refer to Section 22.5 (Formulating Linear Programming Problems), where the steps to define the objective function and constraints are discussed. Remember, the objective function represents the goal (either maximize or minimize), and the constraints represent the limitations or resource availability.
2. **Graphical Method:** Review Section 22.4 (Graphical Method for Solving Linear Programming Problems), especially 22.4.1 (Introduction to Graphical Method)

and 22.4.2 (Steps to Solve Using Graphical Method). Make sure you can graph the constraints, identify the feasible region, and evaluate the objective function at the vertices.

3. **Feasible Region:** Check Section 22.4 (Graphical Method) for understanding how to plot the feasible region, which is formed by the intersection of all the constraints.
4. **Vertices of the Feasible Region:** Understand the importance of vertices, as discussed in Section 22.4.2, where the optimal solution lies at one of the vertices of the feasible region.

CYP 2:

1. **Simplex Method Steps:** Refer to Section 22.6 (Simplex Method) and 22.6.2 (Steps Involved in the Simplex Method) for a step-by-step explanation of solving an LP problem using the Simplex method. Be clear on identifying the entering and leaving variables, pivoting, and updating the tableau.
2. **Duality:** Review Section 22.7 (Duality in Linear Programming), particularly 22.7.1 (Concept of Duality) and 22.7.2 (Primal and Dual Problems) to understand how every LP problem has an associated dual problem and how to interpret dual variables and constraints.
3. **Sensitivity Analysis:** Check Section 22.8 (Sensitivity Analysis), especially 22.8.2 (Importance of Sensitivity Analysis in LP) to learn how changes in the parameters of the objective function or constraints can affect the optimal solution.

22.14 Suggested Readings

1. “Introduction to Operations Research” by Frederick S. Hillier, Gerald J. Lieberman (2015)
2. “Operations Research: An Introduction” by Taha H. A. (2017)
3. “Linear Programming and Network Flows” by Mokhtar S. Bazaraa, John J. Jarvis, Hanif D. Sherali (2010)

4. “Applied Linear Programming” by Vladimir L. Kharitonov (2005)
5. “Operations Research: A Practical Introduction” by Michael P. C. L. Pantea (2019)
6. “Linear Programming: Foundations and Extensions” by Robert J. Vanderbei (2014)
7. “Introduction to Linear Optimization” by Dimitris Bertsimas, John N. Tsitsiklis (1997)

M.A. Economics	Lesson No. 23
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	Unit IV

FORMULATING LINEAR PROGRAMMING PROBLEMS

STRUCTURE

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- 23.2 Introduction
- 23.3 Problem Formulation: Definition
- 23.4 Mathematical Model of a Linear Programming Problem
- 23.5 Graphical Method for Solving LPPs
 - 23.5.1. Type I: Maximisation Problems (all constraints are $\leq$ type)
 - 23.5.2. Type II: Minimisation Problems (All Constraints are $\geq$ type)
 - 23.5.3. Type III: Maximisation and Minimisation Problems (Mixed Type Constraints)
- 23.6 Some Exceptional Cases in Linear Programming under Graphic Method
- 23.7 Let Us Sum Up
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- 23.11 Suggested Readings

23.0 Objectives

The main objectives of this lesson are to enable you to:

- Understand the meaning and significance of Linear Programming in economic decision-making.
- Identify decision variables, objective functions, and constraints in various practical scenarios.
- Translate real-world problems into linear programming models.
- Distinguish between maximization and minimization problems.
- Understand the graphical method of solving LPP with two variables.
- Recognize and interpret special cases in LPP, such as infeasibility, unboundedness, redundancy, and multiple optimal solutions.

23.1 Learning outcomes

After going through this lesson, you will be able to:

- Define and explain the structure of a Linear Programming Problem.
- Formulate LPPs from various real-world economic situations such as diet, production, and investment.
- Solve LPPs using graphical methods when the problem involves two decision variables.
- Identify feasible regions, corner points, and optimal solutions from graphical representations.
- Analyze exceptional cases in LPPs and understand their implications in economic modeling.

23.2 Introduction

In an economy characterized by scarce resources and competing uses, the optimal allocation of limited resources becomes crucial. Linear Programming (LP) is a mathematical technique that aids decision-makers in finding the best possible outcomes

under given constraints. Whether it is a business maximizing profit, a hospital minimizing dietary costs, or a manufacturer optimizing production, LP provides a structured method to achieve efficiency. This lesson introduces the process of formulating linear programming problems (LPP) from real-life scenarios and converting them into mathematical models, laying the groundwork for their solution using graphical and analytical methods.

23.3 Problem Formulation: Definition

Problem formulation is the process of transforming the verbal description of a decision problem into a mathematical form that can be solved. The main steps in the formulation of linear programming problem are as follows:

Step I: Identify the unknown variable (i.e., decision variables) to be determined and assign symbols $x_1, x_2, \dots$, etc., to them.

Step II: Identify the objective function and represent it as a linear function of decision variables.

Step III: Identify all the restrictions or constraints in the problem and express them as linear inequations or inequalities in terms of decision variables.

23.4 Mathematical Model of a Linear Programming Problem

A linear programming problem is abbreviated by LPP. The mathematical model of LPP is given below.

Suppose there are 2 decision variables (x and y) and 3 constraints. The objective function can be formulated as:

$$\text{Maximise } Z = k_1x + k_2y$$

Subject to constraints:

$$a_1x + b_1y \leq c_1$$

$$a_2x + b_2y \leq c_2$$

$$a_3x + b_3y \leq c_3$$

$$\text{and } x, y \geq 0$$

In the above formulation, decision variables are denoted by x and y . The

coefficients of the decision variable of the objective function are denoted by k_1 and k_2 .
The symbol c_1 , c_2 and c_3 represents constraints.

Formulation of minimization problem can be shown as:

$$\text{Minimise } C = k_1x + k_2y$$

Subject to constraints:

$$a_1x + b_1y \geq c_1$$

$$a_2x + b_2y \geq c_2$$

$$a_3x + b_3y \geq c_3$$

$$x, y \geq 0$$

It may be noted that in maximisation problem constraints appear as $\leq$ inequalities, and in minimisation problem the constraints appear as $\geq$ inequalities.

The following examples will make the procedure more clear:

Example A home decorator manufactures two types of lamps, say A and B. Both lamps go through two technicians, first a cutter and then a finisher. Lamp A requires 2 hours of the cutter's time and 1 hour of the finisher's time. Lamp B requires 1 hour of cutter's and 2 hours of finisher's time. The cutter has 104 hours and finisher has 6 hours of time available each month. Profit on Lamp A is Rs. 16.00 and on the B lamp is Rs. 12.00. Formulate the above problem as a linear programming problem.

Solution The given data is shown in the following table:

	Type A Lamp	Type B Lamp	Maximum available time
Time for cutting	2 hrs.	1 hr.	104
Time for finishing	1 hr.	2 hrs.	76
Profit	Rs. 16.00	Rs. 12.00	

Let the decorator manufactures x lamp of Type A and y lamp of Type B.

$$\therefore x \geq 0, y \geq 0$$

Maximum time for cutting = 104 hrs. $\therefore 2x + 1y \leq 104$

Maximum time for finishing = 76 hrs. $\therefore x + 2y \leq 76$

Let Z be the total profit $\therefore Z = 16x + 12y$

$\therefore$ The given L.P. problem reduces to:

Maximise $Z = 16x + 12y$

Subject to the constraints

$$2x + y \leq 104$$

$$x + 2y \leq 76$$

and $x \geq 0, y \geq 0$

Example A diet for a sick person must contain at least 4,000 units of vitamins, 50 units of minerals and 1,400 units of calories. Two foods A and B are available at a cost of Rs. 4 and Rs. 3 per unit respectively. One unit of food A contains 200 units of vitamins, 1 unit of mineral and 40 units of calories and one unit food B contains 100 units of vitamins, 2 unit of minerals and 40 units of calories. Formulate this problem as a linear programming problem for minimising costs.

Solution The given information is shown in the following table:

	Food A	Food B	Maximum Requirement
Vitamins	200	100	4,000
Minerals	1	2	50
Calories	40	40	1,400
Cost	Rs. 4	Rs. 3	

Let x units of Food A and y units of Food B be combined

$$\therefore x \geq 0, y \geq 0$$

The minimum daily requirement of vitamins is 4,000.

$$\therefore 200x + 100y \geq 4,000$$

The minimum daily requirement of minerals is 50.

$$\therefore x + 2y \geq 50$$

The minimum daily requirement of calories is 1,400.

$$\therefore 40x + 40y \geq 1400$$

Let Z be the total cost $\therefore Z = 4x + 3y$

$\therefore$ The given L.P. problem reduces to:

Minimise $Z = 4x + 3y$

Subject to the constraints $200x + 100y \geq 4,000$

$$x + 2y \geq 50$$

$$40x + 40y \geq 1,400$$

and $x \geq 0, y \geq 0$

The above constraints can be simplified as under:

$$2x + y \geq 40$$

$$x + 2y \geq 50$$

$$x + y \geq 35, x \geq 0, y \geq 0$$

Example A dealer deals in only two items: sewing machines and table fans. He has Rs. 5,760 to invest and a space to store at the most 20 items. A sewing machine costs Rs. 360 and a table fan Rs. 240. He can sell a sewing machine at a profit of Rs. 22 and a table fan at profit of Rs. 18. Formulate this problem as a linear programming problem.

Solution The given information is shown in the following table:

	Sewing Machine	Table Fan	Maximum
Cost per piece	Rs. 360	Rs. 240	Rs. 5,760
Profit per piece	Rs. 22	Rs. 18	-

Let the dealer purchases x sewing machine and y fans

$$\therefore x \geq 0, y \geq 0$$

Maximum number of items = 20

$$\therefore x + y \leq 20$$

Maximum investment = Rs. 5,760

$$\therefore 360x + 240y \leq 5,760 \text{ or } 3x + 2y \leq 48$$

Let Z be the total profit,

$$\therefore Z = 22x + 18y$$

$\therefore$ The given problem reduces to:

Maximise $Z = 22x + 18y$

Subject to the Constraints:

$$x + y \leq 20$$

$$360x + 240y \leq 5,760$$

$$\text{and } x \geq 0, y \geq 0$$

The above constraints can be simplified as under:

$$x + y \leq 20, 3x + 2y \leq 48, x \geq 0, y \geq 0$$

23.5 Graphical Method for Solving LPPs

The graphical method for the solution of the linear programming problem is used when the objective function is a linear function of two variables only. In this method, the following steps are involved:

Step I: Consider the constraints as equalities.

Step II: Sketch these linear constraint equation as straight lines. For example to draw line $2x + 3y = 6$, put $x = 0$ find y and which is 2 here. Therefore one point on the line will be (0, 2). Again put $y = 0$ and find x , which is 3 here. Another point on the line is (3, 0). Joining these points, we can find the line represented by the equation $2x + 3y = 6$.

Similarly for all other linear constraint equations sketch the lines.

Step III: Find the feasible region for the values of the variable, which is the region bounded by the lines drawn in Step-II.

To find permissible region for a constraint $2x + 3y \leq 6$.

Substitute origin (0, 0) in the corresponding inequation. If the origin satisfies the inequation (as in this case $0 \leq 6$ is true), then the left side of the line will be the permissible region, where origin lies. If the origin does not satisfy the inequation, then the right side of the line opposite the origin will be the required region.

Generally for greater than or equal to ($\geq$) constraints, the feasible region will be the area which lies above or to the right of the constraint line. For less than or equal to ($\leq$) constraints, this area is generally below or to the left of the line drawn.

Step IV: Find the common region satisfied by all the constraints (if any) and shade it.

Step V: Locate the corner points of the region, which may be obtained by simultaneously solving the two linear equations whose point of intersection is the required point.

Step VI: Calculate the value of the objective function at each corner point obtained in previous step. The set of values corresponding to the maximum or minimum values of the objective function is the solution of the LPP.

The following examples illustrate the graphical method of LPP:

23.5.1 Type I: Maximisation Problems (all constraints are $\leq$ type)

Example Solve graphically the following LPP:

$$\text{Maximise } Z = 4x + 5y$$

Subject to Constraints

$$2x + 3y \leq 12$$

$$2x + y \leq 8$$

$$\text{and } x, y \geq 0$$

Solution Consider the above constraints as equalities:

$$2x + 3y = 12$$

$$2x + y = 8$$

Draw these equations on the graph. Points for drawing lines are calculated as:

$$\text{From equation (i) } 2x + 3y = 12$$

$$\text{When } x = 0, 2(0) + 3y = 12 \Rightarrow 3y = 12 \Rightarrow y = 4 \therefore (0, 4)$$

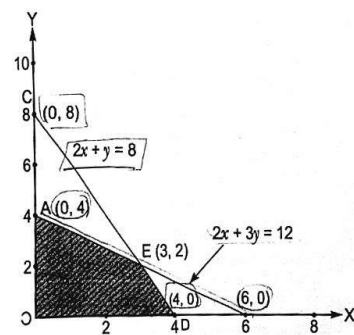
$$\text{When } y = 0, 2x + 3(0) = 12 \Rightarrow 2x = 12 \Rightarrow x = 6 \therefore (6, 0)$$

$$\text{From equation (ii) } 2x + y = 8$$

$$\text{When } x = 0, 2(0) + y = 8 \Rightarrow y = 8 \therefore (0, 8)$$

$$\text{When } y = 0, 2x + 0 = 8 \Rightarrow x = 4 \therefore (4, 0)$$

Now plotting the above equations on the graph as shown below:



The shaded region in the figure indicates the feasible region. This gives four corner points O, A, E and D where O is (0, 0), A is (0, 4), D is (4, 0).

To find E, E is the point of intersection of lines

$$3x + 3y = 12$$

$$2x + y = 8$$

Solving these equations, we get $x = 3$, $y = 2$

Thus, E is (3, 2).

Now, we shall test all feasible solutions for objective function to get the maximum value.

Corner Points	Coordinates of the Corner Points	Objective Function $Z = 4x + 5y$	Value
O	(0, 0)	$Z = 4(0) + 5(0)$	0
A	(0, 4)	$Z = 4(0) + 5(4)$	20
E	(3, 2)	$Z = 4(3) + 5(2)$	22
D	(4, 0)	$Z = 4(4) + 5(0)$	16

Thus, the maximum value of $Z = 22$ when $x = 3$ and $y = 2$.

Example

Solve graphically the following LPP:

Maximise $Z = 3x_1 + 2x_2$

Subject to constraints

$$-2x_1 + x_2 \leq 1$$

$$x_1 \leq 2$$

$$-x_1 - x_2 \geq -2$$

and $x_1, x_2 \geq 0$

Solution

Consider the above constraints into equalities

$$-2x_1 + x_2 = 1 \quad \dots\dots\dots (i)$$

$$x_1 = 2 \quad \dots\dots\dots (ii)$$

$$-x_1 - x_2 \geq -2 \quad \dots\dots\dots (iii)$$

Draw these equations on the graph paper. Points for drawing lines are calculated as follows:

From Equation (i) $-2x_1 + x_2 = 1$

$$\text{When } x_1 = 0 \quad x_2 = 1 \quad \therefore (0, 1)$$

$$\text{When } x_2 = 0 \quad -2x_1 = 1 \Rightarrow x_1 = -\frac{1}{2} \quad \therefore \left(-\frac{1}{2}, 0 \right)$$

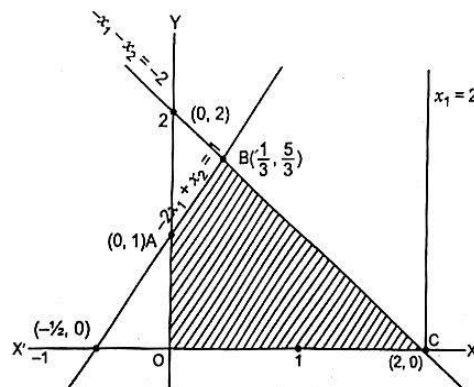
From Equation (ii) $x_1 = 2$

From Equation (iii) $-x_1 - x_2 = -2$

$$\text{When } x_1 = 0 \quad x_2 = 2 \quad \therefore (0, 2)$$

$$\text{When } x_2 = 0 \quad x_1 = 2 \quad \therefore (2, 0)$$

Now plotting the above equations on the graph as shown below:



The shaded region in the figure indicates the feasible region. This gives four points O, A, B and C where O is (0, 0), A is (0, 1) and C is (2, 0).

To find B, B is the point of intersection of lines

$$x_1 + x_2 = 2 \text{ and } -2x_1 + x_2 = 1$$

Solving these equations, we get $x_1 = \frac{1}{3}$, $x_2 = \frac{5}{3}$. Thus, B is

$$\left(\frac{1}{3}, \frac{5}{3} \right).$$

To find C, C is the point of intersection of lines.

$$x_1 + x_2 = 2, x_1 = 2$$

Solving these equations, we get $x_1 = 2$, $x_2 = 0$. Thus, C is (2, 0)

Now we shall test all feasible solution for objective function to get the maximum value.

Corner Points	Co-ordinates of the Corner Points	Objective Function $Z = 3x_1 + 2x_2$	Value
O	(0, 0)	$3(0) + 2(0)$	0
A	(0, 1)	$3(0) + 2(1)$	2
B	$\left(\frac{1}{3}, \frac{5}{3} \right)$	$3\left(\frac{1}{3} \right) + 2\left(\frac{5}{3} \right)$	$\frac{13}{3}$
C	(2, 0)	$3(2) + 2(0)$	6

Hence, Max $Z = 6$ when $x_1 = 2$, $x_2 = 0$

23.5.2 Type II: Minimisation Problems (All Constraints are $\geq$ type)

Example Using graphic method to solve the following LPP:

$$\text{Minimise } Z = 5x_1 + 3x_2$$

Subject to $2x_1 + x_2 \geq 3$

$$x_1 + x_2 \geq 2$$

and $x_1, x_2 \geq 0$

Solution Consider the above constraints as equalities

$$2x_1 + x_2 = 3 \quad \text{..... (i)}$$

$$x_1 + x_2 = 2 \quad \text{..... (ii)}$$

Draw these equations on graph. Points for drawing lines can be calculated as follows:

From equation (i) $2x_1 + x_2 = 3$

When $x_1 = 0$, $2(0) + x_2 = 3 \Rightarrow x_2 = 3 \quad \therefore (0, 3)$

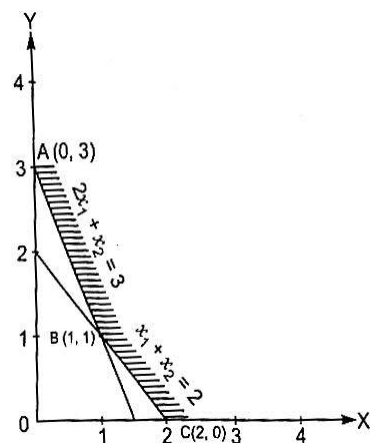
When $x_2 = 0$, $2x_1 + 0 = 3 \Rightarrow x_1 = \frac{3}{2} = 1.5 \quad \therefore (1.5, 0)$

From equation (ii) $x_1 + x_2 = 2$

When $x_1 = 0$, $0 + x_2 = 2 \Rightarrow x_2 = 2 \quad \therefore (0, 2)$

When $x_2 = 0$, $x_1 + 0 = 2 \Rightarrow x_1 = 2 \quad \therefore (2, 0)$

Plotting the above equations on the graph as shown below:



The shaded region indicates the feasible region. This gives three corner points A, B and C where A is (0, 3) and C is (2, 0).

To find B, B is the intersection of lines

$$2x_1 + x_2 = 3 \text{ and } x_1 + x_2 = 2$$

Solving these equations, we get $x_1 = 1$, $x_2 = 1$

Thus, B is (1, 1).

Now we shall test all feasible solutions for objective function to get the minimum values.

Corner Points	Co-ordinates	Objective Function Min. $Z = 5x_1 + 3x_2$	Value
A	(0, 3)	$5(0) + 3 \times 3$	9
B	(1, 1)	$5(1) + 3 \times 1$	8
C	(2, 0)	$5(2) + 3 \times 0$	10

Thus, the minimum value of $Z = 8$ when $x_1 = 1$ and $x_2 = 1$

Example

A diet for a patient must contain at least 4,000 units of Vitamins, 50 units of minerals and 1,400 units of calories. Two foods F_1 and F_2 are available at cost of Rs. 4 and Rs. 3 per unit respectively. If one unit of food F_1 contains 200 units of Vitamins, 1 unit of Mineral and 40 units of Calories and one unit of food F_2 contains 100 units of Vitamins, 2 units of Minerals and 40 units of Calories, find by graphic method, what combination of fooding should be used to have least cost.

Solution

The mathematical formulation of the given problem is as:

$$\text{Minimise } Z = 4x_1 + 3x_2$$

Subject to constraints

$$200x_1 + 100x_2 \geq 4,000$$

$$x_1 + 2x_2 \geq 50$$

$$40x_1 + 40x_2 \geq 1,400$$

and $x_1, x_2 \geq 0$

The above constraints can be simplified as under:

$$2x_1 + x_2 \geq 40, x_1 + 2x_2 \geq 50, x_1 + x_2 \geq 35, x_1, x_2 \geq 0$$

Consider the above constraints as equalities:

$$2x_1 + x_2 = 40 \quad \text{..... (i)}$$

$$x_1 + 2x_2 = 50 \quad \text{..... (ii)}$$

$$x_1 + x_2 = 35 \quad \text{..... (iii)}$$

Draw these equations on the graph. Points for drawing lines can be calculated as follows:

From equation (i) $2x_1 + x_2 = 40$

$$\text{When } x_1 = 0, x_2 = 40 \quad \therefore (0, 40)$$

$$\text{When } x_2 = 0, 2x_1 = 40 \Rightarrow x_1 = 20 \quad \therefore (20, 0)$$

From equation (ii) $x_1 + 2x_2 = 50$

$$\text{When } x_1 = 0, 2x_2 = 50 \Rightarrow x_2 = 25 \quad \therefore (0, 25)$$

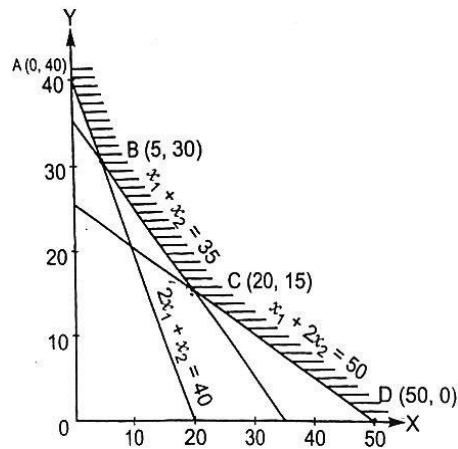
$$\text{When } x_2 = 0, x_1 = 50 \quad \therefore (50, 0)$$

From equation (iii) $x_1 + x_2 = 35$

$$\text{When } x_1 = 0, x_2 = 35 \quad \therefore (0, 35)$$

$$\text{When } x_2 = 0, x_1 = 35 \quad \therefore (35, 0)$$

Plotting the above equations on the graph as shown below:



The shaded region indicates the feasible region. This gives four points A, B, C and D for which A is (0, 40), D is (50, 0).

To find B, B is the point of intersection of the lines.

$$2x_1 + x_2 = 40, x_1 + x_2 = 35$$

Solving these equations, we get $x_1 = 5$ and $x_2 = 30$

Thus, B is (5, 30)

To find C, C is the point of intersection of the lines.

$$x_1 + 2x_2 = 50, x_1 + x_2 = 35$$

Solving these equation, we get $x_2 = 15$, $x_1 = 20$

Thus, C is (20, 15)

Now, we shall test all the feasible solutions for the objective functions to get the minimum value:

Corner Points	Co-ordinates	Objective Function Min. $Z = 4x_1 + 3x_2$	Value
A	(0, 40)	$Z = 4(0) + 3(40)$	120
B	(5, 30)	$Z = 4(5) + 3(30)$	110
C	(20, 15)	$Z = 4(20) + 3(15)$	125
D	(50, 0)	$Z = 4(50) + 3(0)$	200

Thus, the minimum value of $Z = 110$ when $x_1 = 5$ and $x_2 = 30$.

Hence, to have least cost, the diet should contain 5 units of food F_1 and 30 units of food F_2 .

Check your Progress-I

Based on your understanding of the previous section, solve these questions:

Q1. A firm manufacturing two types of products A and B sells them at a profit of Rs. 2.00 on type A and also Rs. 2.00 on type B. Each product is processed on two machines M_1 and M_2 . Type B requires one minute on M_1 and one minute on M_2 . The machine M_1 is available for not more than 6 hours 40 minutes while machine M_2 is available for 10 hours during any working day. Formulate the given problem as a linear programming problem for maximum profits.

Q2. Consider two different types of food stuffs say F_1 and F_2 . Assume that these food stuffs contain vitamins V_1 , V_2 and V_3 respectively. Minimum daily requirements of three vitamins are 1mg of V_1 , 50mg of V_2 and 10mg of V_3 . Suppose that food stuff F_1 contains 1mg of V_1 , 100mg of V_2 and 10 mg of V_3 , whereas the food stuff F_2 contains 1mg of V_1 , 10mg of V_2 and 100mg of V_3 . Cost of one unit of F_1 is Rs. 1 and that of F_2 is Rs. 1.5. Formulate the LPP for minimum cost diet that would supply the body at least the minimum requirements of each vitamin.

Machine	Chair	Table	Time Available
M ₁	3	3	36
M ₂	5	2	50
M ₃	2	6	60

- Q3. A company makes two types of furniture: chairs and tables. The profit for each accounting department is ₹20 per chair and ₹30 per table. Both products are to be processed on three machines M₁, M₂, and M₃. The time required (in hours) by each product and total time available in hours per week on each machine on each machine is as follows:

How should the manufacturer schedule his production in order to maximize profit? Use graphical method to solve this problem.

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-
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- Q4. A manufacturer of furniture makes two products: chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B. A table requires 5 hours on machine A and no time on machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. Profit gained by the manufacturer from a chair and table is ₹0.2 and ₹0.10 respectively. What should be the daily production of each of the two products to have maximum profit?
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- Q5. Vitamin A and B are available in two different foods P and Q. One unit of P contains 2 units of vitamin A and 3 units of vitamin B. One unit of Q contains 5 units of vitamin A and 4 units of vitamin B. The minimum daily consumption of vitamin A and B should be 1000 and 1500 units respectively. One unit of P costs ₹5 and one unit of Q costs ₹6. What should be the intake of P and Q in order to minimise cost? (Use graphic method).

23.5.3 Type III: Maximisation and Minimisation Problems (Mixed Type Constraints)

Example Solve the following LPP by graphic method:

$$\text{Maximise } Z = 2x_1 - 3x_2$$

$$\text{Subject to: } 4x_1 + 5x_2 \leq 40$$

$$2x_1 + 6x_2 \leq 24$$

$$3x_1 - 3x_2 \geq 6$$

$$x_1 \geq 4$$

$$\text{and } x_1, x_2 \geq 0$$

Solution Consider the above constraints as equalities:

$$4x_1 + 5x_2 = 40 \quad \dots\dots\dots (i)$$

$$2x_1 + 6x_2 = 24 \quad \dots\dots\dots (ii)$$

$$3x_1 - 3x_2 = 6 \quad \dots\dots\dots (iii)$$

$$x_1 = 4 \quad \dots\dots\dots (iv)$$

Draw these equations on the graph. Points for drawing lines are calculated as follows:

From Equation (i) $4x_1 = 5x_2 = 40$

$$\text{When } x_1 = 0 \quad 5x_2 = 40 \Rightarrow x_2 = 8 \quad \therefore (0, 8)$$

$$\text{When } x_2 = 0 \quad 4x_1 = 40 \Rightarrow x_1 = 10 \quad \therefore (10, 0)$$

From Equation (ii) $2x_1 + 6x_2 = 24$

$$\text{When } x_1 = 0 \quad 6x_2 = 24 \Rightarrow x_2 = 4 \quad \therefore (0, 4)$$

$$\text{When } x_2 = 0 \quad 2x_1 = 24 \Rightarrow x_1 = 12 \quad \therefore (12, 0)$$

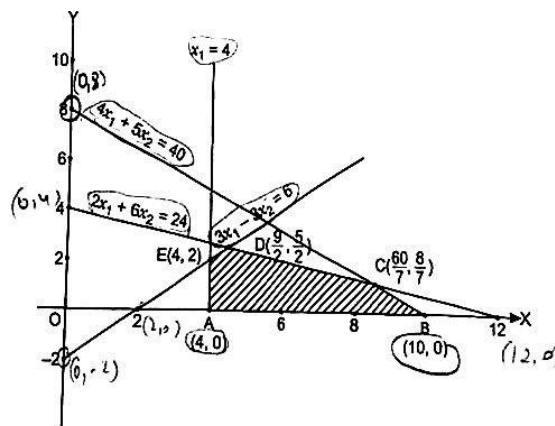
From Equation (iii) $3x_1 - 3x_2 = 6$

$$\text{When } x_1 = 0 \quad -3x_2 = 6 \Rightarrow x_2 = -2 \quad \therefore (0, -2)$$

$$\text{When } x_2 = 0 \quad 3x_1 = 6 \Rightarrow x_1 = 2 \quad \therefore (2, 0)$$

From Equation (iv) $x_1 = 4 \quad \therefore (4, 0)$

Now plotting the above lines on the graph paper as shown below:



The shaded region in the figure indicate the feasible region. This gives us five points A, B, C, D and E where A is (4, 0) and B is (10, 0).

To find C, C is the point of intersection of lines

$$4x_1 + 5x_2 = 40 \text{ and } 2x_1 + 6x_2 = 24$$

Solving these equation, we get $x_1 = \frac{60}{7}$, $x_2 = \frac{8}{7}$

$$\text{Thus, C is } \left(\frac{60}{7}, \frac{8}{7} \right)$$

To find D, D is the point of intersection of lines.

$$3x_1 - 3x_2 = 6 \text{ and } 2x_1 + 6x_2 = 24$$

Solving these equations, we get $x_1 = \frac{9}{2}$, $x_2 = \frac{5}{2}$

$$\text{Thus, D is } \left(\frac{9}{2}, \frac{5}{2} \right)$$

To find E, E is the point of intersection of lines.

$$3x_1 - 3x_2 = 6 \text{ and } x_1 = 4$$

Solving these equations, we get $x_1 = 4$, $x_2 = 2$

Thus, E is (4, 2)

Now we shall test all feasible solutions for objective function to get the maximum value.

Corner Points	Co-ordinates of the Corner Points	Objective Function $Z = 2x_1 + 3x_2$	Value
A	(4, 0)	$(2 \times 4) - (3 \times 0)$	8
B	(10, 0)	$(2 \times 10) - (3 \times 0)$	20
C	$\left(\frac{60}{7}, \frac{8}{7}\right)$	$\left(2 \times \frac{60}{7}\right) + \left(3 \times \frac{8}{7}\right)$	$\frac{96}{7}$
D	$\left(\frac{9}{2}, \frac{5}{2}\right)$	$\left(2 \times \frac{9}{2}\right) + \left(3 \times \frac{5}{2}\right)$	$\frac{3}{2}$
E	(4, 2)	$(2 \times 4) - (3 \times 2)$	2

Thus, Maximise $Z = 20$ when $x_1 = 10$ and $x_2 = 0$.

Example

Solve the following LPP by graphic method:

Maximise $Z = 2.75x_1 + 4.15x_2$

Subject to:

$$2x_1 + 2.5x_2 \leq 100$$

$$7.5x_1 + 5x_2 \geq 150$$

$$4x_1 + 8x_2 \geq 160$$

and $x_1, x_2 \geq 0$

Solution

Consider the above constraints as equalities:

$$2x_1 + 2.5x_2 = 100 \quad \text{..... (i)}$$

$$7.5x_1 + 5x_2 = 150 \quad \text{..... (ii)}$$

$$4x_1 + 8x_2 = 160 \quad \text{..... (iii)}$$

Draw these equations on the graph. Points for drawing lines are calculated as follows:

From equation (i) $2x_1 + 2.5x_2 = 100$

When $x_1 = 0$, $2(0) + 2.5x_2 = 100 \Rightarrow x_2 = 40 \therefore (0, 40)$

When $x_2 = 0$, $2x_1 + 2.5(0) = 100 \Rightarrow x_1 = 50 \therefore (50, 0)$

From equation (ii) $7.5x_1 + 5x_2 = 150$

When $x_1 = 0$, $7.5(0) + 5x_2 = 150 \Rightarrow x_2 = 30 \therefore (0, 30)$

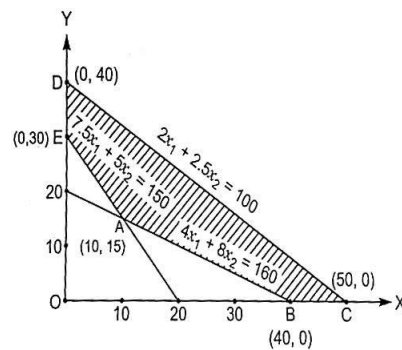
When $x_2 = 0$, $7.5x_1 + 5(0) = 150 \Rightarrow x_1 = 20 \therefore (20, 0)$

From equation (iii) $4x_1 + 8x_2 = 160$

When $x_1 = 0$, $4(0) + 8x_2 = 160 \Rightarrow x_2 = 20 \therefore (0, 20)$

When $x_2 = 0$, $4x_1 + 8(0) = 160 \Rightarrow x_1 = 40 \therefore (40, 0)$

Now plotting the above equations on the graph as shown below:



The shaded region in the figure indicates the feasible region for values of x_1 and x_2 . This gives five corner points A, B, C, D and E where B is (40, 0), C is (50, 0), D is (0, 40) and E is (0, 30).

To find A, A is the point of intersection of lines

$$4x_1 + 8x_2 = 160 \text{ and } 7.5x_1 + 5x_2 = 150$$

Solving these equations, we get $x_1 = 10$, $x_2 = 15$

Thus, A is (10, 15)

Solving these equations, we get $x_1 = 10$, $x_2 = 15$

Thus, A is (10, 15)

Now we shall test all feasible solutions for objective function to get the maximum value.

Corner Points	Co-ordinates of Corner Points	Objective Function $Z = 2.75x_1 + 4.15x_2$	Value
A	(10, 15)	$Z = 2.75(10) + 4.15(15)$	89.75
B	(40, 0)	$Z = 2.75(40) + 4.15(0)$	110.0
C	(50, 0)	$Z = 2.75(50) + 4.15(0)$	137.5
D	(0, 40)	$Z = 2.75(0) + 4.15(40)$	166
E	(0, 30)	$Z = 2.75(0) + 4.15(30)$	124.5

Hence, the maximum value of $Z = 166$ when $x_1 = 0$ and $x_2 = 40$.

23.6 Some Exceptional Cases in Linear Programming under Graphic Method

So far we have discussed only those cases in which a unique optimum solution could be found. But there may exist a few problems where no solution exists or for which the only solution exists is an unbounded one or sometimes multiple solutions may exist for a given problem.

1) Infeasible Solution (or No Solution): Sometimes the system of constraints in a LPP has no common point which satisfies all the constraints. In such cases, the LPP is said to have no feasible solution or infeasible solution.

The following examples illustrate the cases of infeasible solution:

Example Use graphical method to solve the following LPP:

Maximise $Z = x + 4y$

Subject to the constraints:

$$x + y \leq 1$$

$$2x + 2y \geq 4$$

$$x \geq 0, y \geq 0$$

Solution Consider the above constraints as equalities

$$x + y = 1 \quad \dots\dots\dots (i)$$

$$2x + 2y = 4 \quad \dots\dots\dots (ii)$$

Draw the above equations on the graph. Points for drawing lines are calculated as follows:

From equation (i) $x + y = 1$

When $x = 0, y = 1 \quad \therefore (0, 1)$

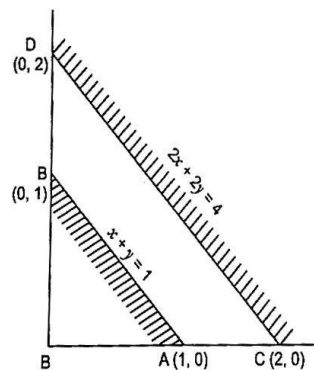
When $y = 0, x = 1 \quad \therefore (1, 0)$

From equation (ii) $2x + 2y = 4$

When $x = 0, 2y = 4 \Rightarrow y = 2 \quad \therefore (0, 2)$

When $y = 0, 2x = 4 \Rightarrow x = 2 \quad \therefore (2, 0)$

Now plotting the above points on the graph as shown below:



The shaded region indicates the feasible region. In this graph, we find that at no point the two constraints intersect each other and thus, they have no point or area in common. The two portions are disjoint. Thus, we cannot find a single point (x, y) which can be in both the regions. Hence the given LPP has no optimal solution.

2) Unbounded solution: A linear programming problem may have unbounded solution which means that it has no limit on the constraints. It simply means that the common feasible region is not bounded in any respect. The primary variables can take any value in the unbounded region. The objective function can be made infinitely large. The following example illustrate the case of unbounded solution.

Example Using graphical method to solve the following LPP:

$$\text{Maximise } Z = 4x_1 + 5x_2$$

Subject to constraints:

$$x_1 + x_2 \geq 1$$

$$-2x_1 + x_2 \leq 1$$

$$4x_1 - 2x_2 \geq 1$$

$$\text{and } x_1, x_2 \geq 0$$

Solution Consider the above constraints as equalities

$$x_1 + x_2 = 1 \quad \dots\dots\dots (i)$$

$$-2x_1 + x_2 = 1 \quad \dots\dots\dots (ii)$$

$$4x_1 - 2x_2 = 1 \quad \dots\dots\dots (iii)$$

Draw these equations on the graph. Points for drawing lines are obtained as follows:

From equation (i) $x_1 + x_2 = 1$

$$\text{When } x_1 = 0, x_2 = 1 \quad \therefore (0, 1)$$

$$\text{When } x_2 = 0, x_1 = 1 \quad \therefore (1, 0)$$

From equation (ii) $-2x_1 + x_2 = 1$

$$\text{When } x_1 = 0, x_2 = 1 \quad \therefore (0, 1)$$

$$\text{When } x_2 = 0, x_1 = -\frac{1}{2} \quad \therefore \left(-\frac{1}{2}, 0 \right)$$

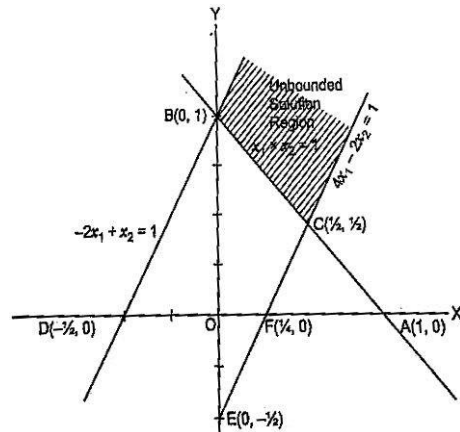
From equation (iii) $4x_1 - 2x_2 = 1$

$$\text{When } x_1 = 0, x_2 = -\frac{1}{2} \quad \therefore \left(0, -\frac{1}{2} \right)$$

$$\text{When } x_1 = 0, x_2 = -\frac{1}{2} \quad \therefore \left(0, -\frac{1}{2} \right)$$

$$\text{When } x_2 = 0, x_1 = \frac{1}{4} \quad \therefore \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

Plotting these points on the graph, as shown below:



3) Multiple Optimum Solution: In certain linear programming problems, situation may arise when there is the possibility of more than more optimum solution. The following example illustrates the case of multiple optimum solution.

Example Use graphical method to solve the following LPP:

$$\text{Maximise } Z = 10x_1 + 6x_2$$

Subject to constraints:

$$5x_1 + 3x_2 \leq 30$$

$$x_1 + 2x_2 \leq 18$$

$$\text{and } x_1, x_2 \geq 0$$

Solution Consider the above constraints as equalities

$$5x_1 + 3x_2 = 30 \quad \dots\dots\dots (i)$$

$$x_1 + 2x_2 = 18 \quad \dots\dots\dots (ii)$$

Draw these equations on the graph. Points for drawing lines are obtained as follows:

From equation (i) $5x_1 + 3x_2 = 30$

When $x_1 = 0$, $3x_2 = 30 \Rightarrow x_2 = 10 \therefore (0, 10)$

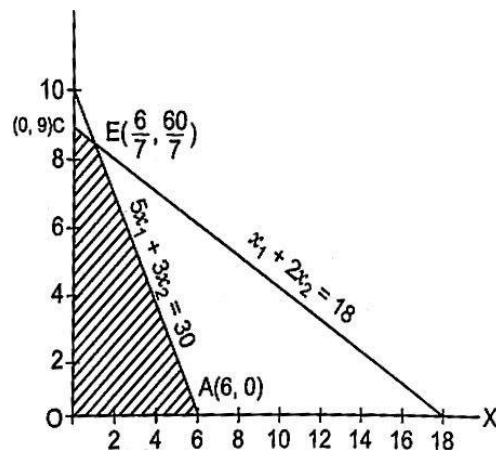
When $x_2 = 0$, $5x_1 = 30 \Rightarrow x_1 = 6 \therefore (6, 0)$

From equation (ii) $x_1 + 2x_2 = 18$

When $x_1 = 0$, $2x_2 = 18 \Rightarrow x_2 = 9 \therefore (0, 9)$

When $x_2 = 0$, $x_1 = 18 \therefore (18, 0)$

Plotting these points on the graph as shown below:



The shaded region indicates the feasible region. Thus, gives points O, C, E and A where O is (0, 0), C (0, 9), A is (6, 0)

To find E, E is the point of intersection of lines

$5x_1 + 3x_2 = 30$ and $x_1 + 2x_2 = 18$

Solving these, we get $x_1 = \frac{6}{7}$, $x_2 = \frac{60}{7}$

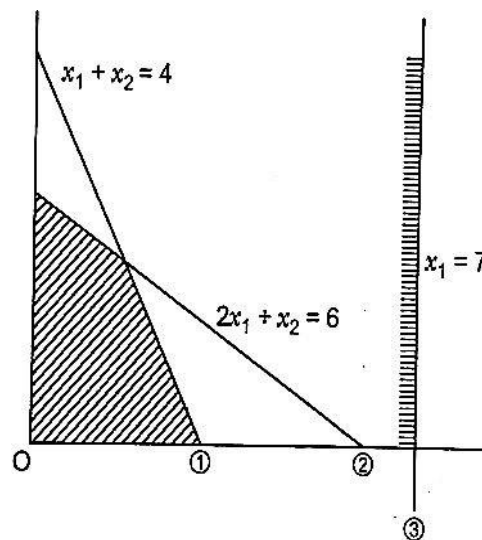
Thus, E is $\left(\frac{6}{7}, \frac{60}{7}\right)$

Now, we shall test all feasible solutions for objective function to get the maximum value.

Corner Points	Co-ordinates of Corner Points	Objective Function Max. $Z = 10x_1 + 6x_2$	Value
C	(0, 9)	$Z = 10(0) + 6(9)$	54
E	$\left(\frac{6}{7}, \frac{60}{7}\right)$	$Z = 10 \times \frac{6}{7} + 6 \times \frac{60}{7}$	60
A	(6, 0)	$Z = 10 \times 6 + 6(0)$	60

Thus, it is clear that at two points E and A, the value of $Z = 60$. Thus, no unique optimal solution exists for this problem. Such a problem is said to have alternative optimal solution. There are in fact, an infinite number of values of x_1 and x_2 that give the same value of the objective function. Thus, any point on the line between A and B give an optimal solution.

4) Redundant Constraint: A linear programming problem may have some time a redundant constraint. A redundant constraint is simply one that does not affect the feasible solution region. Consider the following example:



Maximise	$P = 3x_1 + x_2$	 (i)
Subject to	$x_1 + x_2 \leq 4$	 (ii)
	$2x_1 + x_2 \leq 6$	 (iii)
	$x_1 \leq 7$	

$$x_1, x_2 \geq 0$$

The third constraint $x_1 \leq 7$ is redundant as it has no effect on the possible region obtained.

Check your Progress-II

Based on your understanding of the previous section, solve these questions:

Q1. Solve graphically the following LPP:

$$\text{Maximise } Z = 5x_1 + 4x_2$$

Subject to the constraints

$$2x_1 - 4x_2 \leq 1$$

$$2x_1 + 4x_2 \geq 3$$

where $x_1, x_2 \geq 0$

Q2. Use graphic method to solve the following LPP:

$$\text{Maximise } Z = 10x + 20y$$

Subject to the constraints

$$x + 2y \leq 2$$

$$2x + 6y \geq 18$$

where $x_1, x_2 \geq 0$

-
-
- Q3. Write a note on infeasibility and unboundedness in the graphic method of LPP.

-
-
-
-
- Q4. Use graphically the following LPP:

$$\text{Maximise } Z = 6x_1 + x_2$$

Subject to the constraints

$$2x_1 + x_2 \geq 3$$

$$x_1 + x_2 \geq 2$$

where $x_1, x_2 \geq 0$

23.7 Let Us Sum Up

In this lesson, we studied the formulation of Linear Programming Problems (LPP). We learned how to interpret real-life situations and translate them into mathematical models. The chapter covered both maximization and minimization problems subject to given constraints. Additionally, we explored the methods to identify feasible regions and optimal solutions. The graphical method of solving LPP was also introduced and explained in detail.

23.8 Keywords

1. **Linear Programming (LPP):** A mathematical technique used for optimization, where the objective is to maximize or minimize a linear function subject to a set of linear constraints.
2. **Objective Function:** The function in an LPP that needs to be maximized or minimized. It is expressed as a linear function of decision variables, such as $Z = ax + by$.
3. **Decision Variables:** The unknowns in an LPP whose values are to be determined in order to optimize the objective function.
4. **Constraints:** These are the restrictions or limitations expressed as linear inequalities or equations that the decision variables must satisfy.
5. **Feasible Region:** The common region on the graph that satisfies all the constraints of an LPP, including the non-negativity conditions.
6. **Corner Points (Extreme Points):** The vertices of the feasible region where the maximum or minimum value of the objective function is likely to occur.
7. **Graphical Method:** A method of solving an LPP when there are only two variables, by plotting the constraints on a graph to find the feasible region and evaluating the objective function at the corner points.
8. **Maximization Problem:** An LPP where the objective is to obtain the highest possible value of the objective function.
9. **Minimization Problem:** An LPP where the goal is to obtain the lowest possible value of the objective function.
10. **Infeasible Solution:** A situation where no point satisfies all the constraints simultaneously, resulting in no feasible region.
11. **Unbounded Solution:** When the feasible region is open and extends infinitely in some direction, allowing the objective function to increase or decrease indefinitely.

12. **Multiple Optimal Solutions:** A case where the objective function yields the same maximum or minimum value at more than one point on the boundary of the feasible region.
13. **Redundant Constraint:** A constraint that does not affect the feasible region and can be removed without changing the solution.
14. **Simplified Constraints:** Constraints rewritten into their simplest or more interpretable form for ease of solving graphically.
15. **Standard Form of LPP:** A formulation where the objective function is to be maximized or minimized, subject to linear inequalities and non-negativity constraints on the variables.
16. **Binding Constraint:** A constraint that forms a boundary of the feasible region and affects the optimal solution directly.
17. **Non-negativity Restriction:** A condition in LPP where the decision variables cannot take negative values (i.e., $x \geq 0$, $y \geq 0$).

23.9 Examination Oriented Questions

- Q1. Explain how you would identify the cases of redundant constraints, infeasible solution, multiple optimal solution and unbounded solution from the graph of LPP involving two variables.
- Q2. Given the following LPP:
- Maximise $Z = 4x_1 + 4x_2$
- Subject to the constraints
- $x_1 + 2x_2 \leq 10$
- $6x_1 + 6x_2 \leq 36$
- $x_1 \leq 4$
- where $x_1, x_2 \geq 0$
- Q3. Solve graphically the following LPP:

$$\text{Maximise } Z = 5x_1 + 4x_2$$

Subject to the constraints

$$2x_1 - 4x_2 \leq 1$$

$$2x_1 + 4x_2 \geq 3$$

$$\text{where } x_1, x_2 \geq 0$$

- (a) Solve the above problem graphically.
- (b) Is there any multiple optimal solution? Explain.

Q4. Solve graphically the following LPP:

$$\text{Maximise } Z = x_1 + 4y$$

Subject to the constraints

$$x + y \leq 1$$

$$2x + 2y \geq 4$$

$$\text{Where } x_1, x_2 \geq 0$$

- Q5. A hospital wants to prepare a diet using two food items X and Y. Each unit of X contains 3 units of protein and 5 units of iron. Each unit of Y contains 4 units of protein and 2 units of iron. The minimum daily requirement is 100 units of protein and 80 units of iron. Cost per unit of X is ₹5 and Y is ₹4. Formulate and solve graphically the LPP to minimize cost.
- Q6. A manufacturer makes toys using plastic and metal. A plastic toy gives a profit of ₹5 and a metal toy gives a profit of ₹6. One plastic toy requires 4g of plastic and 2g of metal. One metal toy requires 2g of plastic and 4g of metal. There are 80g of plastic and 60g of metal available. Formulate and solve the LPP to maximize profit.

Q7. A farmer has 1 hectare of land for growing wheat and barley. The profit per hectare is ₹50 for wheat and ₹40 for barley. Wheat requires 3 hours of labor per unit and barley requires 2 hours. Only 100 hours of labor are available. Formulate the LPP to maximize profit.

23.10 Hints to Check your Progress (CYP)

CYP- I

- Q1. Refer Section 23.4
- Q2. Refer Section 23.4
- Q3. Refer Section 23.5.1
- Q4. Refer Section 23.5.1
- Q5. Refer Section 23.5.2

CYP – II

- Q1. Refer Section 23.5.3
- Q2. Refer Section 23.5.3
- Q3. Refer Section 23.6
- Q4. Refer Section 23.5.3

23.11 Suggested Readings

- Aggarwal, C.S. & Joshi, R.C. (2017). Mathematics for Students of Economics. New Academic Publishing Co.
- Allen, R.G.D. (1974). Mathematical Analysis for Economists. Macmillan.
- Anthony, M. & Biggs, N. (1996). Mathematics for Economics and Finance: Methods and Modelling. Cambridge University Press.
- Black, J. & Bradley, J.F. (1987). Essential Mathematics for Economists. John Wiley & Sons.

- Dowling, E.T. (2003). Introduction to Mathematical Economics. Schaum's Outline Series, McGraw-Hill Education.
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- Mehta, B.C. & Madnani, G.M.K. (2014). Mathematics for Economists. Sultan Chand & Sons.

M.A. Economics	Lesson No. 24
C.No. : Eco-103	Semester - 1st
	Unit IV

**SOLVING LINEAR PROGRAMMING PROBLEMS/
QUALITY IN LINEAR PROGRAMMING/APPLICATIONS IN
ECONOMICS AND OPERATIONS RESEARCH
STRUCTURE**

- 24.0 Objectives/Learning Outcomes
- 24.1 Introduction
- 24.2 Graphical Method of Solving LPP (Linear Programming Problem)
- 24.3 Simplex Method for Solving LPP (Linear Programming Problem)
- 24.4 Primal and Duality
 - 24.4.1 Basic Theorems of LPP
- 24.5 Let us Sum Up
- 24.6 Examination Oriented Questions
- 24.7 Hints to check your Progress
- 24.8 Suggested Readings

24.0 Objectives/Learning Outcomes

After going through this lesson, we shall be able to understand :

- ☐ how to solve a LPP graphically.
- ☐ how to solve a LPP by simpler method.
- ☐ what is duality and primal in LPP.

24.1 Introduction

The graphical method for the solution of the LPP is used when the objective function is a linear function of two variables only while simpler method is widely used method for solving a linear programming problem involving two or more variables. It was developed by Dantzing.

24.2 Graphical Method of Solving LPP (Linear Programming Problem)

Rules/Steps to solve LPP by graphic method :

- (i) Formulate the LPP in mathematical form in terms of series of mathematical constraints and an objective function.
- (ii) Construct the graph for the problem so formulated by drawing constraint line. We write each inequality in the constraint equation as equality. Give any arbitrary value to one variable and get the value of other variable by solving the equation. Similarly give another arbitrary value to the variable and determine the corresponding value of other variable. We plot these two sets of values. Join these points by a straight line. We carried out this exercise for each of the constraint equations. So, we have as many as many straight lines as there are equations, each straight line representing one constraint.
- (iii) Identify feasible region (solution space or convex region) and find the vertices (extreme points) of the feasible region (convex region). We determine the area which satisfies all the constraints simultaneously. For greater than and greater than or equal to constraints, the feasible region will be the area lying above the constraint line. For less than and less than or equal to constraints, this area is generally the region below these lines.
- (iv) Now we select the graphic solution techniques and proceed to solve.
 - (a) First we identify each of the convex or extreme points of the feasible region either by visual inspection or by the method of simultaneous equations.
 - (b) Evaluate the value of objective function at these vertices.

- (c) Determine the vertex of which objective function attains its maximum or minimum.
- (d) Interpret the optimum solution so obtained.

Q. Solve by graph.

Maximize $Z = 8x + 6y$,

subject to $6x + 3y \leq 126$

$$2x + 4y \leq 96$$

and $x, y \geq 0$.

Sol. Step I First constraint is

$$6x + 3y \leq 126$$

when $x = 0$, then $y = 42$ [$\square 6(0) + 3y = 126$]

this way, we get first point (A) as (0, 42)

To get second point put $y = 0$

then, $x = 21$ [$\square 6x + 3(0) = 126$]

$\therefore$ Second point (B) is (21, 0)

By joining these we get a straight line AB.

Second Constraint.

When $x = 0$, then $y = 24$ [$\square 2x + 4y = 96$]

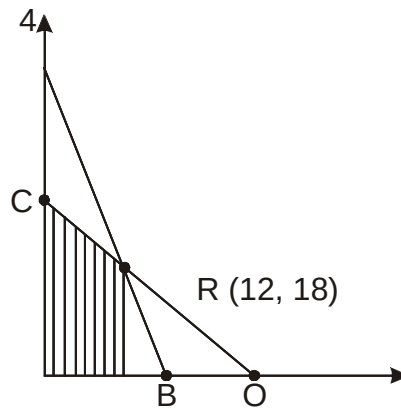
$\therefore$ C (0, 24)

When $y = 0$, then $x = 48$

$\therefore$ Point D will be (48, 0)

By joining these points, we get a straight line CD.

Step II. After plotting the two constraints, the next step is to find the possible region. The possible region is that region in the graph which satisfies all the constraints simultaneously.



The region CRBO region represents the feasible region.

These extreme points are of significance in optimal solution. Optimal solution will always be one of these extreme points.

Point $C(0, 24)$, $x = 0$, $y = 24$

$$\begin{aligned} Z &= 8x + 6y \\ &= 8 \times 0 + 6 \times 24 \\ &= 144 \end{aligned}$$

Point $B(21, 0)$, $x = 21$, $y = 0$

$$\begin{aligned} Z &= 8 \times 21 + 6 \times 0 \\ &= 168 \end{aligned}$$

Point $R(12, 18)$, $x = 12$, $y = 18$

$$\begin{aligned} Z &= 8 \times 12 + 6 \times 18 \\ &= 204 \end{aligned}$$

So, combination $R(12, 18)$ is the optimal and 204 is maximum possible level of profit.

Q. Minimize $C = 6x + 30y$

Subject to $x + 2y \geq 3$

$x + 21y \geq 4$

and, $x \geq 0, y \geq 0$

Sol.

$$A = (0, 1.5)$$

$$B = (3, 0)$$

$$C = (0, 1)$$

$$D = (4, 0)$$

$$R = (2, 5)$$

D(4, 0) gives least cost and ₹24 is the minimum level of cost.

Q. Solve the following LP problem by graphic method.

$$\text{Mini } 2x_1 + 3x_2$$

$$\text{Subject to constraint } x_1 + x_2 \geq 8,$$

$$3x_1 + 5x_2 \geq 30$$

$$2x_1 + x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

Sol.

$$\text{At } A(0, 12)$$

$$Z = 2 \times 0 + 3 \times 12 = 36$$

$$\text{At } N(4, 4)$$

$$Z = 2 \times 4 + 3 \times 4 = 20$$

$$\text{At } M(5, 3)$$

$$Z = 2 \times 5 + 3 \times 3 = 19$$

$$\text{At } D(10, 0)$$

$$Z = 2 \times 10 + 0 = 20$$

Z is min at M(5, 3) and min Z = 19

Q. Solve it :

i) Max. $f = 2x + 5y$ subject to

$$x + 4y \leq 24$$

$$3x + y \leq 21$$

$$x + y \leq 9$$

$$x \geq 0, y \geq 0$$

i) Maximize $z = 8x + 6y$

subject to $6x + 3y \leq 126$

$$2x + 4y \leq 96$$

iii) Maximize $f = 2x + 5y$

subject to

$$x + 4y \leq 24$$

$$3x + y \leq 21$$

$$x + y \leq 9$$

$$x \geq 0, y \geq 0$$

Check Your Progress-I

(i) Maximize $f = 3x + 7y$ subject to

$$x + 4y \leq 28$$

$$3x + y \leq 21$$

$$x + y \leq 9$$

$$x \geq 0, y \geq 0$$

(ii) Maximize $f = 2x_1 + 3x_2$ subject to

$$x_1 + x_2 \geq 10$$

$$3x_1 + 5x_2 \geq 15$$

$$2x_1 + x_2 \geq 14$$

$$x_1, x_2 \geq 0$$

24.3 Simplex Method for Solving LPP (Linear Programming Problem)

Simple method of solving linear programming problems is the most powerful and general method. It was developed by an American Scholar Mr. Dantzig in 1947. It is a repetitive technique and is based on the property that the optimum solution to a linear programming problem if it exists is always one of the basic feasible solutions. The idea of the simplex method is to start with some initial extreme point, compute the value of the objective function and then see whether the latter can be improved upon by moving an adjacent extreme point. For proper understanding of simplex method, let us discuss few definitions.

(a) Standard Form. When all the constraints in a linear programming are written as equalities, it is called standard form. The optimum solution of the standard form of LPP is the same as the optimum solution of the original formulation of the LPP.

(b) Slack variable :- when we add a variable to the left hand side of a less than or equal to constraint to convert the constraint into an equality, the variable is called slack variable. The value of the variable can usually be interpreted as the amount of

unused resources.

Def. Let the constraints of general LPP be $\sum_{j=1}^n a_{ij} x_j \leq b_i, \quad k+1 \leq i <$

Then the non-negative variable S_{n+i} which satisfies.

$\sum_{j=1}^n a_{ij} x_j + S_{n+i} = b_i$ are called slack variables.

(i.e.) S_{n+1} is slack variables.

(c) Surplus Variables. When we subtract a variable from the left hand side of 'a greater than or equal to' constraint to convert the constraint into an equality that variable is called surplus variable.

Let the constraints of general LPP be $\sum_{j=1}^n a_{ij} x_j \geq b_i, \quad 1 + k \leq i \leq$

Then the non-negative variable, S_{n+1} , which satisfies

$\sum_{j=1}^n a_{ij} x_j - S_{n+i} = b_i$ is called surplus variable.

24.3 Simplex Method's Steps to Solve

Max. $a_1x_1 + a_2x_2$ s.t.

$$b_{11}x_1 + b_{12}x_2 \leq c_1$$

$$b_{21}x_1 + b_{22}x_2 \leq c_2$$

$$x_1, x_2 \geq 0$$

Step I. Convert inequalities into equations by adding a slack variables.

$$b_{11}x_1 + b_{12}x_2 + s_1 = c_1$$

$$b_{21}x_1 + b_{22}x_2 + s_2 = c_2$$

$$a_1x_1 + a_2x_2 + 0.s_1 + 0.s_2, \text{ where } x_1, x_2, s_1, s_2 \geq 0$$

Step II. The simplex Tabular

	x_1	x_2	s_1	s_2	
	b_{11}	b_{12}	1	0	c_1
	b_{21}	b_{22}	0	1	c_2
z_1	$-a_1$	$-a_2$	0	0	

Choose the smallest of a_1, a_2 ,

(i.e.) $\min(a_1, a_2)$

that column $\rightarrow$ smallest no. is the privileged column

Suppose, $-a_2 < -a_1$

then a_2 is a privileged column.

Dividing each constraints by the corresponding element of the pre-column.

$$\frac{c_1}{b_{12}}, \frac{c_2}{b_{22}}$$

If $\frac{c_1}{b_{12}} < \frac{c_2}{b_{22}}$ then b_{12} is pivotal element

last column is column of constraints.

Dividing that each row by b_{12} .

Step III. Find privilege column and pivotal element.

Step IV. Divide the row that contains the pivotal element by pivotal element.

Step V. Transfers Row and find the value.

Q. Maximize $z = 20x_1 + 25x_2$

subject to

$$12x_1 + 16x_2 \leq 100$$

$$16x_1 + 8x_2 \leq 80$$

$$x_1, x_2 \geq 0$$

Solve LPP by using simple method.

Sol. Since the LPP is of maximization form, so we use slack variables s_1 , and s_2 for each constraint and equate them to zero.

$$\therefore 12x_1 + 16x_2 + s_1 = 100$$

$$16x_1 + 8x_2 + s_2 = 80$$

our objective function is -

$$z = 20x_1 + 25x_2 + 0s_1 + 0s_2$$

Let's construct a simple table for LPP

	C_i	Basic variables	Solution variables	20	25	0	0	Min Ratio
	C_B			x_1	x_2	s_1	s_2	
R_1	0	s_1	100	12	(16)	1	0	$\frac{100}{16} = 6.25$
R_2	0	s_2	80	16	8	0	1	$\frac{80}{8} = 10$
		Z_i		0	0	0	0	
		$Z_i - C_i$		-20	-25	0	0	

Since -25 is least, so column corresponding to -25 becomes our key column.

To find the minimum ratio, divide the solution variable by the respective element of the key column.

The value which is minimum in min. Ratio is known as min. ratio and the row corresponding to it is "key row".

By intersection of key row and key column, you will get a "key element", i.e., 16.

Now, we will construct interaction - I. Since key element belongs to column x_2 and row s_1 .

So introducing x_2 and replacing s_1 , in table - 2

	C_j	Basic variables	Solution variables	20	25	0	0	Min Ratio
	C_B			x_1	x_2	s_1	s_2	
R_1	25	x_2	$100/16 = 25/4$	$3/4$	1	$1/16$	0	3
R_2	0	s_2	30	(10)	0	$-1/2$	1	3
		Z_i		$75/4$	25	$25/16$	0	
		$Z_i - C_i$		$-5/4$	0	$25/16$	0	

Now, apply the operation on R'_2 as

$$R'_2 = R_2 - 8R'_1$$

5

Since, $-\frac{5}{4}$ in the $z_j - c_j$ is minimum, so we took the column corresponding to this value as key column. Then, dividing the solution variable by the key column to get the minimum value and corresponding to the minimum value and corresponding to the minimum ratio, there will be a key row. Again, the intersection will provide a key element, i.e. 10.

Similarly, do the interaction - II by introducing x_1 and replacing s_2 . In table 3.

Repeat this process, until $z_j - c_j$ becomes zero.

C_j	Basic variables	Solution variables	20	25	0	0
C_B			x_1	x_2	s_1	s_2
25	x_2	4	0	1	$1/10$	$-3/40$
20	x_1	3	1	0	$-1/20$	$1/10$
	Z_i	160	20	25	1.5	18.12
	$Z_i - C_i$		0	0	1.5	18.12

As all $z_j - c_j$ are positive, so we stop here, as $z_j - c_j$ are also '0' for x_1 and x_2 .

$\therefore$, Basic variable and solution variable gives the value of $x_1 = 3$ and $x_2 = 4$.

and, max. z as 160.

$$\begin{aligned}\text{Check : } z &= 20(x_1) + 25(x_2) + 0s_1 + 0s_2 \\ &= 20(3) + 25(4) \\ &= 60 + 100 \\ &= 160\end{aligned}$$

$\therefore$, z is maximum at $x_1 = 3$ and $x_2 = 4$, with the value of 160.

Solve it :

i) $\text{Max } z = 4x_1 + 6x_2$
 subject to constraints

$$\begin{aligned}x_1 + 3x_2 &\leq 240 \\ 3x_1 + 4x_2 &\leq 370 \\ 2x_1 + x_2 &\leq 180 \\ x_1, x_2 &\geq 0\end{aligned}$$

ii) $\text{Max } z = 5x + 8y$
 subject to constraints

$$\begin{aligned}x + 2y &\leq 8 \\ 3x + 4y &\leq 20 \\ x, y &\geq 0\end{aligned}$$

iii) $\text{Min } 8u + 20v \text{ st } 4 + 3v \geq 5$

$$\begin{aligned}2u + 4v &\geq 8 \\ 4.v &> 0\end{aligned}$$

iv) $\text{Min } 8u + 10v \text{ st } 24 + v \geq 1$

$$u + 2v \geq 1$$

Note. In that cases first of all find dual of the function and then find its values through simplex method that are maximising value. In order to find the minimising values, the value of $s_1 = x$ and $s_2 = y$

∴ Consider and put these values in M in function to get min value.

Check Your Progress-II

Solve using simplex method

Maximize $z = 5x + 8y$ subject to

$$x + 2y \leq 8$$

$$3x + 4y \leq 20$$

$$x, y \geq 0$$

24.4 PRIMAL AND DUALITY

In the linear programming, we are to maximise or minimize the given function. The fact every linear programming problem has a counter part called its “dual”. The given problem which is to be maximised or minimized is called primal problem. If the primal problem is one of maximizing the function, then dual problem is one of minimizing the function. Similarly, if the primal is a minimization problem, then the dual is a maximization problem. Every primal problem can be transferred into dual and every dual problem can be transferred into primal problem.

PRIMAL AND DUALITY

Relationship. The primal and dual problems can be derived from each other and we have a unique dual (primal) associated with the primal (dual).

Consider the general linear programming problems,

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$x_1 - 2x_2 + 5x_3 \geq 3,$$

$$4x_1 + 7x_2 - 2x_3 \geq 2, x_1, x_2, x_3 \geq 0$$

Sol. The dual to given primal problem is:

$$\text{Maximize } f = 7y_1 + 4y_2 + 10y_3 + 3y_4 + 2y_5$$

$$\text{subject to, } 3y_1 + 6y_2 + 7y_3 + y_4 + 4y_5 \leq 3$$

$$5y_1 + y_2 - 2y_3 - 2y_4 + 7y_5 \leq -2$$

$$4y_1 + 3y_2 - y_3 + 5y_4 - 2y_5 \leq 2,$$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

24.4.1 Basic Theorems of LPP

Dual of the dual is a primal

Dual Formation Procedure

1. If the primal problem is of minimization, the dual problem is of maximisations and vice-versa.
2. If the primal problem involves $\leq$ signs in the constraints, the dual problem will have $\geq$ signs in its constraints and vice-versa.
3. The coefficients of objective function in the primal problem take the position of constraints and vice-versa.
4. In the constraints inequalities, the coefficient which are found by going from left to right in the primal problem are positioned in the dual problem from top to bottom and vice-versa.
5. A new set of variables appears in the dual.
6. Sign of non-negativity constraints do not change.
7. Neglecting the number of non-negativity constraints, if these are 'n' variables and 'm' constraints in the primal problem, in the dual, there will be 'm' variables and n constraints.

24.5 Let us sum up

We end this lesson by summarizing what we have covered in it :

- i) Graphical methods of solving LPP.
- i) Simplex Methods of solving LPP.
- ii) Primal and Quality.

24.6 EXAMINATION ORIENTED QUESTIONS

- Q.1 What is linear programming ? What are its underlying assumptions ?
- Q.2 Write the steps of solving LPP by simplex method ?.
- Q.3 Use simplex method of solve the following LPP.

Max. $Z = 107x_1 + x_2 + 2x_3$, subject to constraint

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7$$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 \leq 5$$

$$3x_1 + x_2 - x_3 \leq 0$$

$$x_1, x_2, x_3 \geq 0$$

- Q. 4. Solve the following LPP graphically, maximise $Z = 20x_1 + 10x_2$, subject to
 $x_1 + 2x_2 \leq 40$, $3x_1 + x_2 \leq 30$, $4x_1 + 3x_2 \leq 60$, $x_1, x_2 \geq 0$

24.7 HINTS TO CHECK YOUR PROGRESS

CYP-I

Solve using examples as previously solved.

CYP-II

Use examples of solving a LPP on simplex method.

24.8 SUGGESTED READINGS

Black. J. and Bradley. J. F. Essential Mathematics for Economists.

Henderson J. M. and Quandt. R. E. Microeconomic Theory : A Mathematical Approach.

Aggarwal, C.S & R.C Joshi : Mathematics for students of Economics (New Academic Publishing Co.)

Kandoi B. : Mathematics and Economic Will Applications (Himalaya Publishing House).

Aggarwal S.C., Rana R.K., Gupta Leena : Mathematics for Economics. (VK Global Publications Pvt. Ltd.)

M.A. ECONOMICS

LESSON NO. 25

C.No. : Eco-103

Semester - 1st

UNIT-IV

BASICS OF INPUT-OUTPUT ANALYSIS

STRUCTURE

- 25.0 Objectives/Learning Outcomes
- 25.1 Introduction
- 25.2 Input-output analysis
 - 25.2.1 Main features of input-output analysis
 - 25.2.2 Assumptions of input-output analysis
 - 25.2.3 Types of input-output models
- 25.3 Let us sum up
- 25.4 Examination Oriented Questions
- 25.5 Hints to Check Your Progress
- 25.6 Suggested Readings

25.0 OBJECTIVES/LEARNING OUTCOMES

The basic objective of their lesson is to study the input output model and its reference in the decision making across sectors.

25.1 INTRODUCTION

Input and output analysis is a method of calculating income and employment multipliers which takes account of differences in technology between industries and of the linkages between industries. Its analysis seeks to explain how one industry sector affects others

in the same nation or region. The analysis illustrates that the output of one sector can in turn become an input for another sector, which results in an interlinked economic system. The analysis is represented as a matrix, where different rows and columns are filled with values representing the inputs and outputs of various sectors.

25.2 INPUT-OUTPUT ANALYSIS

Output analysis developed by the 20th century Russian born U.S economist Wassily W. Leontief, in which the interdependence of an economy's various productive sectors is observed by viewing the product of each industry both as a commodity demanded for final consumption and as a factor in the production of itself and other goods. The analysis usually involves constructing a table in which each horizontal row describes how one industry's total product is divided among various production processes and final consumption. Each vertical column denotes the combination of productive resources used within one industry.

25.2.1 Main features of input-output analysis

- (i) The input-output analysis handles a problem of purely technological nature. It studies the production structure or nature of output mix of an economy consisting a number of producing sector.
- (ii) This technique doesn't deal with demand analysis.
- (ii) This analysis is based on empirical study i.e., the basic thrust of input-output analysis is empirical investigation rather than theoretical explanations.
- (iv) The input-output analysis incorporates the nature of inter-dependence between various sectors/industries in an economy. It consists of solving a set of equation.

25.2.2 Assumptions of input-output analysis

The input-output analysis is based on the following assumptions :

- (i) The economy is divisible into two sectors the inter-industry sector and the final demand sector. Each sector can be further sub-divided.
- (ii) Each sector/industry produces only one commodity and not two products are produced jointly.

- (ii) All inputs are used in rigidly fixed proportion. It implies that model is static in nature.
- (iv) There is no substitution between inputs.
- (v) Prices of commodities as well as factors of production are given.
- (vi) Amount of factor services as well as nature of consumer demand is given.
- (vi) There are constant returns to scale.
- (vii) There are no external economies and diseconomies of production.
- (ix) Both inputs and outputs can be expressed in monetary units.
- (x) There is perfect competition in the producing sector.

25.2.3 Types of input-output models

Broadly speaking, there are two types of input-output models viz., closed and open.

- (i) **Closed Input-Output Model :** In this model, the entire production is consumed by those participating in the production i.e., production is consumed internally. In the closed model, final demand and primary inputs do not appear.
- (ii) **Open Input-Output Model :** In this model, some of the production is consumed by those who produce it and the rest of the production is consumed by external agencies such as households, govt. and so on. There is, thus, an additional sector of final demand in the open model.

CHECK YOUR PROGRESS

- (i) Discuss the main features of input-output analysis.

-
- (ii) What are the main assumptions of input-output analysis.

25.3 LET US SUM UP

We end this lesson by summarising what we have covered in it :

- (i) Input-Output Analysis
- (ii) Types of input-output models
- (iii) Assumptions of input-output analysis

25.4 EXAMINATION ORIENTED QUESTIONS

- Q.1. Discuss in detail the input-output analysis along with its assumptions.
- Q.2. Explain the different types of input-output models.

25.5 HINTS TO CHECK YOUR PROGRESS

- (i) Go through section 25.2.1.
- (ii) Go through section 25.2.2.

25.6 SUGGESTED READINGS

Black. J. and Bradley. J. F. Essential Mathematics for Economists.

Henderson J. M. and Quandt. R. E. Microeconomic Theory : A Mathematical Approach.

Aggarwal, C.S & R.C Joshi : Mathematics for students of Economics
(New Academic Publishing Co.)

Kandoi B. : Mathematics and Economic Will Applications (Himalaya Publishing House).

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M.A. Economics

Lesson No. 26

C.No. : Eco-103

Semester - 1st

Unit IV

STRUCTURE AND ASSUMPTIONS OF INPUT-OUTPUT MODELS

STRUCTURE

- 26.0 Objectives
- 26.1 Learning Outcomes
- 26.2 Introduction
- 26.3 Structure of Input-Output Models
 - 26.3.1 Definition and Components
 - 26.3.2 Key Components
 - 26.3.3 Balanced System
- 26.4 Assumptions of Input-Output Models
- 26.5 Applications of Input-Output Models
- 26.6 Limitations of Input-Output Models
- 26.7 Practical Examples
- 26.8 Let's Sum Up
- 26.9 Keywords
- 26.10 Examination-Oriented Questions
- 26.11 Hints to Check Your Progress (CYP)

26.12 Suggested Readings

26.0 Objectives

The main objectives of this lesson are to enable you to:

1. Discuss the structure of input-output models.
2. State the key components involved in input-output models.
3. Examine how input-output models are used to analyze economic relationships.
4. Explore the assumptions underlying input-output models.
5. Apply input-output models in real-world economic scenarios.
6. State the limitations of input-output models.
7. Gain insights into the practical uses of input-output analysis in economic planning and policy formulation.

26.1 Learning Outcomes

After going through this lesson, you will be able to:

1. Define and describe the components of an input-output model.
2. Discuss the assumptions that underpin the input-output analysis.
3. Apply input-output models to solve economic problems.
4. Examine the applications and limitations of input-output models.
5. Interpret the results of input-output analysis in various sectors.
6. Describe the economic concepts of interdependencies and multipliers.
7. Formulate input-output models for practical economic situations.

26.2 Introduction

Input-output models are an essential tool in economics that analyze the interdependencies between different sectors of an economy. Developed by economist

Wassily Leontief in the 1930s, these models provide a framework to understand how industries and sectors interact, how the output of one sector serves as input to another, and how economic activities are connected across the entire economy. The power of input-output analysis lies in its ability to break down complex economic systems into manageable, quantitative relationships, making it possible to forecast economic outcomes, allocate resources efficiently, and assess the impact of policy changes.

An input-output model is represented by an input-output matrix, which details the flow of goods and services between different sectors in an economy. In this matrix, each sector produces goods that are both used by other sectors as inputs and sold to final consumers. By using these models, economists can trace how changes in one sector can affect other sectors and the economy as a whole, allowing them to predict the ripple effects of economic decisions, such as government spending or changes in demand.

Input-output models are particularly useful in economic planning and policy analysis, as they provide a clear representation of how resources are distributed across sectors and how the demand for certain goods and services influences overall economic output. Governments and businesses alike rely on these models to make informed decisions about investments, infrastructure projects, and industrial development.

One of the most powerful applications of input-output models is their ability to evaluate the multiplier effect—the concept that an increase in spending in one sector can lead to a larger overall increase in economic activity. For example, when the government invests in infrastructure projects, the construction sector not only directly benefits but also boosts demand for materials, labor, and services in other sectors. By analyzing these relationships through input-output models, economists can better understand how investments in specific sectors influence broader economic growth.

While these models provide a comprehensive view of an economy's structure, it's essential to acknowledge that input-output models rely on certain assumptions. For example, they assume that input-output ratios remain constant over time, which may not always be the case in a dynamic economy where technological advances or changes in consumer preferences could alter production processes. Furthermore, these models tend to treat economic sectors as homogeneous, ignoring the nuances of individual producers or markets.

Despite these assumptions and limitations, input-output models have found wide-ranging applications. From helping policymakers design economic strategies to allowing businesses to make more informed decisions regarding supply chain management, these models offer invaluable insights into the flow of goods, services, and capital across economies.

In this lesson, we will explore the **structure of input-output models**, the **assumptions** that underpin them, and how they can be applied in real-world economic scenarios. Understanding these concepts will allow you to analyze the complex interactions within an economy and make better decisions based on data-driven insights.

26.3 Structure of Input-Output Models

Input-output models serve as a fundamental analytical tool in economics to study how different sectors of an economy are interrelated. These models represent the flow of goods and services between sectors in a given economy, providing insights into how the output of one industry serves as an input to another, and how this cyclical exchange affects overall economic activity. Developed by Wassily Leontief in the 1930s, input-output analysis has since become a vital component of economic planning, forecasting, and policy analysis. This section aims to provide a detailed breakdown of the structure of input-output models, exploring the definition and components, the key components of an input-output system, and the concept of a balanced system.

26.3.1 Definition and Components

An input-output model is a quantitative economic model that represents the relationships between different sectors of an economy. The model uses a matrix to show how the output of one industry (sector) serves as an input to another industry. The input-output matrix captures these interconnections and allows analysts to track how goods and services are transferred from one sector to another. By using this model, economists can predict the effects of changes in one sector on other sectors of the economy.

Key Elements of Input-Output Models:

1. **Decision Variables:** These are the amounts of goods or services that

are produced or consumed in various sectors. The decision variables represent the quantities that are determined through the model.

2. **Output and Input Coefficients:** The core of an input-output model is the representation of the relationship between outputs and inputs. Output coefficients describe how much output from a given sector is used as input by other sectors. Input coefficients specify how much input from other sectors is needed to produce one unit of output in a particular sector.
3. **Final Demand:** The final demand for products represents the external demand for goods and services that are not used as inputs by other sectors. This includes consumer spending, government expenditure, exports, and other external purchases.
4. **Technical Coefficients (A):** In the input-output matrix, the technical coefficient matrix (denoted as **A**) represents the amount of input from one sector required to produce one unit of output in another sector. It defines the input-output relationship within the system.
5. **Leontief Matrix:** The Leontief matrix is used to express the system in a more compact form, often used to analyze the flow of goods and services across industries. It represents the direct input-output relationships between industries.
6. **Exogenous Variables:** These are variables that are determined outside the model, such as changes in government policy, foreign demand, or technological advancements. They are inputs to the model but not determined by it.
7. **Endogenous Variables:** The endogenous variables are the quantities of goods produced by each sector in the economy. They are determined by solving the input-output equations.

An input-output model relies on a system of equations that connects these components to describe the economy's structure. The relationships between different sectors, industries, and the final demand for goods are analyzed to determine how changes in one part of the system will affect the others.

26.3.2 Key Components

Input-output models are composed of several key components that form the backbone of the system. Understanding these components is critical for analyzing and interpreting the data that the model produces. Here, we break down the major components in detail:

1. The Input-Output Matrix

The most fundamental component of an input-output model is the input-output matrix. This matrix shows how the output of each sector (column) is used as an input by other sectors (rows). The matrix is typically denoted as A , where each element a_{ij} represents the amount of output from sector j required as an input for sector i . In other words, a_{ij} tells us how much of the output from sector j is used by sector i .

For example, consider a simple economy with three sectors: agriculture (A), manufacturing (B), and services (C). The input-output matrix might look like this:

$$A = \begin{matrix} & \begin{matrix} a_{11} & a_{12} & a_{13} \end{matrix} \\ \begin{matrix} a_{21} \\ a_{31} \end{matrix} & \begin{matrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{matrix} \end{matrix}$$

Where:

- a_{11} represents the amount of agricultural output used by the agriculture sector.
- a_{12} represents the amount of agricultural output used by the manufacturing sector.
- a_{13} represents the amount of agricultural output used by the services sector, and so on for the other sectors.

The input-output matrix is used to compute the total output of each sector based on the demands for the outputs of other sectors.

2 Final Demand Vector

The final demand vector is a vector representing the total demand for goods and services from outside the economic system. This includes demand from consumers, government, and exports. The final demand vector is often denoted as y .

For example, if the economy has three sectors, the final demand vector might look like this:

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Where:

- y_1 is the final demand for goods from sector 1 (e.g., agriculture),
- y_2 is the final demand for goods from sector 2 (e.g., manufacturing),
- y_3 is the final demand for goods from sector 3 (e.g., services).

The final demand vector helps to understand how much of each sector's output is consumed externally, rather than being used as an input for other sectors.

3. Leontief Inverse Matrix

The **Leontief inverse matrix** is used to calculate the total output required to meet a certain level of final demand. It is derived from the input-output matrix and accounts for both direct and indirect relationships between sectors. The inverse matrix is denoted as $(I-A)^{-1}$, where I is the identity matrix, and A is the input-output matrix.

The Leontief inverse allows analysts to calculate how much total output is needed from all sectors in the economy to meet a given final demand vector. It incorporates the direct and indirect effects of changes in demand across all sectors.

4. Technical Coefficients Matrix (A)

The technical coefficients matrix (A) is a key component that represents the relationship between inputs and outputs within each sector. It contains the coefficients that define how much of the output from one sector is used by another sector. The matrix provides a crucial link between the production process in each industry and the broader economic system.

The matrix allows analysts to assess how much output is required from each sector to meet the total demand from both intermediate and final consumers.

5. Exogenous Variables

Exogenous variables are those that are determined outside of the input-output model. These include things like government policies, external trade, and technological changes that are not part of the immediate system being analyzed. While they are not directly determined by the input-output model, exogenous variables play a critical role in influencing the overall results of the analysis.

For example, an increase in government spending or changes in foreign demand for exports will affect the final demand for goods and services, which in turn will affect the output from various sectors.

6. Endogenous Variables

Endogenous variables, in contrast, are those that are determined by the input-output system itself. These variables represent the output produced by each sector in response to the demand for their goods and services. By solving the system of equations defined by the input-output model, the endogenous variables are calculated, providing the total output levels needed for each sector in the economy.

26.3.3 Balanced System

In the context of input-output models, a balanced system refers to a situation where the total output of an economy is equal to the total demand, both intermediate and final. This means that the sum of all outputs produced within the system must equal the sum of all inputs used within the system. In a balanced system, the economy's output is efficiently allocated to meet both the production needs of various sectors and the final demand for goods and services.

Conditions for a Balanced System:

1. **Equilibrium of Inputs and Outputs:** In a balanced input-output system, the sum of inputs required to produce output must equal the output produced by each sector.
2. **Final Demand Balance:** The total output produced within the economy should meet both the intermediate needs (inputs for other sectors) and final consumption needs (e.g., household consumption, government expenditure, exports).

3. Stable Economy: A balanced input-output system assumes a stable economy with constant input-output coefficients, meaning that the technological relationships between sectors do not change over time.

When the input-output model is balanced, the analysis becomes more straightforward, as the system can be solved using standard algebraic techniques. However, in practice, the system may not always be perfectly balanced due to changes in external factors (like government policy or market conditions), which may require adjustments to the model.

Understanding the structure of input-output models is crucial for analyzing how different sectors of the economy interact and how resources are allocated across various industries. The input-output matrix, final demand vector, and Leontief inverse matrix are the key components of these models, allowing economists to understand the flow of goods and services within an economy. Furthermore, the concept of a balanced system is essential for ensuring that the model accurately reflects the real-world dynamics of an economy. By applying these models, economists can gain valuable insights into how changes in one sector affect the overall economic structure and forecast the potential outcomes of economic decisions.

26.4 Assumptions of Input-Output Models

Input-output models are powerful tools used for understanding the interdependencies between different sectors of an economy. They allow economists and policymakers to analyze how the output of one sector is used as input by other sectors, helping in economic planning, forecasting, and assessing the impact of policy changes. However, the application and results of these models are based on several assumptions that may not always hold true in real-world scenarios. It is important to understand these assumptions to correctly interpret the results of input-output analysis and to recognize the model's limitations.

In this section, we will explore the key assumptions of input-output models in detail, providing an understanding of the simplified structure these models assume about the economy. These assumptions are integral to the model's functionality and serve as the foundation for input-output analysis.

1. Linearity of Relationships (Constant Proportions)

One of the most significant assumptions of input-output models is that the relationships between sectors in the economy are linear. This means that the output of each sector is assumed to be a fixed proportion of the inputs used from other sectors. In other words, the amount of input required from one sector to produce one unit of output in another sector remains constant, regardless of the scale of production.

For example, if Sector A uses 3 units of input from Sector B to produce 1 unit of output, the input-output relationship between these sectors is assumed to remain constant. If Sector A decides to double its production, it will need exactly twice the input from Sector B.

- **Implication:** The assumption of linearity is essential for the simplicity and practicality of input-output models. It allows the use of technical coefficients, which represent the fixed input-output ratios between sectors. However, in reality, production functions may not always exhibit constant returns to scale. For example, as production increases, economies of scale may lead to decreased input requirements per unit of output, which this assumption does not account for.
- **Real-World Issue:** In real economies, relationships between sectors can be non-linear due to changing technologies, resource efficiencies, or diminishing returns to scale. For instance, larger firms might experience lower unit costs of production, which the linear input-output model does not capture.

2. Fixed Coefficients (No Substitution Between Inputs)

Another crucial assumption of input-output models is that input coefficients are fixed. This means that the amount of input required from each sector to produce a unit of output in another sector remains constant over time. This assumption also implies that there is no substitution between inputs—each sector requires specific quantities of inputs, and these inputs cannot be replaced with others.

For example, if Sector A requires 3 units of labor and 5 units of raw materials to produce 1 unit of its output, the input-output model assumes that these coefficients

remain the same, regardless of changes in the production scale or technological advancements.

- **Implication:** The fixed coefficients assumption simplifies the model and makes it computationally feasible. It allows the use of input-output matrices to represent economic transactions. However, this assumption restricts the model's ability to accommodate changes in production technology or input substitution. In reality, technological advancements might lead to more efficient production processes, reducing the reliance on specific inputs.
- **Real-World Issue:** In practice, industries often innovate and substitute between inputs to optimize costs or respond to changes in supply and demand. For instance, a company might replace labor with automation or shift from one material to another based on price fluctuations. Input-output models, with their fixed coefficients, fail to capture such flexibility.

3. Closed Economy (No Trade with External Economies)

Most input-output models assume a closed system, meaning that all economic transactions occur within the economy being modeled, with no external trade or interactions with other economies. This assumption implies that the input-output model does not consider imports or exports, which means that the entire flow of goods and services remains within the domestic economy.

- **Implication:** The assumption of a closed system simplifies the model, as it excludes the complexities of international trade. The model is limited to understanding how goods and services flow within the boundaries of a single economy. This approach is useful when analyzing the internal structure of an economy but does not account for the influence of global trade.
- **Real-World Issue:** Most economies are open economies, meaning they engage in trade with other countries. Input-output models that exclude imports and exports ignore the effects of foreign demand and supply. Changes in international markets—such as fluctuations in global

commodity prices, trade policies, or foreign investment—are not accounted for in a closed system model.

4. Homogeneity of Products (No Product Differentiation)

Input-output models generally assume that the products produced by a sector are **homogeneous**—that is, they are assumed to be identical, regardless of the producer. This assumption implies that all units of output in a given sector are indistinguishable and have the same quality, regardless of the firm producing them.

- **Implication:** The assumption of homogeneity simplifies the input-output model by treating the entire output of a sector as a single, undifferentiated commodity. It allows the model to treat sectors as aggregates and provides a straightforward way to model input-output relationships.
- **Real-World Issue:** In practice, most products within a sector are **heterogeneous**. For example, in the automobile industry, different car models produced by various manufacturers might have different specifications, prices, and qualities. The input-output model does not account for such product differentiation, which can be crucial for industries with multiple firms and varying products.
- **Example:** The agriculture sector might produce a variety of crops, such as wheat, corn, and rice, which differ in terms of quality and demand. By treating these products as homogeneous, the model may overlook crucial distinctions that could affect economic analysis and policy decisions.

5. No Technological Change or Productivity Improvements

Another key assumption of input-output models is that technology and productivity remain constant over time. The model assumes that the input-output coefficients do not change due to technological advancements or improvements in productivity. This means that the relationships between inputs and outputs are fixed, and there is no consideration of how technological innovation might change production processes.

- **Implication:** The assumption of constant technology simplifies the model

by providing a static framework to analyze economic relationships. It allows the model to focus on current production techniques and resource allocation without worrying about the effects of future technological changes.

- **Real-World Issue:** In the real world, technological progress can drastically alter the production functions of industries. For example, the introduction of automation in manufacturing can significantly reduce labor requirements, while improvements in energy efficiency can lower the amount of energy required for production. The input-output model, assuming no technological change, cannot account for such dynamic shifts in production processes.

6. Assumption of Economic Equilibrium

Input-output models assume that the economy is in equilibrium, meaning that the total output produced by all sectors equals the total demand for goods and services (both intermediate and final demand). This implies that there is no excess supply or demand in the economy.

- **Implication:** The equilibrium assumption allows the model to work with a system of equations where inputs and outputs are perfectly balanced. The model assumes that all the goods produced are consumed either as inputs to other sectors or as final demand.
- **Real-World Issue:** In reality, economies are rarely in perfect equilibrium. There may be **supply-demand imbalances** due to factors like market fluctuations, shifts in consumer preferences, or changes in resource availability. Input-output models, assuming equilibrium, do not capture these dynamics and may provide oversimplified conclusions in scenarios where imbalances exist.

7. Static Nature of the Model (No Time Dynamics)

Input-output models are typically static, meaning they represent a snapshot of the economy at a single point in time. The model does not account for temporal dynamics, such as changes over time in production processes, demand patterns, or

technological advancements. It assumes that all relationships between sectors are fixed during the period being analyzed.

- **Implication:** The static nature of input-output models allows for simpler analysis and modeling of economic systems. It focuses on the structure of the economy at a given moment, without the complexity of time-dependent changes.
- **Real-World Issue:** In dynamic economic systems, variables change over time due to factors like technological progress, economic cycles, or policy shifts. A static model cannot account for such changes, making it less effective for long-term forecasting or modeling economic growth over multiple periods.

Input-output models provide valuable insights into the structure and functioning of an economy. However, the assumptions underlying these models—such as linear relationships, fixed coefficients, a closed system, and no technological change—can limit their applicability to real-world situations. While the assumptions simplify the mathematical modeling process, they do not fully capture the complexities and dynamic nature of real economies. For example, the assumption of homogeneity in products, fixed input-output coefficients, and constant technological processes can overlook the nuances of market differentiation, technological innovation, and changing economic conditions.

Understanding these assumptions is crucial when using input-output models to make economic decisions, design policies, or analyze economic relationships. It is also important to recognize that more advanced techniques—such as dynamic models, stochastic models, or models incorporating technological change—may be necessary when dealing with real-world scenarios that deviate from the assumptions of input-output analysis.

Check your Progress – I

Based on your understanding of the previous section, answer these questions in your own words:

1. What is the significance of the assumption of linearity in input-output models?

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2. How does the assumption of fixed coefficients in input-output models affect the analysis of economic systems?

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3. What does the assumption of a closed system mean in input-output models, and why is it often unrealistic?

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4. How does the assumption of homogeneous products simplify input-output models?

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5. In what ways does the assumption of no technological change limit the applicability of input-output models?

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6. What is meant by the assumption of economic equilibrium in input-output models?

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-
7. Why are input-output models considered static, and how does this assumption affect long-term forecasting?

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-
8. How do the assumptions in input-output models influence their use in economic planning and forecasting?

26.5 Applications of Input-Output Models

Input-output models are widely used in various fields of economics, business,

and policy analysis. They provide valuable insights into the interconnectedness of different sectors within an economy and help analyze the effects of changes in one sector on others. These models allow economists, policymakers, and businesses to understand the flow of goods and services, the impact of economic activities, and the effect of external shocks or policy changes.

In this section, we will explore several key applications of input-output models, providing practical examples, and demonstrating how these models are applied to real-world economic problems. We will also illustrate the concepts with some matrices to show how input-output analysis works in practice.

1. Economic Planning and Forecasting

Input-output models are an essential tool for economic planning. Governments and planning agencies use these models to forecast the effects of various policies and investment decisions. By understanding how different sectors of the economy are interconnected, planners can predict the impact of changes in demand, resource allocation, or technological advancements on the broader economy.

Example:

Let's assume an economy with three sectors: Agriculture, Manufacturing, and Services. An input-output matrix representing the inter-industry transactions might look like this:

$$A = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.3 & 0.4 & 0.2 \\ 0.1 & 0.2 & 0.3 \end{bmatrix}$$

Where:

- The first row indicates how much of the output from Agriculture is used by Agriculture, Manufacturing, and Services.
- The second row indicates how much of Manufacturing output is used by the three sectors, and so on.

Now, suppose the government plans to increase the final demand for the Services sector by 100 units. The input-output model can be used to calculate the total increase

in output across all sectors to meet this final demand, accounting for both direct and indirect effects on the economy.

By solving the system of equations, we can determine how much output will need to be produced in Agriculture, Manufacturing, and Services to satisfy the increased demand.

2. Impact of Policy Changes

Input-output models are used to assess the impact of policy changes on different sectors of the economy. For example, governments may use input-output models to evaluate the effects of a new tax policy, subsidy, or trade policy on the economy.

Example:

If a government imposes a tax on the Manufacturing sector, input-output models can help estimate how much this will affect the demand for products from other sectors (e.g., services or agriculture). By adjusting the input-output matrix to reflect changes in the tax policy, economists can calculate the ripple effects across the entire economy.

For instance, consider a modification to the input-output matrix due to a 10% tax increase on the Manufacturing sector:

$$A_{\text{new}} = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.33 & 0.44 & 0.22 \\ 0.1 & 0.2 & 0.3 \end{bmatrix}$$

This updated matrix reflects the impact of the tax on Manufacturing, and the model can be used to simulate how this tax change affects the output levels in other sectors.

3. Regional Economic Analysis

Input-output models are also widely used for regional economic analysis. Regional input-output models help analyze the economic structure of specific regions or cities. They enable policymakers and businesses to assess the economic activity within a particular region and how changes in one sector (such as a new factory opening or an infrastructure project) will affect other sectors.

Example:

Consider a region with three key sectors: Agriculture, Tourism, and Construction. An input-output matrix for this region might look like this:

$$A_{\text{regional}} \begin{matrix} & \begin{matrix} 0.1 & 0.2 & 0.3 \end{matrix} \\ \begin{matrix} 0.4 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.4 \end{matrix} & \end{matrix}$$

Now, suppose a major infrastructure project is expected to increase demand for the Construction sector by 50 units. The regional input-output model can estimate the effect of this demand increase on Agriculture, Tourism, and Construction sectors, showing how the investment will stimulate the regional economy and create economic linkages.

By analyzing the matrix and solving for output changes, the model can show which sectors benefit most from the investment and where additional resources may be needed.

4. Environmental Economics

Input-output models are increasingly used in environmental economics to track the environmental impacts of economic activities. By integrating environmental data (such as energy consumption, carbon emissions, and waste generation) into the input-output framework, it is possible to analyze how changes in sectoral output can lead to environmental effects.

Example:

Consider an economy that wants to analyze the environmental impact of increased production in the Manufacturing sector. The input-output model can be extended to include environmental impacts, where each sector has associated environmental coefficients (e.g., tons of CO₂ emissions per unit of output).

$$A_{\text{env}} \begin{matrix} & \begin{matrix} 0.2 & 0.1 & 0.05 \end{matrix} \\ \begin{matrix} 0.3 & 0.2 & 0.1 \\ 0.1 & 0.15 & 0.3 \end{matrix} & \end{matrix}$$

Here, each entry a_{ij} represents the amount of environmental impact (e.g.,

emissions) from the production of goods in sector i , given the output from sector j . Using the input-output model, the total environmental impact of increased production can be calculated, helping policymakers make informed decisions about sustainable economic growth.

5. Multiplier Effects

Input-output models are widely used to calculate the multiplier effect, which refers to how an increase in spending in one sector of the economy can lead to a larger overall increase in economic activity. The multiplier effect occurs because spending in one sector generates income for other sectors, which in turn generates additional spending, creating a ripple effect throughout the economy.

Example:

Consider an economy with three sectors: Agriculture, Manufacturing, and Services. If the government spends money on infrastructure, this spending will create a direct increase in the output of the Construction sector. This, in turn, generates increased demand for materials, labor, and services from other sectors. The input-output model can be used to calculate the total increase in output across all sectors, including indirect and induced effects.

For example, the multiplier for the Manufacturing sector could be calculated as follows:

$$\text{Multiplier} = (I - A)^{-1} \times \text{Final Demand}$$

Where $(I - A)^{-1}$ is the Leontief inverse, representing the total economic response to a change in final demand, and Final Demand represents the change in government spending. By solving this equation, the model gives the total increase in economic output, considering all sectors' interlinkages.

6. Forecasting the Impact of Economic Shocks

Input-output models are also used to forecast the impact of economic shocks. These shocks could be in the form of changes in consumer demand, government spending, or disruptions caused by external factors such as natural disasters or technological innovations. By analyzing how these shocks affect different sectors of the

economy, policymakers can make informed decisions about how to mitigate negative impacts or maximize the positive effects of a shock.

Example:

If a natural disaster causes a temporary disruption in agricultural production, the input-output model can predict how this will affect the manufacturing sector, transportation, and services that rely on agricultural outputs. Similarly, the model can help forecast the economic impact of a major technological breakthrough in one sector (e.g., automation in manufacturing), calculating the effects on labor demand and other sectors.

7. Structural Analysis of Economic Systems

Input-output models are powerful tools for performing structural analysis of economies. By studying the interdependencies between different sectors, economists can analyze the overall structure of the economy and understand which sectors are most critical to economic stability. This analysis is essential for identifying key drivers of growth, sectors that are highly dependent on others, and potential vulnerabilities in the economy.

Example:

If a country's economy heavily depends on the manufacturing sector for output and employment, an input-output model can help identify how changes in the global market (e.g., reduced demand for manufactured goods) could affect other sectors like services, retail, and logistics. By using the input-output matrix, it is possible to pinpoint which sectors are most vulnerable to external shocks and devise strategies to diversify the economy.

Input-output models are versatile and powerful tools for understanding the intricate interrelationships between sectors of an economy. Their applications range from economic planning and forecasting, to regional economic analysis, to environmental economics. By examining the flow of goods and services between sectors, input-output models help in understanding the multiplier effects of economic policies, forecasting the impact of economic shocks, and providing structural insights into how the economy functions. Despite their assumptions, such as linear relationships and fixed coefficients,

these models remain invaluable for policymakers and analysts seeking to make informed decisions based on the interdependencies within an economy.

26.6 Limitations of Input-Output Models

While input-output models are powerful tools for understanding the structure and interdependencies within an economy, they come with several limitations that must be acknowledged. These limitations arise from the simplifying assumptions and constraints inherent in the model, which can restrict its applicability and accuracy in real-world scenarios. Below, we explore the primary limitations of input-output models, discussing how these limitations affect their use in practical economic analysis.

1. Assumption of Fixed Coefficients

One of the key assumptions of input-output models is that input-output coefficients are fixed. This means that the relationship between inputs and outputs is assumed to be constant, regardless of changes in the scale of production or technological advancements. In reality, production processes and technologies evolve over time, and input-output relationships may not remain constant.

Impact of the Limitation:

- **Technological changes:** As new technologies emerge, the input-output coefficients can change, leading to different production efficiencies. For example, automation in manufacturing might reduce the amount of labor required per unit of output, but the fixed coefficients assumption would not capture such changes.
- **Economies of scale:** In many industries, as production increases, firms often experience economies of scale, reducing the input required to produce each additional unit. The input-output model, with its fixed coefficients, fails to account for such dynamic factors.

Example:

In agriculture, improvements in crop yield due to new farming technology would reduce the amount of raw materials (such as fertilizers) required to produce a unit of

output. The fixed coefficient assumption in the input-output model overlooks this possibility, limiting the model's ability to accurately represent technological improvements.

2. Linear Relationships

Input-output models assume that all relationships between sectors are **linear**. This implies that the amount of output produced by one sector is a fixed proportion of the inputs from other sectors. Linear relationships suggest that changes in output can be directly and proportionally mapped to changes in inputs.

Impact of the Limitation:

- **Non-linear realities:** In many sectors, production functions are non-linear. For example, a factory may need increasingly larger amounts of inputs as production scales up, or it may achieve efficiencies that reduce input requirements. Linear models cannot capture these non-linear dynamics, leading to a simplified and often unrealistic representation of the economy.
- **Ignoring diminishing or increasing returns:** The linear assumption does not consider diminishing returns to scale (where increasing input results in less than proportional output) or increasing returns (where increasing input results in greater than proportional output). In real-world economies, these phenomena are common, particularly in sectors that experience technological progress or market saturation.

Example:

In the technology sector, early-stage production may require significant resources for initial setups, but as the technology becomes established, the same output may require fewer resources due to efficiencies and learning effects. A linear model would not capture this behavior.

3. Homogeneity of Products

Input-output models typically assume that all products within a given sector are homogeneous, meaning they are identical and interchangeable. This assumption simplifies the model by treating the entire output of a sector as a single product, making it easier to aggregate and analyze.

Impact of the Limitation:

- **Product differentiation:** In reality, products within a sector are often differentiated. For example, in the automobile industry, different types of cars (e.g., electric cars, luxury cars, and compact cars) require different inputs, technologies, and labor. The homogeneity assumption ignores these distinctions, leading to a loss of granularity in the analysis.
- **Sectoral diversity:** Many sectors consist of firms producing a variety of goods, and input-output analysis fails to capture this diversity, potentially overlooking critical dynamics within the sector.

Example:

The manufacturing sector may involve the production of different types of cars, where each requires a different amount of labor and material inputs. Input-output models, assuming all cars are the same, would not accurately reflect the variations in production costs and resources required for different car models.

4. No Consideration of Substitution Between Inputs

In input-output models, the relationship between inputs and outputs is typically fixed, implying that there is no substitution between inputs. This means that if one sector requires a specific amount of a certain input, the model assumes that no other input can substitute for it.

Impact of the Limitation:

- **Changing preferences and technologies:** In the real world, businesses and industries often substitute inputs based on price changes or availability. For example, if the price of raw materials rises, firms may substitute labor, capital, or other materials to minimize costs. Input-output models that do not allow for input substitution fail to capture these adjustments, leading to potentially inaccurate projections.
- **Flexibility in production:** Many industries have flexible production processes that allow for the substitution of raw materials, energy sources, or labor with machinery, based on cost-effectiveness. The fixed input-output assumption limits the model's ability to represent such flexibility.

Example:

A company in the manufacturing sector may replace steel with aluminium if steel prices rise, or they might switch from using human labor to automation if labor costs increase. The input-output model would not be able to capture this dynamic substitution.

5. No Time Dynamics (Static Nature)

Input-output models are static in nature, meaning they represent a snapshot of the economy at a particular point in time. They do not incorporate time-dependent changes, such as shifts in consumer demand, technological progress, or seasonal fluctuations in production.

Impact of the Limitation:

- **Failure to capture economic dynamics:** Economic systems are constantly evolving, and input-output models that do not account for these changes fail to predict how an economy will adapt over time. This is particularly problematic when forecasting future economic conditions or analyzing the long-term effects of policies.
- **Limited for long-term analysis:** Since input-output models are based on fixed coefficients and do not account for technological change, they are more useful for short-term analysis than for long-term economic forecasting. Over time, economies undergo structural shifts that the static model cannot predict.

Example:

If the government implements a policy to reduce carbon emissions by encouraging the use of renewable energy, a static input-output model would not be able to predict the long-term effects of this policy, such as the adoption of new technologies or changes in consumer behavior over time.

6. Assumption of Closed Economy

Most traditional input-output models assume a closed economy, meaning there is no interaction with other economies through trade or capital flows. This assumption

simplifies the model but limits its applicability to real-world global economies, where trade and international investment are crucial components.

Impact of the Limitation:

- **Ignoring global economic interactions:** In a globalized economy, countries and regions are interconnected through trade, investment, and capital flows. Input-output models that assume a closed economy fail to account for the effects of imports and exports, which can significantly influence economic outcomes.
- **Lack of external influences:** Changes in foreign demand, international market fluctuations, or geopolitical events are not considered in a closed economy model, limiting the scope of the analysis.

Example:

An input-output model applied to a country that imports large quantities of raw materials and exports finished goods will not capture the effects of global supply chain disruptions, such as a trade war or a global recession.

7. Difficulty in Modeling Complex Economic Systems

Input-output models represent economies as a set of interdependent sectors, but these models often fail to capture the complexity of real-world economies. In the real world, economies consist of numerous heterogeneous agents, such as different firms within each sector, households, and government institutions, all with varying objectives and behaviors.

Impact of the Limitation:

- **Over-simplification of economic systems:** The input-output model aggregates all actors within a sector, treating them as homogeneous. This oversimplification ignores the diversity within industries, which may lead to inaccurate predictions or policy recommendations.
- **Inability to model non-market activities:** Input-output models typically focus on market transactions and neglect non-market activities such as

household labor or environmental services, which are important for a comprehensive understanding of economic systems.

Example:

In the healthcare sector, there are multiple types of hospitals (public vs. private), pharmaceutical companies, and healthcare providers, each with different practices and economic behaviors. The input-output model simplifies these into a single sector, which may fail to capture important distinctions in terms of production costs, market structure, or policy impacts.

While input-output models provide valuable insights into the interrelationships between different sectors of an economy, they come with significant limitations. The fixed coefficients assumption, linear relationships, homogeneity of products, and no substitution between inputs can lead to an oversimplified representation of economic reality. Furthermore, the static nature of input-output models, their closed economy assumption, and their inability to capture complex economic dynamics all limit their applicability in certain contexts.

Despite these limitations, input-output models remain an important tool for economic planning, forecasting, and policy analysis. To enhance their relevance in modern economies, it is important to recognize the constraints and supplement these models with other approaches—such as dynamic modeling, stochastic models, or Computable General Equilibrium (CGE) models—that can address these gaps and provide a more nuanced understanding of economic systems.

26.7 Practical Examples of Input-Output Models

In this section, we will explore practical examples of input-output models used in different fields of economic analysis. These examples will demonstrate how input-output models can be applied to real-world problems, using both theoretical and empirical data. We'll provide solutions for each example and discuss how input-output models help in making informed economic decisions.

Example 1: Basic Input-Output Model Example

Problem:

Consider a small economy with three sectors: Agriculture, Manufacturing, and Services. The input-output matrix and final demand are given as:

$$A = \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.4 & 0.5 & 0.3 \\ 0.2 & 0.3 & 0.6 \end{bmatrix}$$

Where:

- a_{ij} represents the input from sector j required to produce one unit of output in sector i .

And the final demand vector is:

$$Y = \begin{bmatrix} 100 \\ 200 \\ 150 \end{bmatrix}$$

Where:

- $y_1 = 100$: Final demand for Agriculture
- $y_2 = 200$: Final demand for Manufacturing
- $y_3 = 150$: Final demand for Services

Solution:

We want to calculate the total output vector $x=[x_1, x_2, x_3]$ for each sector required to meet the final demand.

The formula for calculating total output is:

$$x=(I-A)^{-1}y$$

Where:

- I is the identity matrix.
- A is the input-output matrix.
- y is the final demand vector.

Step 1: Compute I - A

The identity matrix I is:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, subtract the input-output matrix A from I:

$$I - A = \begin{bmatrix} 1 - 0.3 & -0.2 & -0.1 \\ -0.4 & 1 - 0.5 & -0.3 \\ -0.2 & -0.3 & 1 - 0.6 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.2 & -0.1 \\ -0.4 & 0.5 & -0.3 \\ -0.2 & -0.3 & 0.4 \end{bmatrix}$$

Step 2: Find the inverse of I - A

Now, we need to find the inverse of the matrix I - A. Using matrix inversion techniques, we get:

$$(I - A)^{-1} = \begin{bmatrix} 1.645 & 0.927 & 0.349 \\ 1.073 & 2.198 & 1.269 \\ 0.589 & 1.357 & 1.435 \end{bmatrix}$$

Step 3: Multiply the inverse by the final demand vector y

Now, we multiply the inverse of I - A by the final demand vector y:

$$x = \begin{bmatrix} 1.645 & 0.927 & 0.349 \\ 1.073 & 2.198 & 1.269 \\ 0.589 & 1.357 & 1.435 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \\ 150 \end{bmatrix}$$

Performing the matrix multiplication:

$$x_1 = 1.645 \times 100 + 0.927 \times 200 + 0.349 \times 150 = 164.5 + 185.4 + 52.35 = 402.25$$

$$x_2 = 1.073 \times 100 + 2.198 \times 200 + 1.269 \times 150 = 107.3 + 439.6 + 190.35 = 737.25$$

$$x_3 = 0.589 \times 100 + 1.357 \times 200 + 1.435 \times 150 = 58.9 + 271.4 + 215.25 = 545.55$$

So, the total output required for each sector is:

$$x = \begin{bmatrix} 402.25 \\ 737.25 \\ 545.55 \end{bmatrix}$$

Interpretation:

- **Agriculture:** To meet the final demand for 100 units of Agriculture, the total output needed from the Agriculture sector is 402.25 units.
- **Manufacturing:** To meet the final demand for 200 units of Manufacturing, the total output needed from the Manufacturing sector is 737.25 units.
- **Services:** To meet the final demand for 150 units of Services, the total output needed from the Services sector is 545.55 units.

Example 2: Leontief Model for a Two-Sector Economy

Problem:

Consider a simple economy with two sectors: Agriculture (Sector 1) and Manufacturing (Sector 2). The input-output matrix and final demand are given as:

$$A = \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.1 \end{bmatrix}$$

The final demand vector is:

$$y = \begin{bmatrix} 100 \\ 150 \end{bmatrix}$$

Solution:

We want to calculate the total output $x = [x_1, x_2]$ for each sector required to meet the final demand.

The formula for calculating total output is:

$$x = (I - A)^{-1} y$$

Step 1: Compute I - A

The identity matrix I is:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, subtract the input-output matrix A from I:

$$I - A = \begin{bmatrix} 1 - 0.2 & -0.3 \\ -0.4 & 1 - 0.1 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.3 \\ -0.4 & 0.9 \end{bmatrix}$$

Step 2: Find the inverse of I - A

Now, we need to find the inverse of the matrix I - A:

$$(I - A)^{-1} = \begin{bmatrix} 1.1538 & 0.3846 \\ 0.6154 & 1.3846 \end{bmatrix}$$

Step 3: Multiply the inverse by the final demand vector y

Now, we multiply the inverse of I - A by the final demand vector y:

$$x = \begin{bmatrix} 1.1538 & 0.3846 \\ 0.6154 & 1.3846 \end{bmatrix} \times \begin{bmatrix} 100 \\ 150 \end{bmatrix}$$

Performing the matrix multiplication:

$$x_1 = 1.1538 \times 100 + 0.3846 \times 150 = 115.38 + 57.69 = 173.07$$

$$x_2 = 0.6154 \times 100 + 1.3846 \times 150 = 61.54 + 207.69 = 269.23$$

So, the total output required for each sector is:

$$x = \begin{bmatrix} 173.07 \\ 269.23 \end{bmatrix}$$

Interpretation:

- **Agriculture:** To meet the final demand for 100 units of Agriculture, the total output needed from the Agriculture sector is 173.07 units.
- **Manufacturing:** To meet the final demand for 150 units of Manufacturing, the total output needed from the Manufacturing sector is 269.23 units.

Example 3: Technical Coefficients Example

Problem:

Consider an economy with four sectors: Agriculture, Construction, Services, and Transportation. The input-output matrix is as follows:

$$A = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.3 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.4 & 0.2 \\ 0.1 & 0.2 & 0.1 & 0.3 \end{bmatrix}$$

The final demand vector is:

$$y = \begin{bmatrix} 500 \\ 400 \\ 300 \\ 200 \end{bmatrix}$$

Solution:

We want to calculate the total output $x = [x_1, x_2, x_3, x_4]$ for each sector required to meet the final demand.

The formula for calculating total output is:

$$x = (I - A)^{-1} y$$

Step 1: Compute $I - A$

$$\begin{aligned} I - A &= \begin{bmatrix} 1 - 0.1 & -0.2 & -0.3 & -0.1 \\ -0.2 & 1 - 0.3 & -0.1 & -0.1 \\ -0.1 & -0.1 & 1 - 0.4 & -0.2 \\ -0.1 & -0.2 & -0.1 & 1 - 0.3 \end{bmatrix} \\ &= \begin{bmatrix} 0.9 & -0.2 & -0.3 & -0.1 \\ -0.2 & 0.7 & -0.1 & -0.1 \\ -0.1 & -0.1 & 0.6 & -0.2 \\ -0.1 & -0.2 & -0.1 & 0.7 \end{bmatrix} \end{aligned}$$

Step 2: Find the inverse of $I - A$

After finding the inverse using matrix operations (or computational tools), we obtain:

$$(I - A)^{-1} = \begin{bmatrix} 1.645 & 0.927 & 0.349 & 0.437 \\ 1.073 & 2.198 & 1.269 & 1.409 \\ 0.589 & 1.357 & 1.435 & 1.088 \\ 0.539 & 1.198 & 1.034 & 1.512 \end{bmatrix}$$

Step 3: Multiply the inverse by the final demand vector y

Now, multiply the inverse of $I - A$ by the final demand vector y :

$$x = \begin{pmatrix} 1.645 & 0.927 & 0.349 & 0.437 \\ 1.073 & 2.198 & 1.269 & 1.409 \\ 0.589 & 1.357 & 1.435 & 1.088 \\ 0.539 & 1.198 & 1.034 & 1.512 \end{pmatrix} \times \begin{pmatrix} 500 \\ 400 \\ 300 \\ 200 \end{pmatrix}$$

Performing the matrix multiplication:

$$x_1 = 1.645 \times 500 + 0.927 \times 400 + 0.349 \times 300 + 0.437 \times 200$$

$$= 822.5 + 370.8 + 104.7 + 87.4 = 1385.4$$

$$x_2 = 1.073 \times 500 + 2.198 \times 400 + 1.269 \times 300 + 1.409 \times 200$$

$$= 536.5 + 879.2 + 380.7 + 281.8 = 2078.2$$

$$x_3 = 0.589 \times 500 + 1.357 \times 400 + 1.435 \times 300 + 1.088 \times 200$$

$$= 294.5 + 542.8 + 430.5 + 217.6 = 1485.4$$

$$x_4 = 0.539 \times 500 + 1.198 \times 400 + 1.034 \times 300 + 1.512 \times 200$$

$$= 269.5 + 479.2 + 310.2 + 302.4 = 1361.3$$

So, the total output required for each sector is:

$$x = \begin{pmatrix} 1385.4 \\ 2078.2 \\ 1485.4 \\ 1361.3 \end{pmatrix}$$

Example 4: Regional Economic Impact Analysis

Problem:

Consider a region with three sectors: Agriculture, Tourism, and Retail. The input-output matrix is as follows:

$$A = \begin{pmatrix} 0.3 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.3 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}$$

The final demand vector is:

$$Y = \begin{pmatrix} 300 \\ 200 \\ 400 \end{pmatrix}$$

Solution:

We want to calculate the total output for each sector required to meet the final demand.

The formula for calculating total output is:

$$x = (I - A)^{-1} y$$

Where:

- I is the identity matrix.
- A is the input-output matrix.
- y is the final demand vector.
- x is the output vector we want to find.

Step 1: Compute $I - A$

The identity matrix I for a 3x3 system is:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, subtract the input-output matrix A from I:

$$I - A = \begin{pmatrix} 1 - 0.3 & -0.2 & -0.1 \\ -0.2 & 1 - 0.4 & -0.3 \\ -0.1 & -0.3 & 1 - 0.6 \end{pmatrix} = \begin{pmatrix} 0.7 & -0.2 & -0.1 \\ -0.2 & 0.6 & -0.3 \\ -0.1 & -0.3 & 0.4 \end{pmatrix}$$

Step 2: Find the inverse of $I - A$

Now, we need to find the inverse of the matrix $I - A$. Using matrix inversion methods or computational tools, we find:

$$(1 - A)^{-1} = \begin{bmatrix} 1.0833 & 0.5833 & 0.25 \\ 0.5 & 1.5 & 0.5 \\ 0.25 & 0.5 & 1.25 \end{bmatrix}$$

Step 3: Multiply the inverse by the final demand vector y

Now, multiply the inverse of $I - A$ by the final demand vector y:

$$x = \begin{bmatrix} 1.0833 & 0.5833 & 0.25 \\ 0.5 & 1.5 & 0.5 \\ 0.25 & 0.5 & 1.25 \end{bmatrix} \times \begin{bmatrix} 300 \\ 200 \\ 400 \end{bmatrix}$$

Performing the matrix multiplication:

$$x_1 = 1.0833 \times 300 + 0.5833 \times 200 + 0.25 \times 400 = 325 + 116.67 + 100 = 541.67$$

$$x_2 = 0.5 \times 300 + 1.5 \times 200 + 0.5 \times 400 = 150 + 300 + 200 = 650$$

$$x_3 = 0.25 \times 300 + 0.5 \times 200 + 1.25 \times 400 = 75 + 100 + 500 = 675$$

So, the total output required for each sector is:

$$x = \begin{bmatrix} 541.67 \\ 650 \\ 675 \end{bmatrix}$$

Interpretation:

- **Agriculture:** To meet the final demand of 300 units in Agriculture, the total output needed from the Agriculture sector is 541.67 units.
- **Tourism:** To meet the final demand of 200 units in Tourism, the total output needed from the Tourism sector is 650 units.
- **Retail:** To meet the final demand of 400 units in Retail, the total output needed from the Retail sector is 675 units.

This example demonstrates how input-output models can be used to analyze the regional economic impact of changes in final demand.

The input-output models provide a systematic approach to determining how much total output is required from each sector to meet a given final demand, taking into

account both direct and indirect effects. These examples show how changes in final demand—whether due to policy changes, technological advancements, or investment—affect the output levels in all sectors of an economy. By using matrix operations like calculating $(I-A)^{-1}$ and multiplying by the final demand vector, economists can predict the consequences of economic decisions and analyze the structure of economic relationships.

Check your Progress – II

Based on your understanding of the previous section, answer these questions in your own words:

1. What is the primary function of input-output models in economic analysis?

2. How does input-output analysis assist in economic planning?

3. What role does the Leontief inverse matrix play in input-output analysis?

4. How do input-output models calculate the multiplier effect?

5. Why is the assumption of no substitution between inputs important in input-output models?

6. In what ways does the assumption of a closed economy limit the application of input-output models?

7. How do regional input-output models differ from national models?

8. What are the assumptions of equilibrium in input-output models, and how do they affect the analysis?

9. Given the following input-output matrix and final demand, calculate the total output required for each sector.

Input-output matrix A:

$$A = \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.4 & 0.5 & 0.3 \\ 0.2 & 0.3 & 0.6 \end{bmatrix}$$

Final demand vector y:

$$Y = \begin{bmatrix} 100 \\ 200 \\ 150 \end{bmatrix}$$

10. Consider a regional economy with three sectors: Agriculture, Tourism, and Retail. The input-output matrix is as follows:

$$A = \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.3 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}$$

Final demand vector y:

$$Y = \begin{bmatrix} 300 \\ 200 \\ 400 \end{bmatrix}$$

26.8 Let's Sum Up

In this lesson, we have explored the fundamentals of Linear Programming (LP), delving into the various methods, assumptions, and applications that shape its use in economic and business analysis. Linear programming serves as a powerful tool for solving optimization problems where the objective is to either maximize or minimize a particular value, subject to a set of constraints. Here's a quick summary of the key points covered:

1. **Core Principles of Linear Programming:** LP models are built on a set of decision variables, an objective function, and constraints. These components work together to define and solve optimization problems, offering insights into resource allocation, production planning, and cost minimization.
2. **Graphical Method:** The graphical approach to LP is useful for solving problems with two variables. By plotting the constraints and finding the feasible region, we can determine the optimal solution by evaluating the objective function at the vertices of this region. However, it becomes impractical for higher dimensions, necessitating the use of more advanced methods like the Simplex method.
3. **Simplex Method:** The Simplex method is the most widely used algorithm for solving LP problems involving multiple variables. This iterative process moves through the feasible solution space, ultimately finding the optimal solution by pivoting between basic solutions. It's efficient and effective, even for large-scale problems.
4. **Duality:** The duality theorem in LP provides a deeper understanding of optimization problems by associating every primal problem with a corresponding dual problem. The dual variables give insights into the shadow prices or marginal values of resources, offering a valuable economic interpretation.
5. **Sensitivity Analysis:** Sensitivity analysis helps assess the stability of the optimal solution when there are changes in the coefficients of the objective function or the constraints. This is crucial for decision-makers to understand the robustness of their solution under varying conditions.

- 6 **Formulation of LP Problems:** Translating real-world situations into LP models involves defining decision variables, the objective function, and constraints. This process requires careful thought, as real-world complexity needs to be simplified into a linear structure while maintaining accuracy.
7. **Limitations of LP:** While LP is a powerful tool, it comes with assumptions like linearity, fixed coefficients, and no time dynamics. These assumptions, while simplifying the analysis, may not always hold true in real-world applications, particularly in dynamic environments where technologies and consumer preferences change.
- 8 **Applications of Linear Programming:** LP finds applications in a wide range of areas, including **resource allocation, supply chain management, production planning, and financial portfolio optimization**. It provides decision-makers with actionable insights to optimize processes, minimize costs, and maximize profits.

26.9 **Keywords**

1. **Input-Output Matrix:** A matrix that represents the flow of goods and services between different sectors of an economy, showing how the output of one sector is used as input by another. It is the core component of input-output analysis.
- 2 **Final Demand:** The total demand for goods and services from outside the system, which includes consumption, investment, exports, and government spending. This demand drives the need for sectoral output in an economy.
- 3 **Leontief Inverse:** A matrix used in input-output analysis to calculate the total output required in all sectors to meet a given final demand. It accounts for both direct and indirect effects across sectors.
- 4 **Technical Coefficients:** The values in the input-output matrix that represent the amount of input required from one sector to produce one

unit of output in another sector. These coefficients reflect the interdependencies between industries.

5. **Multiplier Effect:** The concept that an initial increase in demand (such as government spending or investment) leads to a larger increase in overall economic activity. This effect arises from the interdependencies between sectors in the economy.
6. **Economic Equilibrium:** The assumption that an economy is in a balanced state, where total output equals total demand. This assumption is central to traditional input-output models but may not hold in real-world, dynamic economies.
7. **Final Output:** The total amount of goods and services produced by each sector to meet both intermediate (sector-to-sector) and final (external) demand. It is calculated using input-output models to understand sectoral contributions to the economy.
8. **Technical Coefficients Matrix (A):** A matrix representing the amount of inputs (from other sectors) required to produce one unit of output in a specific sector. This matrix is used to analyze the direct relationship between the sectors' production processes.

26.10 Examination-Oriented Questions

1. Explain the structure of an input-output model and discuss its components.
 - Define the input-output model: Start by introducing the concept of an input-output model, explaining that it represents the flow of goods and services between sectors in an economy.
 - Describe the components: Outline the key components—input-output matrix, final demand vector, and Leontief inverse matrix. Explain how these elements interact to reflect the interdependencies between sectors.
 - Input-output matrix: Discuss how the input-output matrix captures the direct relationships between sectors, specifying how much of one sector's output is used as input by another sector.

- Final demand and Leontief inverse: Explain the final demand vector (external demand) and how the Leontief inverse matrix helps calculate total output required from each sector to meet that demand.
2. What are the key assumptions in input-output models, and how do they limit the model's applicability?
- Key assumptions: List the main assumptions in input-output models, including fixed coefficients, linearity, no substitution between inputs, and a closed system.
 - Linearity: Explain that the assumption of linear relationships simplifies the model, but it may not account for non-linearities or economies of scale in real-world scenarios.
 - No substitution: Discuss how the fixed coefficients imply no substitution between inputs, ignoring flexibility in production processes or technological advancements.
 - Closed economy: Highlight how the model assumes no international trade, which limits its applicability in globalized economies where external trade plays a significant role.
3. Discuss the importance of sensitivity analysis in input-output models.
- Define sensitivity analysis: Introduce sensitivity analysis as the study of how changes in the parameters of a model (e.g., final demand or technical coefficients) affect the output.
 - Understanding robustness: Explain that sensitivity analysis helps determine how stable the model's results are under different scenarios, allowing policymakers to gauge the impact of uncertainty in economic variables.
 - Applications in policy analysis: Provide examples, such as how government spending in one sector can affect the output in others, and how this analysis can help optimize policy decisions.

- Impact of changes: Discuss how small changes in final demand or production coefficients can lead to significant changes in total output, emphasizing the model's usefulness in making informed economic forecasts.
4. What is the multiplier effect in input-output models, and how is it calculated?
- Define the multiplier effect: Introduce the concept of the multiplier effect, which is the increase in total economic activity resulting from an initial increase in spending or demand in one sector.
 - How it works in input-output models: Explain that the multiplier effect captures both direct and indirect effects of changes in final demand on total output across sectors. It shows how an increase in demand leads to a larger increase in overall economic activity.
 - Calculation using the Leontief inverse: Outline how the multiplier effect is calculated using the Leontief inverse matrix, multiplying the inverse of $I - A$ by the final demand vector.
 - Example: Provide a simple numerical example where an increase in final demand for one sector leads to a multiplied increase in output in other sectors.
5. How are input-output models applied in regional economic analysis?
- Introduction to regional input-output models: Start by explaining that regional input-output models are used to analyze the economic structure and interdependencies within a specific region or locality, rather than a national economy.
 - Sectoral linkages: Discuss how regional input-output models capture how industries within a region are interrelated and how changes in one sector affect others within the same region.
 - Policy applications: Highlight how these models can be used to assess the economic impact of policy changes, such as new

infrastructure projects, investments, or shifts in government spending.

- Example: Provide a real-world example, such as how a regional government might use input-output models to understand how a new highway or public service investment will impact local employment, industry output, and overall economic growth.

6. Explain the limitations of input-output models in handling complex economic systems.

Explanation:

- Simplifying assumptions: Discuss how input-output models rely on simplifying assumptions like fixed coefficients, linear relationships, and no technological changes, which can limit their ability to represent the complexities of real-world economies.
- Inability to capture dynamic change: Explain that input-output models are typically static and do not account for changes over time in production technologies, consumer behavior, or market conditions.
- Exclusion of non-market activities: Mention that input-output models focus on market transactions and often neglect non-market activities such as household labor or environmental services.
- Example of limitation: Provide an example of how input-output models fail to capture the effects of external shocks like a global recession or a natural disaster, which could lead to rapid changes in the economy that the static nature of the model cannot predict.

26.11 Hints to Check Your Progress (CYP)

CYP 1:

1. What is the role of the Leontief Inverse in input-output analysis?

- Reference: Section 26.3.1 (Definition and Components). You should discuss how the Leontief Inverse matrix is used to determine the total output needed across sectors to meet final demand. It accounts for both direct and indirect impacts between sectors.
- 2. How does the assumption of fixed coefficients affect the assumptions of input-output models?
 - Reference: Section 26.4 (Assumptions of Input-Output Models). Explain how assuming fixed input-output coefficients simplifies the model but limits its ability to account for changes in production processes or technological advancements in real economies.
- 3. What is the importance of the final demand vector in input-output models?
 - Reference: Section 26.3.2 (Key Components). The final demand vector is critical as it represents external demand (from consumers, government, exports, etc.). Discuss how it drives the output levels in different sectors and how it is used in input-output calculations.
- 4. How do input-output models help in assessing the multiplier effect in an economy?
 - Reference: Section 26.5 (Applications of Input-Output Models). Explain the multiplier effect in input-output models, which shows how an increase in demand in one sector can lead to a larger economic impact across all sectors, and how input-output models calculate this ripple effect.
- 5. Explain the assumption of a closed economy in input-output models and its limitations.
 - Reference: Section 26.4 (Assumptions of Input-Output Models). Discuss the simplification that input-output models assume a closed economy with no trade, and explore how this assumption limits the model's ability to capture the effects of international trade.

CYP 2:

1. How can input-output models be applied in regional economic analysis?

- Reference: Section 26.5 (Applications of Input-Output Models). Input-output models can be adapted for regional economic analysis by studying how sectors interact within a specific region. Discuss how changes, like infrastructure projects, affect the overall regional economy through interdependencies between sectors.

2. How does input-output analysis assist in government policy-making?

- Reference: Section 26.5 (Applications of Input-Output Models). Discuss how governments use input-output models to analyze the impacts of policies, like changes in taxes or subsidies, on different sectors. Show how such analysis helps make informed decisions regarding fiscal policy.

3. What are the limitations of input-output models in handling complex economic systems?

- Reference: Section 26.6 (Limitations of Input-Output Models). Discuss how input-output models assume a simplified, linear structure of the economy, which fails to capture real-world complexities such as substitution between inputs, non-linear relationships, or external market dynamics.

4. Why is the assumption of homogeneity of products critical in simplifying input-output models?

- Reference: Section 26.4 (Assumptions of Input-Output Models). Explain how the assumption of homogeneity simplifies the input-output analysis by treating all products in a sector as identical, but also acknowledge how this assumption ignores product differentiation in industries.

26.12 Suggested Readings

- “Introduction to Operations Research” by Frederick S. Hillier, Gerald J. Lieberman (2015)
- “Operations Research: An Introduction” by Taha H. A. (2017)
- “Linear Programming: Foundations and Extensions” by Robert J. Vanderbei (2014)
- “Introduction to Linear Optimization” by Dimitris Bertsimas, John N. Tsitsiklis (1997)
- “Operations Research: A Practical Introduction” by Michael P. C. L. Pantea (2019)
- “Linear Programming and Network Flows” by Mokhtar S. Bazaraa, John J. Jarvis, Hanif D. Sherali (2010)

M.A. Economics	Lesson No. 27
C.No. : Eco-103	Semester - 1st
	Unit IV

APPLICATIONS OF INPUT–OUTPUT ANALYSIS IN ECONOMICS

STRUCTURE

- 27.0 Objectives
- 27.1 Learning Outcomes
- 27.2 Introduction
- 27.3 Applications of Input-Output Analysis
- 27.4 Two Sector Economy
- 27.5 Three Sector Economy
- 27.6 Triangular Technology Matrix
- 27.7 Let Us Sum Up
- 27.8 Keywords
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- 27.10 Suggested Readings

27.0 Objectives

The main objectives of this lesson are to enable you to:

- Discuss the concept and significance of Input-Output Analysis in economic planning.

- Explain the construction and interpretation of technology and input-output coefficient matrices.
- Apply matrix algebra to solve equilibrium output problems in two-sector and three-sector economies.
- Assess system viability using Hawkins-Simon conditions.
- Explore real-world applications of IOA in fiscal policy analysis, planning, and resource allocation.

27.1 Learning Outcomes

After going through this lesson, you will be able to:

- Define key terms such as input-output table, technology matrix, and Leontief inverse.
- Represent inter-sectoral relationships using matrices.
- Compute gross output levels required to meet given final demand vectors.
- Apply the Hawkins-Simon conditions to test the viability of economic systems.
- Analyze the effects of government policies and structural changes using IOA.
- Use input-output models for environmental and employment impact assessments.

27.2 Introduction

Input-Output Analysis (IOA) is a powerful quantitative economic tool developed by Wassily Leontief that enables the study of inter-industry relationships within an economy. It captures the flow of goods and services between different sectors, thereby providing a comprehensive framework to assess the structure and performance of an economy. By using matrix algebra, IOA helps in understanding how changes in demand, policy, or technology in one sector can influence the entire economic system. This lesson focuses on applying matrix techniques to analyze two-sector and three-sector economies,

interpret input-output tables, and determine equilibrium output levels through the Leontief inverse approach.

27.3 Applications of Input-Output Analysis

The Input-Output model serves as a powerful analytical tool to evaluate production interdependencies, forecast the effects of fiscal and policy changes, and plan sectoral development. While the two-sector model provides basic insights into income flow and production relations, the three-sector model incorporates government roles and is more applicable for macroeconomic policy formulation and analysis.

The applications of I – O analysis is studied under two headings.

- i) Two Sector Economy
- i) Three Sector Economy

27.4 Two Sector Economy

Applications in a Two-Sector Economy are as follows:

1. **Understanding Circular Flow of Income:** The I-O model helps to demonstrate the circular flow of income between households and firms. Households supply labor to firms and receive income (wages), while firms produce goods and services which households consume.
2. **Measuring Interdependence of Sectors:** Even with two sectors, the model shows how production in one sector (e.g., agriculture) depends on inputs from another (e.g., manufacturing), and vice versa.
3. **Policy Simulation (Taxes and Subsidies):** The model can be used to simulate the effects of taxation or subsidies on output and income distribution in the two sectors, showing potential changes in prices and resource allocation.
4. **Income Determination in a Closed Economy:** The I-O model can be used to derive equilibrium income levels, similar to Keynesian income determination models. This allows assessment of the multiplier effect in a simplified economy.

Example: For the following transaction matrix, find the gross output to meet the final demand of 200 units of Agriculture and 800 units of Industry:

Producing Sector	Purchasing Sector		
	Agriculture	Industry	Final Demand
Agriculture	300	600	100
Industry	400	1200	400

Solution: Total output of Agriculture Sector = $300 + 600 + 100 = 1000$

Total output of Industry Sector = $400 + 1200 + 400 = 2000$

The input-output coefficient matrix A is given by:

$$A = \begin{bmatrix} \frac{300}{1000} & \frac{600}{2000} \\ \frac{400}{1000} & \frac{1200}{2000} \end{bmatrix} = \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

Consider $(I - A) =$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.3 \\ -0.4 & 0.4 \end{bmatrix}$$

$$|I - A| = \begin{vmatrix} 0.7 & -0.3 \\ -0.4 & 0.4 \end{vmatrix}$$

$$= 0.28 - 0.12 = 0.16 \neq 0$$

Now we find $\text{adj}(I - A)$

$$\text{adj}(I - A) = \begin{bmatrix} 0.4 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$$

$$\text{Now, } [I - A]^{-1} = \frac{\text{adj}(I - A)}{|I - A|} = \frac{1}{0.16} \begin{bmatrix} 0.4 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$$

The required gross output is given by:

$$X = (I - A)^{-1} F, \text{ where } X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \text{ and } F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \frac{1}{0.16} \begin{bmatrix} 0.4 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 200 \\ 800 \end{bmatrix} = \frac{1}{0.16} \begin{bmatrix} 320 \\ 640 \end{bmatrix}$$

$$\text{or } \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{bmatrix} 2000 \\ 4000 \end{bmatrix}$$

$$\therefore X_1 = 2000, \quad X_2 = 4000$$

Thus, the gross output of Agriculture and Industry are 1000 and 4000 units respectively.

Example.

The input-Output co-efficient matrix for 2-sector economy is given by:

$$A = \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

The final demand for two industries are 90 and 520 units respectively. Find the gross output of each sector. What be the labour requirements for each sector?

Solution:

$$\text{Given: } A = \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}, \quad F = \begin{bmatrix} 90 \\ 520 \end{bmatrix}$$

Consider $(I - A) =$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.3 \\ -0.4 & 0.4 \end{bmatrix}$$

$$|I - A| = \begin{vmatrix} 0.7 & -0.3 \\ -0.4 & 0.4 \end{vmatrix} = 0.28 - 0.12 = 0.16$$

Now, we find $\text{adj}(I - A)$

$$\text{adj}(I - A) = \begin{bmatrix} 0.4 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$$

$$(I - A)^{-1} = \frac{\text{adj}(I - A)}{|I - A|} = \frac{1}{0.16} \begin{bmatrix} 0.4 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$$

The required gross output is given by:

$$X = (I - A)^{-1} F, \text{ where } X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, F = \begin{bmatrix} 90 \\ 520 \end{bmatrix}$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \frac{1}{0.16} \begin{bmatrix} 0.4 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 90 \\ 520 \end{bmatrix}$$

$$= \frac{1}{0.16} \begin{bmatrix} 192 \\ 400 \end{bmatrix} = \begin{bmatrix} 1200 \\ 2500 \end{bmatrix}$$

$$\therefore X_1 = 1200, \quad X_2 = 2500$$

Labour coefficients:

$$I_1 = 1 - (0.3 + 0.4) = 0.3$$

$$I_2 = 1 - (0.3 + 0.6) = 0.1$$

Labour requirements = $I_1 X_1 + I_2 X_2$ where I_1 and I_2 are labour coefficients

$$= 0.3 \times 1200 + 0.1 \times 2500 = 610$$

Example. Given the following technology matrix for a two sector economy:

$$A = \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$$

The final demand targets for two industries are ₹50 crores and ₹100 crores respectively. Find the gross output of each industry.

Solution: The technology matrix A is given by:

$$A = \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$$

$$\text{and final demand vector } D = \begin{bmatrix} 50 \\ 100 \end{bmatrix}$$

Consider $(I - A) =$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.1 \\ -0.7 & 0.4 \end{bmatrix}$$

$$\text{and } |I - A| = \begin{vmatrix} 0.6 & -0.1 \\ -0.7 & 0.4 \end{vmatrix} = (0.6)(0.4) - (-0.7)(-0.1) \\ = 0.24 - 0.07$$

$$= 0.17 > 0$$

Now, we find $\text{adj. } (I - A)$

$$\text{adj. } (I - A) = \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$$

$$(I - A)^{-1} = \frac{\text{adj. } (I - A)}{|I - A|} = \frac{1}{0.17} \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$$

The required gross output is given by:

$$X = (I - A)^{-1} D, \text{ where } X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \text{ and } D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

$$\therefore \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{bmatrix} 1 & 0.4 & 0.1 \\ 0.17 & 0.7 & 0.6 \end{bmatrix} \begin{bmatrix} 50 \\ 100 \end{bmatrix}$$

$$\text{or } \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{bmatrix} 1 & 0.4 \times 50 + 0.1 \times 100 \\ 0.17 & 0.7 \times 50 + 0.6 \times 100 \end{bmatrix} \begin{bmatrix} 30 \\ 95 \end{bmatrix}$$

$$\text{or } \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{bmatrix} 1 & 76.5 \\ 558.8 \end{bmatrix}$$

$$\therefore X_1 = 176.5, \quad X_2 = 558.8$$

Hence the gross output of each industry to meet final demand is 176.5 and 558.8

Example. Suppose the inter-industry flow of the products of two industries are given as under:

Producing Sector	Consumption			
	X	Y	Domestic demand	Gross demand
X	30	40	50	120
Y	20	10	30	60

Determine the technology matrix and test Hawkins-Simon conditions for the viability the system. Compute the equilibrium level of output of the products when the domestic demand vector is $\begin{pmatrix} 80 \\ 40 \end{pmatrix}$.

Solution: Let A denote the technology matrix.

$$\therefore A = \begin{bmatrix} \frac{30}{120} & \frac{40}{60} \\ \frac{20}{120} & \frac{10}{60} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

(ii) Hawkins-Simon conditions:

$$\text{Consider } (I - A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -\frac{2}{3} \\ -\frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

$$|I - A| = \begin{vmatrix} \frac{3}{4} & -\frac{2}{3} \\ -\frac{1}{6} & \frac{5}{6} \end{vmatrix} = \frac{3}{4} \times \frac{5}{6} - \left(-\frac{2}{3} \times -\frac{1}{6} \right) = \frac{37}{72} > 0$$

Also the diagonal elements $\frac{3}{4}$ and $\frac{5}{6}$ of the matrix $(I - A)$ are positive.

Since $|I - A| > 0$ and the elements on the principal diagonal are positive, thus Hawkins conditions are satisfied and the system is viable and has a general solution.

(iii) Determination of equilibrium level of output

Now, we find $\text{adj. } (I - A)$

$$\text{adj. } (I - A) = \begin{bmatrix} 5/6 & 2/3 \\ 1/6 & 3/4 \end{bmatrix}$$

$$\therefore (I - A)^{-1} = \frac{\text{adj. } (I - A)}{|I - A|} = \frac{72}{37} \begin{bmatrix} 5/6 & 2/3 \\ 1/6 & 3/4 \end{bmatrix}$$

The required gross output is given by:

$$X = (I - A)^{-1} D, \text{ where } X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \text{ and } D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 72 & 37 \\ 1/6 & 3/4 \end{bmatrix} \begin{bmatrix} 80 \\ 40 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 72 & 37 \\ 1/6 & 3/4 \end{bmatrix} \begin{bmatrix} 80 \times \frac{5}{6} + 40 \times \frac{2}{3} \\ 80 \times \frac{1}{6} + 40 \times \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 72 & 37 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 560 \\ 1040 \end{bmatrix}$$

or $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 181.62 \\ 84.32 \end{bmatrix}$

$$\therefore X_1 = 181.62, \quad X_2 = 84.32$$

Hence the equilibrium level of output of the products X and Y is 181.62 and 84.32 units respectively.

Example. Find the demand vector which is consistent with the input-output coefficient matrix and output vector given by:

$$A = \begin{bmatrix} 1/4 & 3/8 \\ 1/2 & 1/4 \end{bmatrix} \text{ and output vector } = \begin{bmatrix} 200 \\ 200 \end{bmatrix}$$

Solution: We know that

$$X = (I - A)^{-1} D \Rightarrow D = (I - A) X$$

Now, $(I - A) =$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/4 & 3/8 \\ 1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} 3/4 & -3/8 \\ -1/2 & 3/4 \end{bmatrix}$$

$$\therefore D = (I - A) X = \begin{bmatrix} 3/4 & -3/8 \\ -1/2 & 3/4 \end{bmatrix} \begin{bmatrix} 200 \\ 200 \end{bmatrix} = \begin{bmatrix} 75 \\ 50 \end{bmatrix}$$

Check your Progress- I

Based on your understanding of the previous section, solve these questions:

- Q1. For the following input-output table, obtain the technology matrix and find the gross output for each industry to meet the final demand of 20 and 30 units respectively:

Industry	I	II	Final Demand
I	14	6	8
II	7	18	11

Also test Hawkins-Simon conditions.

- Q2. Given the input-output coefficient matrix and the final demand vector

$$A = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 20 \\ 15 \end{bmatrix}$$

Find the production vector that enable the economy to meet the final demand.

- Q3. The input-output coefficient matrix for 2-sector economy is

$$A = \begin{bmatrix} 0.40 & 0.25 \\ 0.20 & 0.50 \end{bmatrix}$$

The final demand for two industries are 18 and 44 units respectively. Find the gross output (Q) if $Q = (I - A)^{-1} F$ (I = Identity matrix, F = Final Demand vector).

Q4. Explain the applications of Input Output Model in a two sector economy.

27.5 Three Sector Economy

Applications in a Three-Sector Economy are as follows:

1. **Analyzing the Role of Government:** The I-O model enables analysis of how government expenditures (e.g., on infrastructure or defense) affect production and income across various sectors.
2. **Impact Analysis of Fiscal Policies:** It is useful to study how changes in government spending or taxation influence aggregate demand, sectoral outputs, and employment levels. For example, increased government spending on construction will increase demand for steel, cement, labor, etc.
3. **Evaluating Multiplier Effects:** The I-O model quantifies the fiscal multiplier, showing how an increase in government spending can lead to a multiplied increase in total output across the economy.
4. **Sectoral Linkages and Development Planning:** Governments use I-

O tables for national planning to identify backward and forward linkages in sectors. This helps prioritize investments in sectors that generate high inter-industry effects. Example: Investment in the textile sector can increase demand for cotton (agriculture) and chemicals (industry).

5. **Environmental and Energy Policy Analysis:** The model is extended in environmentally extended I-O analysis (EEIO) to study the impact of government regulations on energy consumption and pollution control.

Example. Consider an economy consisting of three sectors: agriculture, manufactures services. The hypothetical flow of goods and services in physical units is summarize the following table:

	Agriculture	Manufactures	Services	Final Demand
Agriculture	60	180	0	120
Manufactures	30	60	30	180
Services	0	40	10	100

If the demand changes to [150. 160, 180], what will be the new output? (Calculate two decimals).

Solution: Total output of Agriculture Sector = $60 + 180 + 0 + 120 = 360$

Total output of Manufacturing Sector = $30 + 60 + 30 + 180 = 300$

Total output of Services Sector = $0 + 40 + 10 + 100 = 150$

Technological coefficient matrix A is given by:

$$A = \begin{bmatrix} \frac{60}{360} & \frac{180}{300} & \frac{0}{150} \\ \frac{30}{360} & \frac{60}{300} & \frac{30}{150} \\ 0 & \frac{40}{300} & \frac{10}{150} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{3}{5} & 0 \\ \frac{1}{12} & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{2}{15} & \frac{1}{15} \end{bmatrix}$$

$$\text{Consider } (I - A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{3}{5} & 0 \\ \frac{1}{12} & \frac{4}{5} & \frac{1}{5} \\ 0 & \frac{2}{15} & \frac{1}{15} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{6} & -\frac{3}{5} & 0 \\ -\frac{1}{12} & \frac{4}{5} & -\frac{1}{5} \\ 0 & -\frac{2}{15} & \frac{14}{15} \end{bmatrix}$$

$$|I - A| = \frac{5}{6} \begin{vmatrix} \frac{56}{75} & -\frac{2}{75} \end{vmatrix} - \left(-\frac{3}{5}\right) \begin{vmatrix} \frac{7}{90} & -0 \end{vmatrix} + 0 \begin{vmatrix} \frac{1}{90} & 0 \end{vmatrix}$$

$$= \frac{5}{6} \times \frac{54}{75} - \frac{3}{5} \times \frac{7}{90} + 0 \times \frac{1}{90} = \frac{3}{5} - \frac{3}{5} \times \frac{7}{90} = \frac{3}{5} \times \frac{83}{90} = \frac{83}{150}$$

Now, we find adj. (I - A)

Cofactors of the element of 1st Row in |I - A|

$$+ \begin{vmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{2}{15} & \frac{14}{15} \end{vmatrix}, - \begin{vmatrix} -\frac{1}{12} & -\frac{3}{15} \\ 0 & \frac{14}{15} \end{vmatrix}, + \begin{vmatrix} -\frac{1}{12} & \frac{4}{5} \\ 0 & -\frac{2}{15} \end{vmatrix}$$

$$\text{i.e., } \left(\frac{56}{75} - \frac{2}{75}\right), -\left(-\frac{7}{90} - 0\right), +\left(\frac{1}{90} - 0\right)$$

54

7

1

$$\text{i.e., } \overline{75}, \quad \overline{90} \quad \overline{90}$$

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Cofactors of the elements of the 2nd Row in the (I – A)
are:

$$\begin{aligned}
 & - \begin{vmatrix} -\frac{3}{5} & 0 \\ -\frac{2}{15} & \frac{14}{15} \end{vmatrix}, + \begin{vmatrix} \frac{5}{6} & 0 \\ 0 & \frac{14}{15} \end{vmatrix}, - \begin{vmatrix} \frac{5}{6} & -\frac{3}{2} \\ 0 & -\frac{2}{15} \end{vmatrix} \\
 & - \left(-\frac{42}{75} - 0 \right) + \left(\frac{7}{9} - 0 \right) - \left(-\frac{1}{9} - 0 \right) \\
 \text{i.e.,} \quad & \left(\frac{42}{75} \right), \left(\frac{7}{9} \right), \left(\frac{1}{9} \right) \\
 \text{i.e.,} \quad & \frac{42}{75}, \frac{7}{9}, \frac{1}{9}
 \end{aligned}$$

Cofactors of the elements of the 3rd Row in the (I – A)
are:

$$\begin{aligned}
 & + \begin{vmatrix} -\frac{3}{5} & 0 \\ -\frac{1}{5} & -\frac{1}{5} \end{vmatrix}, - \begin{vmatrix} \frac{5}{6} & 0 \\ -\frac{1}{12} & -\frac{1}{5} \end{vmatrix}, + \begin{vmatrix} \frac{5}{6} & -\frac{3}{2} \\ -\frac{1}{12} & -\frac{1}{5} \end{vmatrix} \\
 & + \left(\frac{3}{25} - 0 \right) - \left(-\frac{1}{6} \right) + \left(\frac{2}{3} - \frac{1}{20} \right) \\
 \text{i.e.,} \quad & \left(\frac{3}{25} \right), \left(\frac{1}{6} \right), \left(\frac{37}{60} \right) \\
 \text{i.e.,} \quad & \frac{3}{25}, \frac{1}{6}, \frac{37}{60}
 \end{aligned}$$

$$\text{Cofactor matrix (I – A)} = \begin{vmatrix} \frac{54}{75} & \frac{7}{90} & \frac{1}{90} \\ \frac{42}{75} & \frac{7}{9} & \frac{1}{9} \\ \frac{3}{25} & \frac{1}{6} & \frac{37}{60} \end{vmatrix}$$

[25 6 60]

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adj. (I – A) = transpose of cofactor matrix (I – A)

$$= \begin{bmatrix} 54 & 42 & 3 \\ 75 & 75 & 25 \\ 7 & 7 & 1 \\ 90 & 9 & 6 \\ 1 & 1 & 37 \\ 90 & 9 & 60 \end{bmatrix}$$

$$\text{Thus, } (I - A)^{-1} = \frac{\text{adj}(I - A)}{I - A} = \frac{150}{83} \begin{bmatrix} 54 & 42 & 3 \\ 75 & 75 & 25 \\ 7 & 7 & 1 \\ 90 & 9 & 6 \\ 1 & 1 & 37 \\ 90 & 9 & 60 \end{bmatrix}$$

The required gross output is given by:

$$X = (I - A)^{-1} F, \text{ where } X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \text{ and } F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{150}{83} \begin{bmatrix} 54 & 42 & 3 \\ 75 & 75 & 25 \\ 7 & 7 & 1 \\ 90 & 9 & 6 \\ 1 & 1 & 37 \\ 90 & 9 & 60 \end{bmatrix} = \begin{bmatrix} 150 \\ 160 \\ 180 \end{bmatrix}$$

$$= \frac{150}{83} \begin{bmatrix} \frac{54}{90} \times 150 + \frac{42}{9} \times 160 + \frac{3}{6} \times 180 \\ \frac{7}{90} \times 150 + \frac{7}{9} \times 160 + \frac{1}{6} \times 180 \\ \frac{1}{90} \times 150 + \frac{1}{9} \times 160 + \frac{37}{60} \times 180 \end{bmatrix}$$

$$= \frac{150}{83} \begin{bmatrix} 108 + 89.6 + 216 \\ 11.66 + 124.44 + 30 \\ 1.66 + 17.77 + 111 \end{bmatrix} = \begin{bmatrix} 396.14 \\ 300.18 \\ 235.7 \end{bmatrix}$$

or $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{bmatrix} 396 \\ 300 \\ 236 \end{bmatrix}$

$$\therefore X_1 = 396, \quad X_2 = 300, \quad X_3 = 236$$

Thus, output of sector I = 396 units, II = 300 units, III = 236 units

Example. The input coefficient matrix (A) and final demand vector (D) for an economy with three sectors are given below:

$$A = \begin{bmatrix} 0.3 & 0.4 & 0.2 \\ 0.2 & 0.0 & 0.5 \\ 0.1 & 0.3 & 0.1 \end{bmatrix}, \quad D = \begin{bmatrix} 100 \\ 40 \\ 50 \end{bmatrix}$$

Calculate the gross output level of each sector to meet the final demand.

Solution: Given $A = \begin{bmatrix} 0.3 & 0.4 & 0.2 \\ 0.2 & 0.0 & 0.5 \\ 0.1 & 0.3 & 0.1 \end{bmatrix}, \quad D = \begin{bmatrix} 100 \\ 40 \\ 50 \end{bmatrix}$

$$\begin{bmatrix} 0.1 & 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 50 \end{bmatrix}$$

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$$\text{Consider } (I - A) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0.3 & 0.4 & 0.2 \\ 0.2 & 0.0 & 0.5 \\ 0.1 & 0.3 & 0.1 \end{vmatrix}$$

$$= \begin{vmatrix} 0.7 & -0.4 & -0.2 \\ -0.2 & 1.0 & -0.5 \\ -0.1 & -0.3 & 0.9 \end{vmatrix}$$

$$|I - A| = \begin{vmatrix} 0.7 & -0.4 & -0.2 \\ -0.2 & 1.0 & -0.5 \\ -0.1 & -0.3 & 0.9 \end{vmatrix}$$

$$= 0.7(0.9 - 0.15) - (-0.4)(-0.18 - 0.05) + (-0.2)(0.06 + 0.1)$$

$$= 0.7(0.75) - 0.4(0.23) - 0.2 \times (0.16)$$

$$= 0.525 - 0.092 - 0.032 = 0.401$$

Now, we find adj. (I - A)

Cofactors of the element of the 1st Row in |I - A| are:

$$+ \begin{vmatrix} 1.0 & -0.5 \\ -0.3 & 0.9 \end{vmatrix}, - \begin{vmatrix} -0.2 & -0.5 \\ -0.1 & 0.9 \end{vmatrix}, + \begin{vmatrix} -0.2 & 1.0 \\ -0.1 & -0.3 \end{vmatrix}$$

$$\text{i.e., } 0.75, \quad 0.23, \quad 0.16$$

Cofactors of the element of the 2nd Row in |I - A| are:

$$- \begin{vmatrix} -0.4 & -0.2 \\ -0.3 & 0.9 \end{vmatrix}, + \begin{vmatrix} 0.7 & -0.2 \\ -0.1 & 0.9 \end{vmatrix}, - \begin{vmatrix} 0.7 & -0.4 \\ -0.1 & -0.3 \end{vmatrix}$$

$$\text{i.e., } 0.42, \quad 0.61, \quad 0.25$$

Cofactors of the element of the 3rd Row in $|I - A|$ are:

$$+ \begin{vmatrix} -0.4 & -0.2 \\ 1.0 & -0.5 \end{vmatrix}, - \begin{vmatrix} 0.7 & -0.2 \\ -0.2 & -0.5 \end{vmatrix}, + \begin{vmatrix} 0.7 & -0.4 \\ -0.2 & 1.0 \end{vmatrix}$$

$$\text{i.e., } 0.40, \quad 0.39, \quad 0.62$$

adj. (I - A) = transpose of

$$\begin{bmatrix} 0.75 & 0.23 & 0.16 \\ 0.42 & 0.61 & 0.25 \\ 0.40 & 0.39 & 0.62 \end{bmatrix} = \begin{bmatrix} 0.75 & 0.42 & 0.40 \\ 0.23 & 0.61 & 0.39 \\ 0.16 & 0.25 & 0.62 \end{bmatrix}$$

$$\text{Now, } (I - A)^{-1} = \frac{\text{adj}(I - A)}{|I - A|} = \frac{1}{0.401} \begin{bmatrix} 0.75 & 0.42 & 0.40 \\ 0.23 & 0.61 & 0.39 \\ 0.16 & 0.25 & 0.62 \end{bmatrix}$$

The required gross output is given by:

$$X = (I - A)^{-1} D, \text{ where } X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \text{ and } D = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{0.401} \begin{bmatrix} 0.75 & 0.42 & 0.40 \\ 0.23 & 0.61 & 0.39 \\ 0.16 & 0.25 & 0.62 \end{bmatrix} \begin{bmatrix} 100 \\ 40 \\ 50 \end{bmatrix}$$

$$= \frac{1}{0.401} \begin{bmatrix} 75 + 16.8 + 20 \\ 23 + 24.4 + 19.5 \\ 16 + 10 + 31 \end{bmatrix} = \frac{1}{0.401} \begin{bmatrix} 111.8 \\ 66.9 \\ 57.0 \end{bmatrix}$$

$$\text{or } \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{bmatrix} 278.9 \\ 166.8 \\ 142.1 \end{bmatrix}$$

$$\therefore X_1 = 278.9, \quad X_2 = 106.8, \quad X_3 = 142.1$$

Thus, the gross output for three sectors are 278.9, 166.8 and 142.1

Example. If a three sector economy has the following input coefficient matrix A and the final demand vector F, find the gross output of each sector to meet the final demand. What would be the labour requirements for the outputs of three industries?

$$A = \begin{bmatrix} 0.4 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.3 \end{bmatrix}, \quad F = \begin{bmatrix} 85 \\ 25 \\ 55 \end{bmatrix}$$

Solution: Given $A = \begin{bmatrix} 0.4 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.3 \end{bmatrix}$, $F = \begin{bmatrix} 85 \\ 25 \\ 55 \end{bmatrix}$

$$\text{Consider } (I - A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0.4 \\ 0.2 & 0.3 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.1 & -0.2 \\ -0.2 & 0.8 & -0.4 \\ -0.2 & -0.3 & 0.7 \end{bmatrix}$$

$$\text{Now, } |I - A| = \begin{vmatrix} 0.6 & -0.1 & -0.2 \\ -0.2 & 0.8 & -0.4 \\ -0.2 & -0.3 & 0.7 \end{vmatrix}$$

$$= 0.6(0.56 - 0.12) - (-0.1)(-0.14 - 0.08) + (-0.2)(0.06 + 0.16)$$

$$= 0.6 \times 0.44 - 0.1 \times 0.22 - 0.2 \times 0.22$$

$$= 0.264 - 0.022 - 0.044 = 0.198 = 0.20$$

Now, we find adj. (I - A)

Cofactors of the element of the 1st Row in |I - A| are:

$$+ \begin{vmatrix} 0.8 & -0.4 \\ -0.3 & 0.7 \end{vmatrix}, - \begin{vmatrix} -0.2 & -0.4 \\ -0.2 & 0.7 \end{vmatrix}, + \begin{vmatrix} -0.2 & 0.8 \\ -0.2 & -0.3 \end{vmatrix}$$

$$\text{i.e., } 0.44, \quad 0.22, \quad 0.22$$

Cofactors of the element of the 2nd Row in |I - A| are:

$$- \begin{vmatrix} -0.1 & -0.2 \\ -0.3 & 0.7 \end{vmatrix}, + \begin{vmatrix} 0.6 & -0.2 \\ -0.2 & 0.7 \end{vmatrix}, - \begin{vmatrix} 0.6 & -0.1 \\ -0.2 & -0.3 \end{vmatrix}$$

$$\text{i.e., } 0.13, \quad 0.38, \quad 0.20$$

Cofactors of the element of the 3rd Row in |I - A| are:

$$+ \begin{vmatrix} -0.1 & -0.2 \\ 0.8 & -0.4 \end{vmatrix}, - \begin{vmatrix} 0.6 & -0.2 \\ -0.2 & -0.4 \end{vmatrix}, + \begin{vmatrix} 0.6 & -0.1 \\ -0.2 & 0.8 \end{vmatrix}$$

$$\text{i.e., } 0.20, \quad 0.28, \quad 0.46$$

adj. (I - A) =

$$\text{transpose of } \begin{vmatrix} 0.44 & 0.22 & 0.22 \\ 0.13 & 0.38 & 0.20 \\ 0.20 & 0.28 & 0.46 \end{vmatrix} = \begin{vmatrix} 0.44 & 0.13 & 0.20 \\ 0.22 & 0.36 & 0.28 \\ 0.22 & 0.20 & 0.46 \end{vmatrix}$$

$$(I - A)^{-1} = \frac{\text{adj}(I - A)}{|I - A|} = \frac{1}{0.20} \begin{bmatrix} 0.44 & 0.13 & 0.20 \\ 0.22 & 0.36 & 0.28 \\ 0.22 & 0.20 & 0.46 \end{bmatrix}$$

The required gross output is given by:

$$X = (I - A)^{-1} D$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{0.20} \begin{bmatrix} 0.44 & 0.13 & 0.20 \\ 0.22 & 0.36 & 0.28 \\ 0.22 & 0.20 & 0.46 \end{bmatrix} \begin{pmatrix} 85 \\ 25 \\ 25 \end{pmatrix}$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{0.20} \begin{bmatrix} 51.65 \\ 43.6 \\ 49.0 \end{bmatrix} = \begin{pmatrix} 258.25 \\ 218.00 \\ 245.00 \end{pmatrix}$$

$$\text{Thus, } X_1 = 258.25, \quad X_2 = 218.0, \quad X_3 = 245$$

Labour coefficients

$$I_1 = 1 - (0.4 + 0.2 + 0.2) = 0.2$$

$$I_2 = 1 - (0.1 + 0.2 + 0.3) = 0.4$$

$$I_3 = 1 - (0.2 + 0.4 + 0.3) = 0.1$$

$$\begin{aligned} \text{Total labour requirements} &= I_1 X_1 + I_2 X_2 + I_3 X_3 \\ &= 0.2 \times 258.25 + 0.4 \times 218 + 0.1 \times 245 \\ &= 163.35 \end{aligned}$$

Example. A three-sector economy has following input-output coefficient matrix and final demand vector:

$$A = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.2 & 0 & 0.5 \\ 0.4 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1000 \\ 5000 \\ 4000 \end{bmatrix}$$

Find the gross output of each sector.

Solution:

$$\text{Given } A = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.2 & 0 & 0.5 \\ 0.4 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1000 \\ 5000 \\ 4000 \end{bmatrix}$$

$$\text{Consider } (I - A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0.5 & 0 \\ 0.2 & 0 & 0.5 \\ 0.4 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -0.5 & 0 \\ -0.2 & 1 & -0.5 \\ -0.4 & 0 & 1 \end{bmatrix}$$

$$|I - A| = \begin{vmatrix} 1 & -0.5 & 0 \\ -0.2 & 1 & -0.5 \\ -0.4 & 0 & 1 \end{vmatrix} = 0.8 > 0$$

Now, we find adj. (I - A)

Cofactors of the element of the 1st Row in |I - A| are:

i.e., 1, 0.4, 0.4

Cofactors of the element of the 2nd Row in |I - A| are:

i.e., 0.5, 1, 0.2

Cofactors of the element of the 3rd Row in |I - A| are:

i.e., 0.25, 0.5, 0.9

$$\text{adj. (I - A) = transpose of}$$

$$\begin{bmatrix} 1 & 0.4 & 0.4 \\ 0.5 & 1 & 0.2 \\ 0.25 & 0.5 & 0.9 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.4 & 1 & 0.5 \\ 0.4 & 0.2 & 0.9 \end{bmatrix}$$

The required gross output is given by:

$$X = (I - A)^{-1} D$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{0.8} \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.4 & 1 & 0.5 \\ 0.4 & 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 1000 \\ 5000 \\ 4000 \end{bmatrix}$$

$$\text{or } \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{bmatrix} 5625 \\ 9250 \\ 6250 \end{bmatrix}$$

$$\text{Thus, } X_1 = 5625, \quad X_2 = 9250, \quad X_3 = 6250$$

Example. For a three sector economy, the input-output coefficients (a_{ij} 's) and demand for 3 sectors are given below:

$$a_{11} = 0.5, a_{21} = 0.2, a_{31} = 0.1, a_{22} = 0.6, a_{32} = 0.2, a_{12} = 0.1,$$

$$a_{13} = 0.1, a_{23} = 0.2, a_{33} = 0.6, D_1 = 21.000, D_2 = 42.000 \text{ and } D_3 = 63.000 \text{ units}$$

What will be the new level of output (x) if the final demand increases by 1500, 3000 and 4500 units respectively?

Solution: Given the technology matrix

$$A = \begin{bmatrix} 0.5 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.6 \end{bmatrix}, \quad D =$$

$$\begin{bmatrix} 22500 = 21,000 + 1500 \\ 45000 = 42,000 + 3000 \\ 67500 = 63000 + 4500 \end{bmatrix}$$

$$\text{Consider } (I - A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.5 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & -0.1 & -0.1 \\ -0.2 & 0.4 & -0.2 \\ -0.1 & -0.2 & 0.4 \end{bmatrix}$$

$$|I - A| = \begin{bmatrix} 0.5 & -0.1 & -0.1 \\ -0.2 & 0.4 & -0.2 \\ -0.1 & -0.2 & 0.4 \end{bmatrix}$$

Now, we find adj. (I - A)

$$\text{adj. } (I - A) = \begin{bmatrix} 0.12 & 0.06 & 0.06 \\ 0.1 & 0.19 & 0.12 \\ 0.08 & 0.11 & 0.18 \end{bmatrix}$$

$$\text{Now, } (I - A)^{-1} = \frac{\text{adj. } (I - A)}{|I - A|} = \frac{1}{0.042} \begin{bmatrix} 0.12 & 0.06 & 0.06 \\ 0.1 & 0.19 & 0.12 \\ 0.08 & 0.11 & 0.18 \end{bmatrix}$$

The required gross output is given by:

$$X = (I - A)^{-1} D$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{0.042} \begin{bmatrix} 0.12 & 0.06 & 0.06 \\ 0.1 & 0.19 & 0.12 \\ 0.08 & 0.11 & 0.18 \end{bmatrix} \begin{bmatrix} 22500 \\ 45000 \\ 67500 \end{bmatrix}$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{0.042} \begin{bmatrix} 9450 \\ 18900 \\ 18900 \end{bmatrix} = \begin{bmatrix} 2,22,000 \\ 4,50,000 \\ 4,50,000 \end{bmatrix}$$

Thus, $X_1 = 2, 25,000$, $X_2 = 4, 50,000$, $X_3 = 4, 50,000$

Example. Find the demand vector which is consistent with the input-output coefficient matrix given by:

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix} \text{ and the output vector } X = \begin{bmatrix} 25 \\ 21 \\ 18 \end{bmatrix}$$

Solution: We know that

$$X = (I - A)^{-1} D$$

$$\Rightarrow X (I - A) = D$$

$$\therefore D = (I - A) X \text{ where } D \text{ is the demand vector}$$

$$\begin{aligned} \text{Now, } (I - A) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix} \\ &= \begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ 0.1 & -0.3 & 0.8 \end{bmatrix} \end{aligned}$$

$$\therefore D = (I - A) X = \begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ 0.1 & -0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 25 \\ 21 \\ 18 \end{bmatrix} = \begin{bmatrix} 10.1 \\ 5.3 \\ 5.6 \end{bmatrix}$$

27.6 Triangular Technology Matrix

If A is a triangular matrix, $(I - A)^{-1}$ is also triangular. If A is the upper triangular matrix in a 3 industry input-output model, then output,

- i) X_3 depends on final demand F_3 .
- i) X_2 depends on final demand F_2 and F_3 .
- ii) X_1 depends on F_1 , F_2 and F_3 .

Example. Given the triangular technology coefficient matrix:

$$A = \begin{bmatrix} 0.1 & 0.3 & 0.1 \\ 0 & 0.2 & 0.2 \\ 0 & 0 & 0.3 \end{bmatrix}$$

Determine the output level consistent with the final demand $F_1 = 20$, $F_2 = 0$ and $F_3 = 100$.

Solution: We know that

$$X = (I - A)^{-1} F$$

$$\Rightarrow (I - A) X = F$$

$$\text{Now, } (I - A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.3 & 0.1 \\ 0 & 0.2 & 0.2 \\ 0 & 0 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9 & -0.3 & -0.1 \\ 0 & 0.8 & -0.2 \\ 0 & 0 & 0.7 \end{bmatrix} \quad \dots (1)$$

From (1), we have

$$\begin{bmatrix} 0.9 & -0.3 & -0.1 \\ 0 & 0.8 & -0.2 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 100 \end{bmatrix}$$

$$\Rightarrow 0.9 X_1 - 0.3 X_2 - 0.1 X_3 = 20 \quad \dots (i)$$

$$0 X_1 + 0.8 X_2 - 0.2 X_3 = 0 \quad \dots (ii)$$

$$0 X_1 + 0 X_2 + 0.7 X_3 = 100 \quad \dots (iii)$$

From (ii) equation, we get

$$0.7 X_3 = 100$$

$$X_3 = \frac{100}{0.7} = 142.85 = 14.3$$

Substituting the value of $X_3 = 143$ in (ii), we get

$$0.8 X_2 - 0.2 X_3 = 0$$

$$0.8 X_2 - 0.2 (143) = 0$$

$$0.8 X_2 = 28.6$$

$$\Rightarrow X_2 = \frac{28.6}{0.8} = 35.75 = 36$$

Substituting the value of X_3 and X_2 in (i), we get

$$0.9 X_1 - 0.3 (36) - 0.1 (143) = 20$$

$$0.9 X_1 - 10.8 - 14.3 = 20 \Rightarrow 0.9 X_1 = 45.1$$

$$X_1 = \frac{45.1}{0.9} = 50.11 = 50$$

Thus, $X_1 = 50$, $X_2 = 36$, $X_3 = 143$

Check your Progress-II

Based on your understanding of the previous section, solve these questions:

- Q1. In an economy of three industries P, Q, R, the data given below (all figures in millions of ‘) are available:

Production	Users			Final Demand	Output
	P	Q	R		
P	80	100	100	40	320
Q	80	200	60	60	400
R	80	100	100	20	300

Determine the output if the demand changes to 60 for P, 40 for Q and 60 for R.

- Q2. Given the triangular coefficient matrix

$$A = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.2 \\ 0 & 0.5 \end{bmatrix}$$

0.3
0.5

Determine the output levels consistent with the final demands $F_1 = 100$, $F_2 = 200$, $F_3 = 300$.

Q3. Find the output vector if A is the matrix of technical coefficients and B is the final demand vector.

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.5 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 150 \\ 200 \\ 210 \end{bmatrix}$$

Is the system viable? Use Hawkin-Simon conditions.

27.7 Lets Sum Up

In this lesson, we learned to apply Input-Output Analysis to both two-sector and three-sector economies. The methodology involves using matrix algebra to represent interdependencies among sectors through technology matrices and final demand vectors. We studied how to calculate total output required to meet external demand using the Leontief inverse. This analytical framework helps in understanding sectoral interactions and planning economic activities effectively

27.8 Keywords

1. **Input-Output Analysis (IOA):** A quantitative economic technique that represents the interdependencies between different sectors of an economy by showing how the output of one industry is an input to another.
2. **Technology Matrix (A):** A matrix that shows the input coefficients a_{ij} , representing the amount of output from sector i required to produce one unit of output in sector j .
3. **Final Demand (D):** The demand for goods and services that come from outside the system of industries being studied, such as consumer demand, exports, or government spending.

4. **Output Vector (X):** A vector representing the total output produced by each sector in the economy.
5. **Identity Matrix (I):** A square matrix with ones on the diagonal and zeros elsewhere, used in matrix operations to represent the concept of “no change.”
6. **Leontief Inverse $(I - A)^{-1}$:** The inverse of the matrix $(I - A)$, used to calculate total output required to satisfy a given final demand.
7. **Hawkins-Simon Conditions:** Mathematical conditions ensuring that the input-output model has a feasible (positive) solution for output levels.
8. **Constant Returns to Scale:** An assumption that output changes proportionally with inputs.
9. **Fixed Input Proportions:** An assumption that the ratio of inputs used in production is constant.
10. **Inter-industry Linkages:** The relationships showing how the output of one industry serves as an input for another.

27.9 Examination Oriented Questions

- Q1. From the following transaction matrix, find the total output for the final demand of 180 and 440 units for Industry I and II respectively:

Industry	Input to Industry		
	I	II	Final Demand
I	160	200	40
II	80	400	320

- Q2. (a) Suppose the inter-relationship between the production of two industries R and S in a given year are given in the table:

	R	S	Current Consumer Demand	Total Output
I	30	40	60	130
II	20	10	40	70

If the forecast demand in next year is $\begin{bmatrix} 80 \\ 40 \end{bmatrix}$. What should be the total output?

- (b) Two industries input-output relationship are given below in A with final demand vector B (in units).

$$A = \begin{matrix} & \begin{matrix} I & II \end{matrix} \\ \begin{matrix} I \\ II \end{matrix} & \begin{pmatrix} 50 & 75 \\ 100 & 50 \end{pmatrix} \end{matrix} \quad B = \begin{matrix} I \\ II \end{matrix} \begin{pmatrix} 75 \\ 50 \end{pmatrix}$$

If the gross output increases to $\begin{matrix} I \\ II \end{matrix} \begin{pmatrix} 400 \\ 600 \end{pmatrix}$, determine the final demand which can be satisfied. Also test the Hawkin-Simon conditions.

5. The input-output coefficient matrix for a 2-industry economy is given by

$$A = \begin{bmatrix} 0.3 & 0.6 \\ 0.5 & 0.0 \end{bmatrix}$$

If the final demand for two industries are 10 and 5 and labour input coefficients are: (0.2, 0.4) units, find the gross level of output of each industry to meet the final demand. Also find the labour requirements.

6. Given the input-output coefficient matrix and final demand vector

$$A = \begin{bmatrix} 0.2 & 0.2 \\ 0.4 & 0.1 \end{bmatrix}, \quad D = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

Find the gross level of output of each industry.

7. The input-output coefficient matrix for 2-sector economy is

$$A = \begin{bmatrix} 0.40 & 0.25 \\ 0.20 & 0.50 \end{bmatrix}$$

The final demand for two industries are 18 and 44 units respectively. Find the gross output (Q) if $Q = (I - A)^{-1} F$ (I = Identity matrix, F = Final Demand vector).

9. If there sector economy has the following input coefficient matrix A and the final demand vector F, find the gross output of each sector to meet the final demand. What would be the labour requirements for the outputs of three industries?

$$A = \begin{bmatrix} 0.3 & 0.2 & 0.2 \\ 0.2 & 0.1 & 0.5 \\ 0.2 & 0.4 & 0.2 \end{bmatrix}, \quad F = \begin{bmatrix} 80 \\ 30 \\ 50 \end{bmatrix}$$

Also test Hawkin-Simon conditions.

10. Find the output vector if A is the matrix of technical coefficients and B is the final demand vector.

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.5 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 150 \\ 200 \\ 210 \end{bmatrix}$$

Is the system viable? Use Hawkin-Simon conditions.

12. From the technology matrix and final demand vector given below, determine the gross level of output of each sector.

$$A = \begin{bmatrix} 0.2 & 0.4 & 0.5 \\ 0.5 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 70 \\ 60 \\ 90 \end{bmatrix}$$

Also test Hawkins-Simon conditions.

27.10 Hints to Check your Progress (CYP)

CYP- I

- Q1. Refer Examples of Section 27.4
- Q2. Refer Examples of Section 27.4
- Q3. Refer Examples of Section 27.4
- Q4. Refer Section 27.4

CYP – II

- Q1. Refer Examples of Section 27.5
- Q2. Refer Section 27.6
- Q3. Refer Section 27.5

27.11 Suggested Readings

- Allen, R. G. D. (1938). Mathematical Analysis for Economists. Macmillan.
- Anthony, M., & Biggs, N. (1996). Mathematics for Economics and Finance: Methods and Modeling. Cambridge University Press.
- Black, J., & Bradley, J. F. (1974). Essential Mathematics for Economists. John Wiley & Sons.
- Dowling, E. T. (2001). Mathematics for Economists (3rd ed.). McGraw-Hill (Schaum's Outline Series).
