Directorate of Distance Education

UNIVERSITY OF JAMMU
JAMMU

SELF LEARNING MATERIAL

For

B.A. SEMESTER—I

SUBJECT : MATHEMATICS | **Unit: I-V**

Course No. : MA - 101 | **LESSON No. 1-13**

Course Co-ordinator
STANZIN SHAKYA

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DESCRIPTIVE MATHEMATICS MA - 101

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MATHEMATICS

There shall be one written paper of 80 marks and of three hours duration and 20 marks shall be reserved for internal asessment. Paper will be set for 80 marks. In case of regular students, internal assessment received from the colleges will be added to the marks obtained by them in the university examination and in each case of private candidates, marks obtained by them in the Univesity examination shall be increased proportionately in accordance with the Statutes / Regulations.

SYLLABUS

SEMESTER - I
CALCULUS

Total Marks : 100

Credits - 04 **Theory Examination : 80**

Course No. MA - 101 **Internal Assessment : 20**

Unit-I Function of two variables, their limit and continuity. Partial derivatives and Euler's theorem for homogeneous functions. Total derivatives and equality of f_{xy} (x, y) and f_{yx} (x, y), double points, concavity, convexity and points of inflexion (11 lectures)

Unit-II Asymptotes in cartesian forms, Envelopes of one and two parameter family of curves. Indeterminate forms, L' Hospital rule, Curve tracing in cartesian co-ordinates. (10 lectures)

Unit III Ordinary and partial derivatives of vector-valued functions, Directional derivatives of vector-valued function of several variables, the operator ∇, Gradient of scaler function, divergence and curl of vector functions, seond order derivative of functions, the Laplacian operator ∇^2, Line Integral. (14 lectures).

Unit IV Polar co-ordinates and their relationship with cartesian coordinates, Angle between radius vector and tangent at a point on the curve and the angle of intersection of two curves, curve sketching in polar co-ordinates such as r = a+b cosθ, a+b sinθ, a cosn θ, a sin n θ (for n=2 and 3 only). (13 lectures)

Unit-V Reduction formulae of Rectification of plane curve in cartesian form only, Volume and surface of revolution of curves in cartesian form. (13 lectures)

NOTE FOR PAPER SETTING

1. Each lecture will be of one hour duration.

2. The question paper shall contsist of 10 questions, two questions from each unit. The candidate will be required to do five questions selecting exactly one question from each unit.

BOOKS RECOMMENDED

1. Differential Calculus by Shanti Narayan, Dr. P. K. Mittle, Pub. S. Chand

2. Vector Calculus by Shanti Narayan, Dr. P. K. Mittle, Pub. S. Chand

3. Integral Calculus by Shanti Narayan, Dr. P. K. Mittle, Pub. S. Chand

B.A.		Semester-I
Unit-I	**MATHEMATICS**	**Lesson No.-1**

Prof. Vijay Rattan,
M.A.M. College, Jammu

Objectives

To develop the knowledge of partial differentiation.

Structure

i) Function of Several Variables.

ii) Partial Differentiation.

iii) Euler's theorem on Homogenous functions.

Introduction

In this lesson you will be made :

i) Familiar with functions of two or more variables.

ii) Finding derivatives of a function w.r.t. one variable keeping the othervariables constant.

iii) application of Euler's theorem to homogenous functions.

Objectives

To develop the knowledge of partial differentiation which is necessary in studying many other chapters of differential calculus **prescribed in your course**.

Functions of Several Variables

Definition : If to each pair (x, y) of values of two independent variable quantities x and y, there corresponds a definite value (unique) of the quantity z, then we say z is a function of two independent variables x and y.

A function z of two variables x and y is symbolically written as z=f(x, y), z=F(x,y) and so forth.

Examples

1. Let V be the volume of a box of length x units, breadth y units and height z units, then V=xyz

Volume is a function of length, breadth and height.

Definition : The collection of triads (x, y, z) of values of x, y and z for which V is defined is called domains of functions V,

where V=f(x, y, z)

2. $u = \frac{x^2 + y^2 + z^2 + t^2}{\sqrt{1-x^2}}$ is a function of four variables x, y, z and t.

Partial Differential Coefficients

Let z=f(x, y) be a function of two variables x and y, then $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or fx orfx(x, y) is called first order partial derivative of z w.r.to x and is defind as :

$$\frac{\partial z}{\partial x} = \underset{\partial x \to 0}{\mathrm{Lt}} \frac{f(x+\partial x, y) - f(x, y)}{\partial x}$$

Where as $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or fy or fy(x, y) is called first order partial derivative of z w.r. to y and is defined as

$$\frac{\partial z}{\partial y} = \underset{\partial y \to 0}{\text{Lt}} \frac{f(x, y+\partial y) - f(x, y)}{\partial y}$$

Second Order Partial Derivatives : for Z=f(x, y) are :

$$\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2} \text{ and } \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x} \text{ where}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial^2 z}{\partial x}\right]; \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial^2 z}{\partial y}\right], \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]$$

and $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]$. Similarly third order partial derivatives can also be found.

Example

1. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, then show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$

Sol. $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$ ...... (1)

Differentiate on both side partially w.r.t. y we get

$$\frac{\partial u}{\partial y} = x^2 . \frac{1}{1+(y/x)^2} . \left(\frac{1}{x}\right) - \left[2y \cdot \tan^{-1}\left(\frac{x}{y}\right) + y^2 \frac{1}{1+(x/y)^2}\left(\frac{-x}{y^2}\right)\right]$$

$$=\frac{x^3}{x^2+y^2}-2y\ \tan^{-1}\left(\frac{x}{y}\right)+\frac{xy^2}{x^2+y^2}$$

$$\Rightarrow\ \frac{\partial u}{\partial y}=\frac{x\left(x^2+y^2\right)}{x^2+y^2}-2y\ \tan^{-1}\left(\frac{x}{y}\right)$$

$$\Rightarrow\ \frac{\partial u}{\partial y}=x-2y\ \tan^{-1}\left(\frac{x}{y}\right)$$

Differentiate on both sides partially w.r.t. x we get

$$\frac{\partial^2 u}{\partial x\partial y}=1-2y.\frac{1}{1+\left(x/y\right)^2}.\frac{1}{y}$$

$$\Rightarrow\ \frac{\partial^2 u}{\partial x\partial y}=1-2y.\frac{y^2}{x^2+y^2}.\frac{1}{y}=1-\frac{2y^2}{x^2+y^2}$$

$$\Rightarrow\ \frac{\partial^2 u}{\partial x\partial y}=\frac{x^2+y^2-2y^2}{x^2+y^2},$$

Hence $\frac{\partial^2 u}{\partial x\partial y}=\frac{x^2-y^2}{x^2+y^2}$

2. If $z=\tan^{-1}\left(\frac{y}{x}\right)$, then show that :

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$$

Sol. $z=\tan^{-1}\left(\frac{y}{x}\right)$ .... (1)

Diff. on both side partially w.r.t. x, we get

$$\frac{\partial z}{\partial x}=\frac{1}{1+(y/x)^2}.\left(-y/x^2\right), \qquad \Rightarrow \frac{\partial z}{\partial x}=\frac{-y}{x^2+y^2}$$

$\Rightarrow \frac{\partial z}{\partial x}=-y(x^2+y^2)^{-1}$ Diff. again partiallyw.r.t. x, we get

$$\frac{\partial^2 z}{\partial x^2}=(-y)(-1)(x^2+y^2).(2x)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}=\frac{2xy}{\left(x^2+y^2\right)^2} \qquad \text{....... (2)}$$

Diff. (1) partially w.r.t. y , we get

$$\frac{\partial z}{\partial y}=\frac{1}{1+(y/x)^2}.\frac{1}{x}, \quad Þ \ \frac{\partial z}{\partial y}=\frac{x}{x^2+y^2}$$

$\Rightarrow \frac{\partial z}{\partial y}=x(x^2+y^2)^{-1}$, Diff. again partially w.r.t. y , we get

$$\frac{\partial^2 z}{\partial y^2}=x(-1)(x^2+y^2)^{-2}.(2y),$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2}=\frac{-2xy}{\left(x^2+y^2\right)^2} \qquad (3)$$

Adding equations (2) and (3), we get

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$$

3. If $u=e^{xyz}$, then show that $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=(1+3xyz+x^2y^2z^2)e^{xyz}$

Sol. Here $u=e^{xyz}$

Diff. on both side partially w.r.t. z , we get

$$\frac{\partial u}{\partial z}=xye^{xyz}$$

Diff. again partially w.r.t. y , we get

$$\frac{\partial^2 u}{\partial y\,\partial z}=x[1.e^{xyz}+y.(xz)e^{xyz}]$$

$$\Rightarrow \frac{\partial^2 u}{\partial y\,\partial z}=(x+x^2yz)e^{xyz}$$

Diff. partially w.r.t. x , we get

$$\Rightarrow \frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=(1+2x.yz)e^{xyz}+(x+x^2yz).(yz)e^{xyz}$$

$$=[(1+2xyz)+(x+x^2yz).(yz)]e^{xyz}$$

Hence $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=[(1+3xyz+x^2y^2z^2)]e^{xyz}$.

4. If z=f(x+ay)+f(x–ay), then prove that $\frac{\partial^2 z}{\partial y^2}=a^2.\frac{\partial^2 z}{\partial x^2}$

Sol. Here z=f(x+ay)+f(x–ay) ......(1)

Diff. (1) partially w.r.t. x , we get

$$\frac{\partial z}{\partial x}=f'(x+ay).1+f'(x-ay).1$$

Diff. again partially w.r.t. x , we get

$$\frac{\partial^2 z}{\partial x^2}=f''(x+ay)+f''(x-ay) \quad \text{........ (2)}$$

Diff. (1) partially w.r.t. y , we get

$$\frac{\partial z}{\partial y}=f'(x+ay).a+f'(x-ay).(-a)$$

Diff. again partially w.r.t. y , we get

$$\frac{\partial^2 z}{\partial y^2}=a^2f''(x+ay)+(-a)^2.f''(x-ay)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2}=a^2[f''(x+ay)+f''(x-ay)] \quad \text{........ (3)}$$

From equations (2) and (3), we get $a^2\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial y^2}$

Activity 1 : If u=log ($x^3+y^3+z^3-3xyz$), then show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z}$

Activity 2 : If $u=e^x(x \cos y - y \sin y)$, then prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

Activity 3 : If $u=\sqrt{x^2+y^2+z^2}$ then prove that $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=1$

Activity 4 : If $\frac{1}{v}=\sqrt{x^2+y^2+z^2}$ then show that $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0$

Homogenous function : A function z in x and y is said to be a homogenous function of degree n if it can be put in the form

$z=x^n$.**[**A function of (y/x)**]**

Equivalently $z=f(x, y)$ is called a homogenous function of degree x if $f(tx, ty)=t^n f(x, y)$.

Example

1. $z=x^3+y^3+3x^2y+2xy^2$ is a homogenous function of degree 3 in x and y as

   $z=x^3$**[**$1+(y/x)^3+3(y/x)+2(y/x)^2$**]**

   or $z=x^3$**[**A function of (y/x)**]**

2. $z=\frac{x^{1/2}+y^{1/2}}{x^{1/3}-y^{1/3}}$ is a homogenous function of degree $\frac{1}{6}$ in x and y as proved below. Replacing x by tx and y by ty we get :

$$\frac{(tx)^{1/2}+(ty)^{1/2}}{(tx)^{1/3}-(ty)^{1/3}}=\frac{t^{1/2}\left(x^{1/2}+y^{1/2}\right)}{t^{1/3}\left(x^{1/3}-y^{1/3}\right)}=t^{1/6}\left[\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right]$$

Activity 1 : Find the degree of homogenous functions.

(i) $z=\dfrac{x^4+y^4}{x-y}$ (ii) $z=\dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$

Euler's Theorem on Homogenous Functions

Statement : If u is a homogenous function of degree n in x and y, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$$

Proof : Suppose that $u=x^n f(y/x)$ .... (1)

Diff. (1) partially w.r.t. x , we get

$$\frac{\partial u}{\partial x}=nx^{n-1}f(y/x)+x^n.f'(y/x).(-y/x^2)$$

Multiplying by x on both sides, we get

$$x.\frac{\partial u}{\partial x}=nx^n\ f(y/x)-x^{n-1}y.f'(y/x) \qquad \text{........ (2)}$$

Diff. (2) partially w.r.t. y , we get

$$\frac{\partial u}{\partial y}=x^n.f'(y/x).(1/x)$$

Multiplying by y on both sides, we get

$$y.\frac{\partial u}{\partial y}=x^n y\ f'(y/x) \qquad \text{........ (3)}$$

Adding corresponding sides of equations (2) and (3), we get

$$x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}=n.x^n\ f\ (y/x)$$

Hence $x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}=n.u$

Example

1. If $u=\tan^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, then apply Euler's theorem to prove that

$$x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}=\frac{1}{4}\sin 2u$$

Sol. $u=\tan^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, $\Rightarrow$ $\tan u=\frac{x+y}{\sqrt{x}+\sqrt{y}}=z$ (Say)

$\Rightarrow$ $z=\tan u$ and $z=\frac{x+y}{\sqrt{x}+\sqrt{y}}$

Now $z=\frac{x+y}{\sqrt{x}+\sqrt{y}}$, $\Rightarrow$ $z=\frac{x\left[1+(y/x)\right]}{x^{1/2}\left[1+\sqrt{y}/x\right]}$

$\Rightarrow$ $z=x^{1/2}$ **[**A function of (y/x)**]**

$\Rightarrow$ z is a homogenous function of degree ½ in x and y, so by Euler's theorem, we get :

$$x.\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{1}{2}z, \qquad \text{Put } z=\tan u$$

$$\Rightarrow x.\frac{\partial}{\partial x}(\tan u)+y.\frac{\partial}{\partial y}(\tan u)=\frac{1}{2}(\tan u)$$

Dividing by $\sec^2 u$ on both sides, we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\left(\frac{\sin u}{\cos u}\right)\frac{1}{\sec^2 u}$$

$$=\frac{1}{4}\frac{\sin u}{\cos u}\times\frac{2\cos^2 u}{1}=\frac{1}{4}\sin 2u$$

Hence $x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}=\frac{1}{4}\sin 2u$

2. If $u=\log\frac{x^4-y^4}{x-y}$, then apply Euler's theorem to show that

$x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}=3.$

Sol. Here $u=\log\frac{x^4-y^4}{x-y}$ $\Rightarrow$ $\frac{x^4-y^4}{x-y}=e^u$

Let $\frac{x^4-y^4}{x-y}=e^u=z,$

then $z=e^4$, $z=\frac{x^4-y^4}{x-y}$

Now $z=\frac{x^4-y^4}{x-y}=\frac{x^4\left[1-(y/x)^4\right]}{x\left[1-(y/x)\right]}$

$\Rightarrow$ $z=x^3$ **[**A function of (y/x)**]**.

This shows that z is a homogenous function of degree 3 in x and y, so by Euler's theorem, we get

$$x.\frac{\partial z}{\partial x}+y.\frac{\partial z}{\partial y}=3z \qquad \text{Put} \quad z=e^u$$

$$\Rightarrow \ x.\frac{\partial}{\partial x}(e^u)+y.\frac{\partial}{\partial y}(e^u)=3(e^4)$$

$$\Rightarrow \ x(e^u)\frac{\partial u}{\partial x}+y(e^u)\frac{\partial u}{\partial y}=3e^u.$$

Dividing by e^u on both sides, we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$$

which is the required result to be proved.

3. Verify Euler's Theorem for the function $u=e^{-x/y}$

Sol. Here $u=e^{-x/y}$ ...... (1)

$$\Rightarrow \ u=x^0.\left[e^{-\frac{1}{(y/x)}}\right]$$ which shows that u is a homogenous function of degree 0 in x and y, so by Euler's theorem, we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.4$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0 \qquad \text{....... (2)}$$

Diff equation (1) on both side partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = -\frac{1}{y}.e^{-x/y}$$

Multiplying by x on both side, we get

$$x\frac{\partial u}{\partial x} = -\frac{x}{y}e^{-x/y} \qquad \text{....... (3)}$$

Diff. equation (1) on both side partially w.r.t. y, we get

$$\frac{\partial u}{\partial y} = \frac{x}{y^2}e^{-x/y}$$

Multiplying on both side by y, we get

$$y\frac{\partial u}{\partial y} = \frac{x}{y}e^{-x/y} \qquad \text{....... (4)}$$

Adding corresponding sides of *equations* (3) and (4), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

which is the same result as obtained by Euler's theorem in *equation* (2). This completes the verification of Euler's theorem.

4. Verify Euler's Theorem for the function :

$$Z = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$$

Sol. Here $Z = \dfrac{x^{1/4}\left[1 + (y/x)^{1/4}\right]}{x^{1/5}\left[1 + (y/x)^{1/5}\right]}$

$\Rightarrow$ $Z=x^{1/20}$[A function of (y/x)]

$\Rightarrow$ Z is a homogenous function of degree $1/20$ in x and y, so by Euler's theorem, we get

$$x.\frac{\partial z}{\partial x}+y.\frac{\partial z}{\partial y}=\frac{1}{20}Z \qquad \text{----- (1)}$$

Put $D_1=x^{1/4}+y^{1/4}$ and $D_2=x^{1/5}+y^{1/5}$

Then $Z=\frac{\Delta_1}{\Delta_2}$

Diff. on b.s.p.w.r.t. *x*

We get

$$\frac{\partial z}{\partial x}=\frac{\Delta_2.\frac{1}{4}x^{-3/4}-\Delta_1.\frac{1}{5}x^{-4/5}}{(\Delta_2)^2}$$

Multiplying by *x* on both sides, we get

$$x.\frac{\partial z}{\partial x}=\frac{1}{\Delta_2^2}\left[\frac{1}{4}\Delta_2 x^{1/4}-\frac{1}{5}\Delta_1 x^{1/5}\right] \qquad \text{...... (2)}$$

Similarly by inter-changing *x* and *y* we get

$$y.\frac{\partial z}{\partial y}=\frac{1}{\Delta_2^2}\left[\frac{1}{4}\Delta_2 y^{1/4}-\frac{1}{5}\Delta_1 y^{1/5}\right] \qquad \text{...... (3)}$$

Adding corresponding sides of equations (2) and (3), we get

$$x.\frac{\partial z}{\partial x}+y.\frac{\partial z}{\partial y}=\frac{1}{\Delta_2^2}\left[\frac{1}{4}\Delta_2\left(x^{1/4}+y^{1/4}\right)-\frac{1}{5}\Delta_1\left(x^{1/5}+y^{1/5}\right)\right]$$

$$\Rightarrow x.\frac{\partial z}{\partial x}+y.\frac{\partial z}{\partial y}=\frac{1}{\Delta_2^2}\left[\frac{1}{4}\Delta_2(\Delta_1)-\frac{\Delta_1}{5}\ (\Delta_2)\right]$$

$$=\frac{\Delta_1\Delta_2}{(\Delta_2)^2}\left(\frac{1}{4}-\frac{1}{5}\right)=\frac{1}{20}\left(\frac{\Delta_1}{\Delta_2}\right)$$

$\Rightarrow\ x.\frac{\partial z}{\partial x}+y.\frac{\partial z}{\partial y}=\frac{1}{20}z$ which is same as the result obtained by Euler's Theorem in equation (1).

This complets the verification of Euler's Theorem.

Activity 1

Use Euler's Theorem to prove that $x.\frac{\partial z}{\partial x}+y.\frac{\partial z}{\partial y}=nz$, where $z=x^n \log (y/x)$

Activity 2

Verify Euler's Theorem for functions

a) $Z=\frac{x^{1/2}+y^{1/3}}{x^{1/3}-y^{1/3}}$

b) $Z=\frac{xy}{x+y}$

Activity 3

If $V=\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, then show that $x.\frac{\partial V}{\partial x}+y.\frac{\partial V}{\partial y}=-\frac{1}{2}\cot V$

Exercises

a) If u=f(y/x), then show that $x.\frac{\partial u}{\partial x}y.\frac{\partial u}{\partial y}=0$

b) If $u=\sin^{-1}\dfrac{x^2+y^2}{x+y}$, then show that $x.\dfrac{\partial u}{\partial x}+y.\dfrac{\partial u}{\partial y}=\tan u$

c) If $u=(x^2+y^2+z^2)^{-½}$, then show that $x.\dfrac{\partial u}{\partial x}+y.\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=-u$

Exercises

1. If $u=\tan^{-1}\dfrac{xy}{\sqrt{1+x^2+y^2}}$, then prove that $\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{1}{\left(1+x^2+y^2\right)^{3/2}}$

2. If $u=\log(\tan x+\tan y)$, then prove that $\sin 2x.\dfrac{\partial u}{\partial x}+\sin 2y.\dfrac{\partial u}{\partial y}=2$

Exercise

Verify Euler's Theorem for $u=\sqrt{x^2+y^2}$

Prof. Vijay Rattan

(M.A.M. College, Jammu)

Objectives

To read the behaviour of the curve.

Structure

i) Total derivatives

ii) Double points

iii) Convexity and concavity of a curve

Introduction

In this lesson we shall learn about the total derivatives of composite functions. Finding of nature and position of double points on the given curve. We shall also find the intervals of convexity and concavity on a curve besides the points of inflexion on the curve.

Composite Functions and Total Derivative (Def.)

If u is given to be a function of the two variables x and y and further x and y are functions of a variable t, then u is called a composite function of the variable t.

Therefore the relations $u=f(x, y)$; $x=\phi(t)$; $y=\psi(t)$ defines u as a composite function of t and $\frac{du}{dt}$ is called **total derivative of u w.r. to t**.

Mathematically $\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$

Formulae involving total derivative

1. If u=f(x, y) where y=g(x) i.e. y is a function of x, then $\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}$

2. If u=f(x, y) where y=g(t), x=h(t) then $\frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$

3. If u=f(x, y), where x= ϕ(r, s), y= ψ(r, s)

Then $\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial r}$

and $\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial s}$

Example

1. Find the total derivative of u w.r.to t when $u=xy^2+x^2y$, where $x=at^2$, $y=2at$

Sol. $u=xy^2+x^2y$ $x=at^2$ $y=2at$

$\Rightarrow \frac{\partial u}{\partial x}=y^2+2xy,\quad \frac{\partial u}{\partial y}=2xy+x^2,\quad \frac{dx}{dt}=2at \quad \frac{dy}{dt}=2a$

Using $\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$, we get

$$\frac{du}{dt}=(y^2+2xy)(2at)+(2xy+x^2)(2a)$$

Putting values of x and y, we get

$$\frac{du}{dt}=2a[\{4a^2t^2+2(at^2)(2at)\}t+\{2(at^2)(2at)+a^2t^4\}]$$

$$=2a[4a^2t^3+4a^2t^4+4a^2t^3+a^2t^4]$$

$$\Rightarrow \frac{du}{dt} = 2a[4a^2t^4 + 8a^2t^3 + a^2t^4]$$

$$\Rightarrow \frac{du}{dt} = 2a^3t^3[5t+8] \text{ Ans.}$$

2. If $z=f(x, y)$ where $x=e^u+e^{-v}$, $y=e^{-u}-e^v$ then show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x.\frac{\partial z}{\partial x} - y.\frac{\partial z}{\partial y}$

Sol. $z=f(x, y)$, $x=e^u+e^{-v}$, $y=e^{-u}-e^v$

$$\therefore \quad \frac{\partial x}{\partial u} = e^u, \quad \frac{\partial x}{\partial v} = -e^{-v}, \quad \frac{\partial y}{\partial u} = -e^{-u}, \quad \frac{\partial y}{\partial v} = -e^v$$

Using the equation, $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$ we get

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u) + \frac{\partial z}{\partial y}.(-e^{-u}) \qquad \text{........ (1)}$$

Again using *equation*, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$ we get

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v) \qquad \text{......... (2)}$$

Subtracting (2) from (1), we get

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\{e^u + e^{-v}\} - \frac{\partial z}{\partial y}(e^{-u} - e^v)$$

$$= \frac{\partial z}{\partial x}(x) - \frac{\partial z}{\partial y}(y)$$

Hence $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x.\frac{\partial z}{\partial x} - y.\frac{\partial z}{\partial y}$

3. If w=f(x, y), x=r cos θ, y= r sin θ then show that

$$\left(\frac{\partial w}{\partial r}\right)^2+\left(\frac{\partial w}{\partial \theta}\right)^2 \cdot \frac{1}{r^2}=\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2$$

Sol. w= f(x, y) x=r cos θ, y=r sin θ

$$\therefore \quad \frac{\partial x}{\partial r}=\cos\theta, \quad \frac{\partial x}{\partial \theta}=-r\sin\theta, \quad \frac{\partial y}{\partial r}=\sin\theta, \quad \frac{\partial y}{\partial \theta}=r\cos\theta$$

Using equation $\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$, we get

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}(\cos\theta)+\frac{\partial w}{\partial y}(\sin\theta) \quad \text{......... (1)}$$

Again using equation $\frac{\partial w}{\partial \theta}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}$, we get

$$\frac{\partial w}{\partial \theta}=\frac{\partial w}{\partial x}(-r\sin\theta)+\left(\frac{\partial w}{\partial y}\right)(r\cos\theta)$$

Dividing by r on both side, we get

$$\frac{1}{r}\cdot\frac{\partial w}{\partial \theta}=-\frac{\partial w}{\partial x}(\sin\theta)+\frac{\partial w}{\partial y}(\cos\theta) \quad \text{......... (2)}$$

Squaring and adding equations. (1) and (2), we get

$$\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2=\left(\frac{\partial w}{\partial x}\right)^2(\cos^2\theta+\sin^2\theta)+\left(\frac{\partial w}{\partial y}\right)^2(\sin^2\theta+\cos^2\theta)$$

$$\Rightarrow \left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2=\left(\frac{\partial w}{\partial x}\right)^2(1)+\left(\frac{\partial w}{\partial y}\right)^2(1)$$

Hence $\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2=\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2$ $(\because \sin^2\theta+\cos^2\theta=1)$

$$\Rightarrow\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2=\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2 \qquad \text{(Since } w = f\text{)}$$

Activity 1

Find the differential coefficient of $Z=x^2y$ with respect to x when x and y are connected by the relation $x^2+xy+y^2=1$

Activity 2

Find the total derivative of u with respect to t, when $u=e^x \sin y$ and $x=\log t$, $y=t^2$.

Activity 3

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, when x and y are connected by the relation :

$x^3+y^3-3axy=0$

Double Points

Def. A point P(x, y) on a curve y=f(x) is called a double point if two branches of the curve pass through it.

There are in general two tangents at a double point which may real and distinct or real and coincident or imaginary.

Classification of Double Points

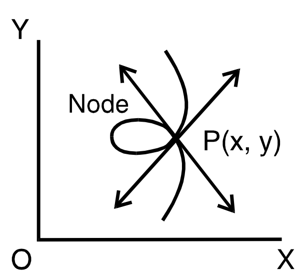

Fig. (i)

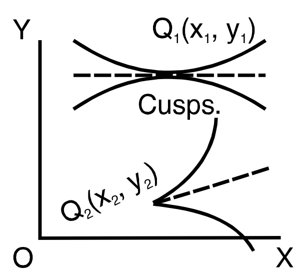

Fig. (ii)

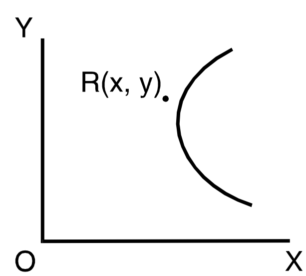

Fig. (iii)

There are three kinds of double points.

(a) Node : It is a point on the curve through which two pass two real branches of the curve and the two tangents at which are real and distinct. As shown above in fig. (i) P(x, y) is a node.

(b) Cusp : A point on the curve through which two real branches of the curve pass and the two tangents at which are real and coincident is called a cusp. As shown above in fig. (ii) $Q_1(x_1, y_1)$ and $Q_2(x_2, y_2)$ are both cusps.

(c) Conjugate point : A conjugate point or an isolated point on a curve is a point in the neighbourhood (nbd.) of which there is no other real point of the curve.

The two tangents at conjugate point are generally imaginary but some time, they may be real.

Art. Show that the necessary and sufficient conditions for any point (x, y) on f(x, y)=0, be a multiple point are that fx(x, y)=0 and fy(x, y)=0

Proof : The equation of the curve is f(x, y)=0

Diff. on both sides of (1) with respect to x keeping y as a function of x we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0 \qquad \text{.........} \ (2)$$

where $\frac{dy}{dx}$ gives the slope of the tangent at the point P(x, y). If P(x, y) is a multiple point, then there must be at least two tangents which may be real, coincident or imaginary. Thus $\frac{dy}{dx}$ must have at least two values at P(x, y). But (2) is a first degree equation in $\frac{dy}{dx}$ and is satisfied by at least two values of $\frac{dy}{dx}$ which is possible only if it becomes an identity. Thus $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$.

Also (x, y) lies on the curve f(x, y)=0

∴ The necessary and sufficient conditions for any point (x, y) on the curve f(x, y)=0 to be a multiple point are $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$

i.e. fx(x, y)=0 and fy(x, y)=0

Hence the result.

Working rule for finding nature and position of double points :

Let f(x, y)=0 be the equation of given curve.

1. Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}$ or $\frac{\partial^2 f}{\partial y \partial x}$

2. Solutions of the equations $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ in the form (x, y) which satisfy f(x, y)=0 gives the position of double points.

3. A point (double) is a node, cusp or conjugate point according as

$$\left(\frac{\partial^2 f}{\partial y \partial x}\right)^2 - \left(\frac{\partial^2 f}{\partial y^2}\right)\left(\frac{\partial^2 f}{\partial x^2}\right) \gtreqless 0 \quad \text{respectively.}$$

Note : For double pts. $\frac{\partial^2 f}{\partial x \partial y}=\frac{\partial^2 f}{\partial y \partial x}$.

Example

Prove that the curve $y^2=(x-a)^2(x-b)$ has at x=a, a node if a>b, cusp if a=b and a conjugate pt. if a<b.

Sol. $y^2=(x-a)^2(x-b)$

$\Rightarrow$ $(x-a)^2(x-b)-y^2=0$

Let $f(x, y)=(x-a)^2(x-b)-y^2$

$\therefore$ $\frac{\partial f}{\partial x}=2(x-a)(x-b)+(x-a)^2=(x-a)[2(x-b)+(x-a)]$

$\Rightarrow$ $\frac{\partial f}{\partial x}=(x-a)(3x-2b-a)$, $\frac{\partial f}{\partial y}=-2y$, $\frac{\partial^2 f}{\partial y^2}=-2$

$\frac{\partial^2 f}{\partial x^2}=1.(3x-2b-a)+(x-a).3$

$\Rightarrow$ $\frac{\partial^2 f}{\partial x^2}=6x-4a-2b$, $\frac{\partial^2 f}{\partial y\,\partial x}=0$

For double pts. $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ give

$(x-a)(3x-2b-a)=0$ and $-2y=0$

$\Rightarrow$ $x-a=0$ or $3x-2b-a=0$ and $y=0$

$\Rightarrow$ $x=a$, $x=\frac{a+2b}{3}$ and $y=0$

Thus possible double points are (a, 0) and $(\frac{a+2b}{3}, 0)$. But only (a, 0) lie on the curve.

At (a, 0)

$$\left(\frac{\partial^2 f}{\partial y\,\partial x}\right)^2-\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)$$

$=(0)^2-[6x-4a-2b][-2a]$ Put $x=a$

$=0+2a(6a-4a-2b)=4(a-b)$

Hence at $x=a$, there is node if $4(a-b)>0$ or if $a-b>0$, or if $a>b$.

At $x=a$, there is a cusp if $4(a-b)=0$

$\Rightarrow$ $a-b=0$, or if $a=b$

At $x=a$, there is a conjugate point if

$4(a-b)<0$ or if $a-b<0$ or if $a<b$

2. Prove that only singular point on the curve $(y-b)^2=(x-a)^3$ is a cusp and find its co-ordinates.

Sol. The given curve is $(x-a)^3-(y-b)^2=0$

Let $f(x, y)=(x-a)^3-(y-b)^2=0$

$$\frac{\partial f}{\partial x}=3(x-a)^2, \qquad \frac{\partial f}{\partial y}=-2(y-b), \qquad \frac{\partial^2 f}{\partial x^2}=6(x-a)$$

$$\frac{\partial^2 f}{\partial x \partial y}=0, \quad \frac{\partial^2 f}{\partial y^2}=-2$$

For double points $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ give

$3(x-a)^2=0$ and $-2(y-b)=0$

$\Rightarrow$ $x-a=0$ or $x=a$ and $y-b=0$, $\Rightarrow$ $y=b$

Therefore (a, b) is the only double point which also lies on given curve.

At (a, b)

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2-\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)$$

$=(0)^2-[6(x-a)][-2]$ Put $x=a$

$=12(a-a)=0$

Hence (a, b) is a cusp on the given curve.

3. Show that the origin is a node, cusp or a conjugate point on the curve $y^2=ax^2+bx^3$ according as a > = < 0 respectively.

Sol. Let $f(x, y) = ax^2+bx^3-y^2=0$

Then $\frac{\partial f}{\partial x}=2ax+3bx^2$, $\frac{\partial f}{\partial y}=-2y$, $\frac{\partial^2 f}{\partial y^2}=-2$

$\frac{\partial^2 f}{\partial x^2}=2a+6bx$, $\frac{\partial^2 f}{\partial x \partial y}=0$

For double points $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$

give $x(2a+3bx)=0$ Þ $x=0$, $x=-2a/3b$

and $-2y=0$ Þ $y=0$

But (0, 0) is the only double points.

At (0, 0)

$$\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2-\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)$$

$= (0)^2-(2a+6bx)(-2)$ Put $x=0$

$= 4a$

Hence (0, 0) is a node if $4a>0$ or if $a>0$

(0, 0) is a cusp if $4a=0$, or if $a=0$

(0, 0) is a conjugate pt. if $4a<0$ or if $a<0$.

Activity 1

Examine the existence and nature of double points on the curve $x(x^2+y^2)=ay^2$.

Activity 2

Show that the origin is a cusp on the curve $x^4-ax^2y+axy^2+a^2y^2=0$.

Activity 3

Find the nature of double points on the curve $a^2y^2=x^4(2x^2-3a^2)$.

Convexity and Concavity : Consider the curve $y=f(x)$ in the interval [a, b]. Let it be continuous and possess tangents at every point in (a, b). Draw a tangent to the curve at any pt. P(c, f(c)) on the curve which is not || y–axis.

If portion of the curve on both sides of P however small it may be, lies above the tangent at P_1 then we say curve is **convex downwords or concave upwards at P.**

If a portion of the curve on both sides of P*, however small it may be lies below the tangent at P_1^* then we say curve is **concave downwards or convex upwords at P^*.**

Points of inflexion : A point on a curve at which curve changes from convexity to concavity or vice versa is called point of inflexion. The following figures depict the above concepts.

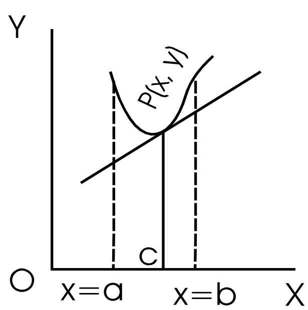

Fig. 1

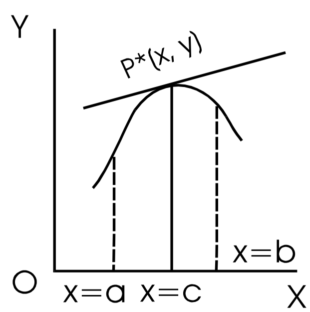

Fig. 2

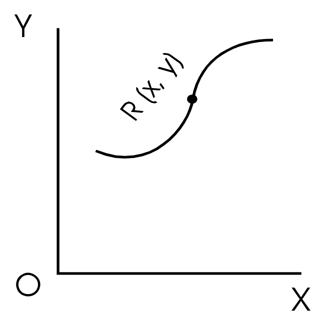

Fig. 3

In Fig (i) curve is concave upwards at P(x, y) and satisfies $\frac{\partial^2 y}{\partial x^2} > 0$.

In Fig. (ii) curve is convex upwards at P*(x, y) and satisfies $\frac{\partial^2 y}{\partial x^2} < 0$.

In Fig. (iii) R(x, y) is a pt. of inflexion and satisfies $\frac{\partial^2 y}{\partial x^2} = 0$.

1. Find the values of a and b, so that the curve $y=ax^3+3bx^2$ has a point of inflexion at (–1, 2).

Sol. $y=ax^3+3bx^2$ .... (1)

has a point of inflexion at (–1, 2), so (1) is satisfied by it

$\Rightarrow$ $2=a(-1)^3+3b(-1)^2$

$\Rightarrow$ $2=-a+3b$

$\Rightarrow$ $a=3b-2$ .... (2)

Diff. equation (1) on both sides w.r.t. x we get

$\frac{dy}{dx} = 3ax^2 + 6bx$, $\Rightarrow$ $\frac{dy^2}{dx^2} = 6ax + 6b$

Now for double points $\frac{d^2y}{dx^2} = 0$

$\Rightarrow$ $6ax + 6b = 0$, Put $x = -1$

$\Rightarrow$ $6a(-1) + 6b = 0$, $\Rightarrow$ $6a = 6b$, $\Rightarrow$ $b = a$.

Putting b = a in 2. we get

$b = 3b - 2$, $\Rightarrow 2b = 2$

Hence $b = 1$, $\Rightarrow$ $a = 1$.

2. Show that the points of inflexion of the curve $y^2 = (x-a)^2 (x-b)$ lie on 3 x + a = 4b.

Solution :- Given equation is $y^2 = (x-a)^2 (x-b)$.

Taking Sq. root of b.s. we get $y = (x-a)(x-b)^{1/2}$

$$\therefore \frac{dy}{dx} = 1.(x-b)^{1/2} + (x-a)\frac{1}{2(x-b)^{1/2}}$$

$$\Rightarrow \frac{2\left(\sqrt{x-b}\right)^2 + (x-a)}{\sqrt[2]{x-b}}, \quad \Rightarrow \frac{dy}{dx} = \frac{3x-a-2b}{2\sqrt{x-b}}$$

$$\frac{dy}{dx} = \frac{3x-a-2b}{2\sqrt{x-b}}, \text{Differentiate again w.r.t. x}$$

$$\Rightarrow \frac{d^2}{dx^2} = \frac{\left[2\sqrt{x-b}\right](3) - (3x-a-2b)\left[2\frac{1}{2\sqrt{x-b}}\right]}{4(x-b)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{6\left(\sqrt{x-b}\right)^2 - (3x-a-2b)}{4(x-b)\sqrt{x-b}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{3x-4b+a}{4(x-b)^{3/2}}$$

For points of inflexion $\frac{d^2y}{dx^2} = 0$

$$\Rightarrow \frac{3x-4b+a}{4(x-b)^{3/2}} = 0, \quad \Rightarrow 3x - 4b + a = 0$$

Hence pt. of inflexion lie on $3x + a = 4b$

Activity 1 :

Prove that the curve

i) $y = e^x$ is concave upwards every where.

ii) $y = \log x$ is convex upwards every where

Activity 2 :

Find the points of inflexion on the curve

$$x = (\log y)^3$$

Activity 3 :

Find the intervals of convexity and concavity for the curve

$y = x^3 - 3x^2 - 9x + 9$

Books Recommended for Studying

i) Differential calculus by K.C. Saxena and K.C. Gupta

ii) A text book of calculus by Chopra and Kochhar.

iii) A text book of calculus by B.L. Raina and Ram Krishan.

iv) The spectrum of calculus by R.L. Sharma & M.S. Baloria.

v) Hand book of Mathematical formulae by V.N. Indurkar.

vi) Advanced Diff. Calculus by Goyal and Gupta

vii) College Calculus by Larson & Hostetles.

viii) Elements of Differential and integral calculus by Granville.

ix) Differential Calculus by D.M. & Gupta.

x) Mathematics Quiz by M.G. Wells.

xi) Teach yourself calculus by P. Abbott.

xii) Differential Calculus by Saxena and Kapoor.

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Objective

To find envelopes and asymptotes to curves.

Structure

(i) Family of curves and Parameters.

(ii) Asymptotes parallel to axes.

(iii) Oblique asymptotes.

(iv) Envelopes.

Introductions

Above concepts involve the developing the knowledge of family of curves having one and two parameters and then finding the envelopes of some selected family of curves such as tangents to OS, parabola, ellipse and hyperbola.

We shall also learn about the infinite branches of the curves and then finding asymptotes which are parallel to co-ordinate axes and non parallel or oblique asymptotes to the infinite branches of the curves. It should be noted that a 3rd degree curve in x and y can have at the most three asymptotes.

Family of Curves

Consider the circle $(x-\alpha)^2+y^2=a^2$ having centre at $(\alpha,0)$ on x-axis and radius (fixed) a. If we give different values to α keeping 'a' fixed, then we get a system of circles whose centres lie on different points on x-axis. This system of circles is called a family of circles and α is called the parameter.

A system of curves $f(x, y, \alpha)=0$, where a is the parameter is called a family of curves.

Def : (Envelope). The envelope of a family of curves is the locus of the limiting positions of the point of intersection of any two consecutive members of the family when one of them tends to coincide with the others which is kept fixed.

Method of finding envelope of $f(x, y, \alpha) = 0$,

where α is the parameter.

Equation of given family of curves is

$$f(x, y, \alpha) = 0 \qquad \text{...........(i)}$$

Diff (i) partially w.r.t. parameter α, we get.

$$\frac{\partial f}{\partial \alpha} = 0 \qquad \text{...........(ii)}$$

Eleminating a from (i) and (ii) we get equation of envelope.

Envelopes of some special curves

(a) If $A \cos\theta + B \sin\theta = c$, where A, B and C are functions of x and y only and θ is the parameters, then equation of envelope is $A^2+B^2=C^2$.

Proof $A \cos\theta + B \sin\theta = C$ ....(i)

Diff on b. sides partially w.r.t. θ we get

$$A (-\text{Sin}\theta) + B (\text{Cos}\theta) = 0 \qquad \text{............(ii)}$$

Squaring and adding corresponding sides of above two equations we get

$$A^2 [(\text{Cos}\theta)^2 + (-\text{Sin}\theta)^2] + B^2 [(\text{Sin}\theta)^2 + (\text{Cos}\theta)^2] = C^2 + O$$

$\Rightarrow$ $A^2 (I) + B^2 (I) = C^2$, or $A^2+B^2=C^2$ is the required equation of the envelope.

(b) Equation of a family of curves which is quadratic in the parameters is **Discriminant=0**.

PF. Let $A\alpha^2+B\alpha+C = 0$ be the family of curves where α is the parameter. Here A, B, and C are functions of x and y only.

$$A\alpha^2+B\alpha+C = 0 \qquad \text{........(i)}$$

Diff (i) partially w.r.t. α we get

$$2\,A\alpha+B = 0, \quad \Rightarrow \quad \alpha=\frac{-B}{2A} \qquad \text{.......(ii)}$$

Eleminating a from (i) and (ii) we get

$$A\left(\frac{-B}{2A}\right)^2 + B\left[\frac{-B}{2A}\right] + C = 0$$

Multiplying by 4A on both sides we get

$$B^2 -2B^2 + 4AC = 0 \qquad \text{or} \qquad -B^2+4AC = 0$$

$\therefore$ $B^2-4AC = 0$ Or Discriminant = 0 in equation of the envelope.

Example (1) Find the envelope of the family of st lines $y=mx \pm a\sqrt{1+m^2}$, where m is the parameter.

Solution : $y=mx \pm a\sqrt{1+m^2}$ ....(i) is given family of straight lines.

$$\Rightarrow (y-mx) = \pm\sqrt{1+m^2}\,.\,a.$$

Squaring on both sides, we get

$$m^2x^2-2mny + y^2 = a^2\,(1+m)^2$$

$$\Rightarrow \quad (x^2-a^2)\,m^2 - 2mxy + (y^2-a^2) = 0 \qquad \text{......(ii)}$$

which is quadratic in the parameters m.

Hence equation of the envelope is Discriminant = 0

$\Rightarrow$ $[-2xy]^2 - 4(x^2-a^2)(y^2-a^2) = 0$

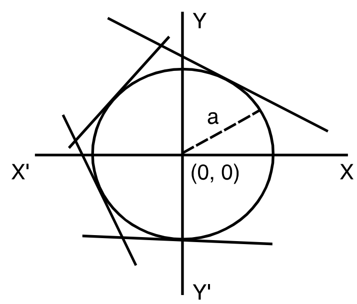
Dividing by 4 on both sides, we get

$x^2y^2-(x^2y^2-a^2x^2-a^2y^2+a^4) = 0$

$\Rightarrow$ $a^2x^2+a^2y^2 - a^4 = 0$

Dividing by a^2 on both sides we get

$x^2+y^2-a^2 = 0$ or $x^2+y^2=a^2$



which is reqd. = n of the envelope. It is a circle having centre at (0,0) and radius a. The given family of straight lines are tangents to the circle.

Example (2) Find the envelope of the family of curves $\frac{a^2}{x}$ Cosθ $-\frac{b^2}{y}$ Sin θ = c where θ is the parameter.

Solution :—The given family of curves is (a^2/x) Cosθ $-$ $\frac{b^2}{y}$ Sinθ =C ...(i)

Diff on both sides partially w.r.t. θ we get

$$\left(\frac{a^2}{x}\right)(-\text{Sin}\theta) - \frac{b^2}{y}(\text{Cos}\theta) = 0 \qquad .....(ii)$$

Squaring and adding above two equations, we get

$$\left(\frac{a^2}{x}\right)^2 [(\text{Cos}\theta)^2 + (-\text{Sin}\theta)^2] + (b^2/y)^2 [\text{Sin}^2+\text{Cos}^2\theta] = c^2+0$$

$$\Rightarrow \left(\frac{a^2}{x}\right)^2 (1) + (b^2/y)^2 (1) = C^2$$

Hence $\frac{a^4}{x^2}+\frac{b^4}{y^2} = c^2$ is equation of envelope.

Example (3) Find the envelope of the family of curves $\frac{ax}{\cos\alpha}-\frac{by}{\sin\alpha} = a^2-b^2$, where α is the parameter.

Solution : $\frac{ax}{\cos\alpha}-\frac{by}{\sin\alpha} = a^2-b^2$

or $(ax).\ \text{Sec}\ \alpha - (by).\ \text{cosec}\ \alpha = a^2-b^2$ .....(i)

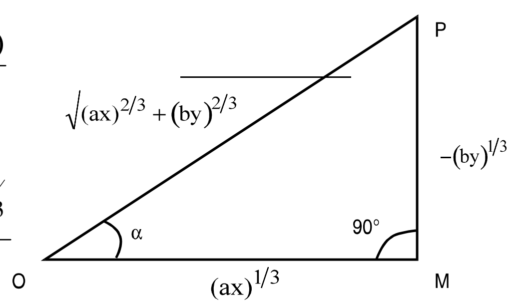
Diferentiating on both sides partially w.r.t. α we get

$\Rightarrow$ $(ax).\ \text{Sec}\ \alpha\ \tan\alpha - (by)\ (-\text{Cosec}\ \alpha.\ \text{Cot}\ \alpha) = 0$

$\Rightarrow$ $(ax).\ \frac{1}{\text{Cos}\alpha}\frac{\text{Sin}\ \alpha}{\text{Cos}\alpha} = -\ (by).\ \frac{1}{\text{Sin}\alpha}\frac{\text{Cos}\alpha}{\text{Sin}\alpha}$

$\Rightarrow$ $\frac{\text{Sin}^3\alpha}{\text{Cos}^3\alpha}=\frac{-by}{ax}, \Rightarrow \left(\frac{\text{Sin}\alpha}{\text{Cos}\alpha}\right)^3 = \frac{-(by)}{(ax)}$

$\therefore$ $\tan^3\alpha = \frac{-(by)}{(ax)}, \Rightarrow \tan\alpha = \frac{-(by)^{1/3}}{(ax)^{1/3}}$



Put $\Delta = \sqrt{(ax)^{2/3}+(by)^{2/3}}$

$\therefore$ $\text{Sec}\ \alpha = \frac{\Delta}{(ax)^{1/3}}$ and $\text{Cosec}\ \alpha = \frac{-\Delta}{(by)^{1/3}}$

Putting the values of Sec α and Cosec α in (i), we get

$$(ax)\frac{\Delta}{(ax)^{1/3}} - (by)\times\frac{\Delta}{(by)^{1/3}} = a^2 - b^2$$

$\Rightarrow \Delta\,[(ax)^{2/3} + (by)^{2/3}] = a^2 - b^2$

$\Rightarrow \Delta\,[\Delta^2] = a^2 - b^2$ or $\Delta^3 = (a^2 - b^2)$

Raising power 2/3 on both sides we get

$[\Delta^3]^{2/3} = (a^2 - b^2)^{2/3}$, $\Rightarrow$ $\Delta^2 = (a^2 - b^2)^{2/3}$

Hence $(ax)^{2/3} + (by)^{2/3} = (a^2 - b)^{2/3}$ is the equation of the envelope.

Activity (1) Find the envelope of family of curves

(i) $(x-a)^2 = y^2 = 4a$

(ii) $(x-a)^2 + y^2 = 4$

where a is the parameter

Activity (2). Find the envelope of family of curves x $\cos\theta + \sin\theta = \sin\theta\cos\theta$. Where θ is the parameter.

Activity (3) Find the equation of envelope of the family of tangents $y = mx + \frac{a}{m}$ to the parabola where m is a parameter.

Envelope of two parameter family of curves :—

Let $f(x, y, \alpha, \beta) = 0$ be a two parameter family of curves where a and b are parameters connected by the relation $f(\alpha, \beta) = 0$ ....(II)

Diff. these two equations partially with respect to a keeping b as a function of θ, we get

$$\frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial \beta} \times \frac{\partial \beta}{\partial \alpha} = 0 \qquad ....(III)$$

$$\frac{\partial\phi}{\partial\alpha}+\frac{\partial\phi}{\partial\beta}\times\frac{\partial\beta}{\partial\alpha} = 0 \qquad \text{........(IV)}$$

Eleminating a and b from above four equations we get equation of the envelope.

Example (1) Find the equation of envelope of the family of straight lines $\frac{x}{a}+\frac{y}{b} = 1$ where a and b are parameters connected by the relation $ab=c^2$.

Solution : $\frac{x}{a}+\frac{y}{b} = 1$ or $x.a^{-1} + y.b^{-1} = 1$ —(I)

$$ab=c^2 \qquad \text{—(II)}$$

Diff (I) and (II) partially w.r.t. a keeping b as a function of a.

$$x.(-1)a^{-2} + y.(-1).\ b^{-2}.\frac{db}{da} = 0, \ \Rightarrow \frac{-x}{a^2}=\frac{y}{b^2}.\frac{db}{da}$$

$$\text{or} \quad \frac{db}{da}=\frac{-b^2x}{a^2y} \qquad \text{—(III)}$$

$$I.b + a.\frac{db}{da} = 0, \Rightarrow a.\frac{db}{da} = -b, \text{ or } \frac{db}{da} = \frac{-b}{a} \qquad \text{—(IV)}$$

From equations (III) and (IV), we get

$$\frac{-b^2x}{a^2y}=\frac{-b}{a}, \ \Rightarrow \frac{x}{a}=\frac{y}{b}, \Rightarrow \frac{x/a}{1}=\frac{y/b}{1}$$

Using ratio and proportion, we get

$$\frac{x/a}{1}=\frac{y/b}{1}=\frac{x/a+y/b}{1+1} \text{ using equation (1), we get}$$

$\frac{x}{a} = \frac{y}{b} = \frac{1}{2}, \Rightarrow \frac{x}{a} = \frac{1}{2}, \Rightarrow a = 2x$

Also $\frac{y}{b} = \frac{1}{2}, \Rightarrow$ b=2y.

Putting values of a and b in (II), we get $(2x)(2y) = C^2$

$\Rightarrow$ $4xy = C^2$ is = n of envelope

Activity (1) Find the envelope of the family of ellipses $x^2/a^2 + y^2/b^2 = 1$, where parameters a and b are connected by the relation $a^2+b^2 = C$.

Activity (2). Find the envelope of the family of parabolas $(x/a)^{1/2} + (y/b)^{1/2} = 1$, where parameters a and b are connected by the relation $ab=C^2$.

Asymptotes

Infinite Branches of a curve. If in an equation y=f(x) y has two or more values for every value of x then we suppose that in such a case we are given two or more distinct functions. But generally we regard the curves corresponding to these functions, not as different curves but as different branches of one curve.

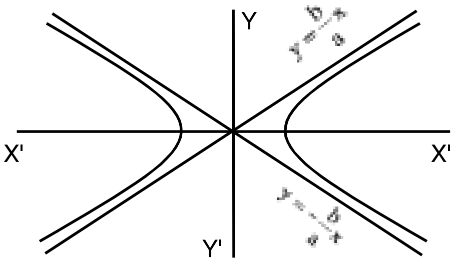
Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solving for y we get $y = \pm \frac{b}{a} \sqrt{x^2 - a^2}$.

Here for every value of x, y has two values and as $x \to \pm \propto$, y also tends to $\pm \propto$. Thus hyperbola has two infinite branches as shown in adjoining fig.

Here $y=\pm\frac{b}{a}x$ are two two oblique asymptotes to hyperbola.

Asymptote (Def.) A straight line at a finite distance from the origin, is said to be an asymptote of an infinite branch of a curve, if the perpendicular distance of a point P on that branch from the straight line tends to zero as P tends to infinity along the branch of the curve.



Asymptotes parallel to co-ordinate axces :—

If equation of the curve is of degree n in x and y where x^n (y^n) is absent in the equation, then coefficient of next higher power of x(y) present in the equation when equated to zero gives asymptote or asymptotes parallel to x-axis (y-axis) provided this coefficient is not merely a constant or gives rise to imaginary lines.

Example (1) Find the asymptotes || to axes for the curve. $xy^3+x^3y-9^4 = 0$.

Solution : $xy^3+x^3y-9^4 = 0$ .....(i)

If is of degree 4 in x and y in which both x^4 and y^4 are absent.

Coefficient of $x^3=y$, so $y=0$ is asymptote parallel to x-axis.

Coeff. of $y^3 = x$, so $x = 0$ is asymptote || x-axis.

Example (2) Find asymptotes parallel to axes for the curves (a) $x^2y^2 = 9(x^2+y^2)$, (b) $y^3=(x-1)^2$.

Solution : (a) The given equation is $x^2y^2-9x^2-9y^2=0$ ......(i)

Which is of degree 4 in x and y in which both x^4 and y^4 are absent.

Coeff. of $x^2=y^2-9$, \ $y^2-9=0$ or $y = \pm 3$ are two asymptotes || x-axis.

Coeff. of $y^2 = x^2-9$, so $x^2-9=0$, or $x=\pm 3$ are two asymptotes || y–axis.

Solution : (b) $y^3 = (x-1)$ or $y^3 = x^2 - 2x+1$ Þ $y^3-x^2 + 2x - 1 = 0$ is of degree 3 in x and y in which y^3 is present, so no asymptotes || to›y–axis. Although x^3 is absent but coefficient of x^2 is –1 which is constant. Hence no asymptote || to x-axis as well.

Activity (1) Find asymptotes parallel to axes for the curves.

(a) $x^2y^2 = a^2 (x^2-y^2)$ (b) $(x^2+y^2) x=ay^2$

(b) $y^3-xy^2=x^2+1$ (d) $x^2y^2+y^2=1$

Oblique asymptotes :—An asymptote to a curve which is not parallel to any of the two co-ordinate axes is called on oblique asymptote.

Method of finding oblique asymptotes :—

(i) Let y=mx+c be an oblique asymptote to a 3rd degree curve in x and y.

$f(x, y) = 0$ .... (1)

(2) Put x=1, y=m in 3rd, 2nd and 1st degree terms in L.H.S. of (1) seprately. Let the expressions obtained be respectively called as $\phi_3(m)$, $\phi_2(m)$ and $\phi_1(m)$.

(3) The values of m for y=mx+c are the roots of equation $\phi_3(m)=0$.

(4) For non repeated values of m, c is given by $c=\dfrac{-\phi_2(m)}{\phi_3'(m)}$

(5) For repeated roots of $\phi_3(m) = 0$, c is given by

$$\frac{c^2}{2!}\,\phi''_3(m) + c.\,\phi^1_2(m) + \phi_1(m)=0$$

Putting the corresponding values of m and c in y=mx+c we get the *equations* of asymptotes

Example (1) Find the asymptotes of the curve.

$3x^3+2x^2y-7xy^2+2y^3-14xy+7y^2+4x+5y=0$

Solution : Let y=mx+c be an oblique asymptote. Put, x=1, y=m in 3rd and 2nd degree terms on the L.H.S. of (1) we get

$\phi_3(m) = 3+2m-7m^2+2m^3$; $\phi_2(m) = -14m+7m^2$

Now $\phi'_3(m) = 2-14m+6m^2$.

For values of m, $\phi_3(m)=0$ gives $2m^3-7m^2+2m+3=0$

Here m=1 is a root, so by synthetic division we get

$$\begin{array}{cccc|c} 2 & -7 & 2 & 3 & 1 \\ & 2 & -5 & -3 & \\ \hline & -5 & -3 & 0 & \end{array}$$

$\therefore$ Reduced equation is $2m^2-5m-3=0$

$\Rightarrow$ $2m^2-6m + m-3 = 0$, $\Rightarrow$ $2m(m-3) + 1.(m-3) = 0$

$\Rightarrow$ $(m-3)(2m+1) = 0$, $\Rightarrow$ $m=3, \frac{1}{2}, 1$

Now $C=-\phi_2(m)/\phi_3(m)$, $\Rightarrow$ $C= -\frac{[-14m+7m^2]}{[2-14m+6m^2]}$

For m=1, $C= -\frac{[-14+7]}{[2-14+6]} = \frac{-7}{6}$

For m=3, $C = \frac{-[-42+63]}{[2-42+54]}$, $\Rightarrow$ $C= \frac{-21}{14} = \frac{-3}{2}$

For m=–½, $C=\frac{[7+7/4]}{[2+7+6/4]} = \frac{-(35/4)}{(42/4)}$, $\Rightarrow$ $C=\frac{-5}{6}$

Therefore three asymptotes are $y=(1)x+(-7/6)$

$y=(3)x+(-5/6)$ and $y=(-1/2)x+(-5/6)$

Example (2) Find all asymptotes of the curve $x^2+x^2y-xy^2-y^3-3x-y-1=0$

Solution : $x^3+x^2y-xy^2-y^3-3x-y-1 = 0$ ......(1)

Let $y=mx+c$ be an oblique asymptote to (1)

Put x=1, y=m in 3rd, 2nd and 1st degree terms on the L.H.S. of equation (I) and the expressions obtained be respectively taken as

$\phi_3(m) = 1+m-m^2-m^3$, $\phi_2(m) = 0$, $\phi_1(m) = -3-m$

$\phi_3(m) = 1-2m-3m^2$, $\phi''_3(m) = -2-6m$, $\phi'_2(m) = 0$

For values of m, $\phi_3(m) = 0$ gives

$1+m-m^2-m^3=0$, $\Rightarrow$ $1.(1+m)-m^2(1+m) = 0$

$\Rightarrow$ $(1+m)(1-m^2) = 0$, $\Rightarrow$ $1+m=0$ or $1-m^2 = 0$

$\Rightarrow$ $m=-1$ or $m^2=1$, $\Rightarrow$ $m=\underline{+}1$

For m=1, $C=\dfrac{-\phi_2(m)}{\phi_3'(m)}=\dfrac{0}{1-2m-3m^2}=\dfrac{0}{1-2-3}=0$

For m =–1 C is obtained from the equation $\dfrac{C^2}{2!}\phi''_3(m)+C\,\phi'_2(m)+\phi_1(m) = 0$

$\Rightarrow$ $\dfrac{C^2}{2!}(-2-6m) + C(0) + (-3-m) = 0$, Put $m=-1$

$\Rightarrow$ $\dfrac{C^2}{2\times 1}[-2-6\times(-1)] + (-3+1) = 0$

$\Rightarrow$ $\dfrac{C^2}{2}(4) + (-2) = 0$ or $C^2-1=0$, $\Rightarrow$ $C= \pm 1$

Thus three asymptotes are $y = (1)\,x+0$; $y=(-1)\,x+1$, $y= -x-1$

or $x-y=0$, $x+y-1=0$, $x+y+1=0$

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Objective : To solve indeterminate limits and trace the curves.

Structure

(i) Tracing of cartesian curves.

(ii) L-Hospital Rule and its applications :

Introduction

In the chapter of tracing of certesian curves we shall find the symmetry of the curves about x-axis, y-axis, about the lines y=x, y=–x. We shall also find intersection of curves with axes, double points on the curve. We shall also find asymptotes and region of the curve.We will not find any other points on the curves.

L-Hospital rule is used to evaluate special types of limits which are in $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0.\infty$, ∞-∞, 1^∞, 0^∞, 0°, ∞^0 or ∞^∞ form. Such limits cannot be evaluated without the application of this rule.

Method for tracing of cartesian curves

Symmetry (1) If equation of given curve remains unchanged when y is changed to –y, then the curve is symmetrical about x-axis.

(2) If equation of the curve remains unchanged when x is changed to–x, then the curve is symmetrical about y-axis.

(3) If equation of the curve remains unchanged when x and y are inter changed, then curve is symmetrical about the line y=x.

(4) If equation of the curve remains unchanged x and y are changed to –x and –y respectively then the curve is symmetrical in opposite quadrants.

ORIGIN : Given curve passes through the origin if x=0 and y=0 satisfy its equation. Such curve does not contain a constant term. If the curve passes through the origin then find tangents at the origin by equating lowest degree terms (present in the equation) to zero.

The origin will be a node, cusp or conjugate point according as two tangents are real and distinct, coincident or imaginary.

AXES : Find the points of intersection of the given curve with axes by solving it with equations x=0 and y=0.

ASYMPTOTES : Find the asymptotes of the given curve if any. These help in tracing of curve.

REGION : Find out if there is any region in which the curve lies or does not lie. This is done by solving for y (or for x as the case may be). Also find the values of x (y) which makes y (x) imaginary.

POINTS OF INFLEXION : Find the points of inflexion and double pts on the curve if any. Also find nature of double pts on it.

Then we draw a rough sketch of the curve.

Example (1) Trace the curve $y^2 (a+x) = x^2 (3a-x)$

Solution : The equation of the curve is $y^2(a+x) = x^2 (3a-x)$ ..... (i)

(1) The curve is symmetrical about x-axis because it contains only even powers of y.

(2) The curve passes through the origin (0, 0).

From (1), $ay^2+xy^2-3ax^2+x^3=0$ ...... (ii)

$\therefore$ Tangents at origin are given by

$ay^2-3ax^2=0 \qquad \Rightarrow \qquad y^2=3x^2$

$\Rightarrow$ $y = \pm\sqrt{3}\,x$ are two real and distinct tangents at the origin. Hence origin is a node on the given curve.

(3) From equation (ii) which is 3rd degree in x and y, the term containing y^3 is absent. Coefficient of $y^2 = x+a$.

$\therefore$ $x + a = 0$ or $x=-a$ is an asymptote to the curve parallel to y-axis.

(4) Putting $y = 0$ in equation (i) we get

$x^2 (3a-x) = 0$, $\Rightarrow$ $x=0, 0, 3a$.

Thus curve meets x-axis at (0, 0) and (3a, 0)

Put $x=0$ in (i), then $y^2(a+0) = 0$, $\Rightarrow$ $y = 0$

So curve meets y-axis only at (0, 0)

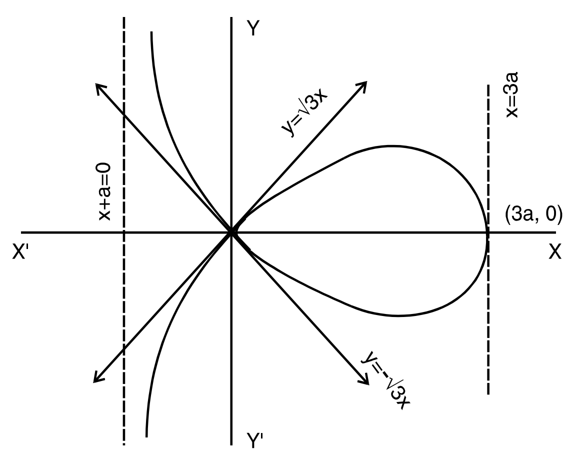
The curve has a loop between $x=0$ and $x=3a$.

(5) **Region :** From given equation (i)

$$y^2 = \frac{x^2(3a-x)}{a+x}, \quad \Rightarrow y = \pm x\sqrt{\frac{3a-x}{a+x}}$$

Now if $3a-x < 0$, y is imaginary is if $x > 3a$, then y is imaginary, so no part of the curve lies beyond the line $x=3a$.

Also if $a+x < 0$, then y is imaginary i.e. if $x<-a$, then y is imaginary, so curve does not lie on left side of line $x=-a$. Hence shape of the curve is as shown below.



Example (2) Trace the cartesian curve $x^{2./3}+y^{2./3} = a^{2./3}$

Solution : The equation of curve is $x^{2/3}+y^{2/.3} = a^{2/3}$

Dividing by $a^{2/3}$ on both sides we get $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{a}\right)^{2/3} = 1$ ....(i)

I. Since (i) contains only even powers of both x and y, so it is symmetrical about both axes.

II. Equation of the curve does not change when x and y are inter changed or when x and y are changed to –x and –y respectively, so the curve is also symmetrical about the line y=x and in opposite quardrants also.

III. The curve does not pass through the origin (0,0).

IV. Putting y = 0 in 1 we get $\left(\frac{x}{a}\right)^{\frac{2}{3}} = 1$

$$\Rightarrow \left(\frac{x}{a}\right)^2 = (1)^3, \Rightarrow \left(\frac{x}{a}\right)^2 = 1, \Rightarrow \frac{x}{a} = \pm 1$$

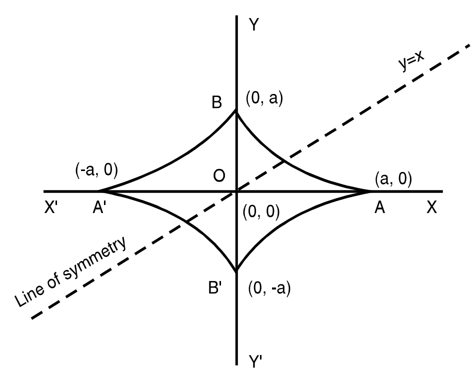
Hence curve meets x–axis at (a,0), (–a, 0). Similarly by taking x = 0, we find the curve meets y–aseis at (0,a) and (0,–a).

V. **Region :** From equation I we get

$$\left(\frac{y}{a}\right)^{2/3} = 1-\left(\frac{x}{a}\right)^{2/3}.$$

Therefore if –a > x > a, then y is imaginary so no part of the curve lies beyond the lines x = a and x = –a. By symmetry no part of the curve lie beyond the lines y = a and y = –a. Hence the curve lies within a square bounded by four straight lines x = a, x = –a, y=a, y = –a.

VI. The curve has no asymptotes, no double pts and no point of inflexion. Hence shape of the curve is as shown in the adjoining figure.



Example 3. Trace the cartesian curve $y^2 (a^2 - x^2) = x^4$.

Solution : The equation of the curve is $y^2 (a^2 - x^2) = x^4$ ....(i)

1. It is symmetrical about both axes as it contains only even powers of x and y. The curve is symmetrical in opposite quadrants as its = x does not change when x and y are changed to –x and –y respectively.

2. The curve passes through the origin (0,0) and tangents at origin are given by $a^2 y^2 = 0$

$\Rightarrow$ $y^2 = 0$, So $y = 0$ and $y = 0$ are two real and coincident tangents at the origin. Hence origin is a cusp on the curve.

3. As y^4 is absent in 4th degree = n and coefficient of $y^2 = a^2 - x^2$

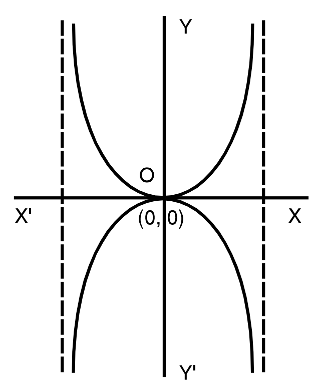
$\therefore$ $a^2 - x^2 = 0$ or $x = a$, $x = -a$ are two asymptotes to the curve.

4. From = n we get $y^2 = \dfrac{x^4}{a^2 - x^2}$

$\Rightarrow$ $y = \pm \dfrac{x^2}{\sqrt{a^2 - x^2}}$, This shows that x can not exceed a or decrease from –a, otherwise y is imaginary.

Hence curve does not lie on the right side of $x = a$ and on the left side of $x = -a$.

The branches of the curve tend to be infinite when x is very close to a or –a. Hence the shape of the curve is as shown in the adjoining figure.



Activity 1 Trace the cartesian curve $y^2 = x (x+1)^2$.

Activity 2 Trace the parabola $y^2 = 4 ax$

Activity 3 Trace the cartesian curve

$$y^2 (a-x) = x^2 (a+x)$$

L - Hospital's Rule

Def. 1 Let f(x) and g(x) be two functions of x in [a, b] such that $\underset{x\to a}{\text{Lt}}\ f(x)\ \underset{x\to a}{\text{Lt}}\ g(x)=0$ then.

$$\underset{x\to a}{Lt}\ \frac{f(x)}{g(x)} = \underset{x\to a}{Lt}\ \frac{f'(x)}{g'(x)}$$ provided the latter limit exists whether finite or infinite.

2. If $\underset{x\to a}{\text{Lt}}\ f(x)=\underset{x\to a}{\text{Lt}}\ g(x)=\infty$, even then $\underset{x\to a}{Lt}\ \frac{f(x)}{g(x)} = \underset{x\to a}{Lt}\ \frac{f'(x)}{g'(x)}$, provided the latter limit exists whether finite or infinite.

Remarks : 1. If given limit is in $\infty\times 0$ form or $\infty - \infty$ form, then we first change it into $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ and then apply L – Hospital rule.

2. If given limit is in 0^{∞}, 0°, ∞^{∞}, ∞ or 1^{∞} form, then given limit is evaluated by taking log of both sides.

Example 1 : Prove that $\underset{x\to 0}{\text{Lim}}\ \frac{e^x - e^{-x}}{\text{Sin}\ x} = 2$

Solution : Let $l = \underset{x\to 0}{\text{Lim}}\ \frac{e^x - e^{-x}}{\text{Sin}\ x}$ | $\frac{0}{0}$ form; $\because e^{\circ} = 1$ and $\sin 0^{\circ} = 0$

Using L - Hospital's rule we get

$$\Rightarrow l = \underset{x\to 0}{\text{Lim}}\ \frac{e^x + e^{-x}}{\cos x} = \frac{e^0 + e^0}{\cos 0^0} = \frac{1+1}{1} = 2$$ Q.E.D.

Hence $l = 2$.

Example 2 : Prove that $\underset{x\to 0}{\text{Lim}}\ \frac{x\,e^x - \log(1+x)}{x^2} = 3/2$.

Solution : Let $l = \lim_{x \to 0} \frac{xe^x - \log(1+x)}{x^2} \Big|\frac{0}{0}$ form

Since log 1 = 0 and $e^0 = 1$, so using L- Hospital rule, we get

$$L = \lim_{x \to 0} \frac{x\ e^x + e^x - \frac{1}{1+x}}{2x} \Big|\frac{0}{0} \text{ form}$$

Again using L - Hospital rule, we get

$$L = \lim_{x \to 0} \frac{e^x + x\ e^x + e^x + \frac{1}{(1+x)^2}}{2} = \frac{e^0 + o.e^0 + e^0 + \frac{1}{(1+0)^2}}{2}$$

Hence $l = \frac{1+0+1+1}{2}; \Rightarrow l = 3/2$ Ans. Q.E.D.

Example 3. Prove that $\mathop{Lt}_{x \to 0} \frac{(1+x)^{1/x} - e}{x} = \frac{-1}{2}e$

Solution : Let $L = \mathop{Lt}_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$

Put $y = (1+x)^{1/x}$, Taking log of both sides, we get $\log y = \frac{1}{x} \log(1+x)$.

$$\Rightarrow \log y = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - ....\right) = 1 - \frac{x}{2} + \frac{x^2}{3} - .....$$

$$\Rightarrow \log y = 1 + \left(\frac{-x}{2} + \frac{x^2}{3} - .....\right) = 1 + z \quad \text{(say)}$$

Where $z = \frac{-x}{2} + \frac{x^2}{3} - ...., Þ \log y = 1 + z$

Now $y = e^{1+z}, \Rightarrow y = e.e^z$

$$\Rightarrow y = e.\left[1 + z + \frac{z^2}{2} + ...\right].$$

$$\Rightarrow y = e.\left[1 + \left\{\frac{-x}{2} + \frac{x^2}{3} - ......\right\} + \frac{1}{2}\left\{\frac{-x}{2} + \frac{x^2}{3} - ...\right\}^2 + ...\right]$$

$$\Rightarrow y = e\left[1 + \left(-\frac{1}{2}\right)x + \left(\frac{1}{3} + \frac{1}{8}\right)x^2 - ......\right]$$

$$\Rightarrow y = e\left[1 - \frac{x}{2} + \frac{11}{24}x^2 + ...\right]$$

Hence $l = \underset{x \to 0}{\text{Lt}} \frac{e\left[1 - \frac{x}{2} + \frac{11}{24}x^2 + ...\right] - e}{x}$

$$\Rightarrow \quad l = \underset{x \to 0}{\text{Lt}} \left[\frac{e\left(-\frac{x}{2} + \frac{11}{24}x^2 + ...\right)}{x}\right]$$

$$l = \underset{x \to 0}{\text{Lt}} e\left[\frac{1}{2} + \frac{11}{24}x + ...\right] = e\left[\frac{1}{2} + \frac{11}{24}(0) + ...\right]$$

Hence $l = \frac{-1}{2}$ Ans. Q.E.D.

Example 4 : Prove that $\underset{x \to 0}{Lt} \dfrac{\log \tan nx}{\log \tan x} = 1$

Solution :- Let $l = \underset{x \to 0}{Lt} \left[\dfrac{\log \tan nx}{\log \tan x}\right] \quad \left| \dfrac{\infty}{\infty} \right.$ from $\begin{matrix} \log 0 = \infty \\ \tan 0^0 = 0 \end{matrix}$

Using L- Hospital's rule we get

$$l = \underset{x \to 0}{Lt} \left[\frac{\left(\frac{1n}{\tan nx}\right). \sec^2 nx}{\frac{1}{\tan x}.\sec^2 x}\right] = \underset{x \to 0}{Lt} \left(\frac{nx}{\tan nx}\right)\left(\frac{\tan x}{x}\right)\left(\frac{\sec^2 nx}{\sec^2 x}\right)$$

Hence $l = (1)\,(1)\left(\dfrac{1}{1}\right) = 1, \Rightarrow L = 1$

Since $\underset{\theta \to 0}{Lt} \left(\dfrac{\tan \theta}{\theta}\right) = 1$ and $Sec^2 0^\circ = 1$

Example 5 : Prove that $\underset{x \to 0}{Lt} \left(\dfrac{1}{x} - \cot x\right) = 0$

Solution Let $l = \underset{x \to 0}{Lt} \left(\dfrac{1}{x} - \cot x\right) \quad \left| \begin{matrix} \infty - \infty \\ \text{form} \end{matrix} \right.$

$$\Rightarrow l = \underset{x \to 0}{Lt} \left(\frac{1}{x} - \frac{\cos x}{\sin x}\right) = \underset{x \to 0}{Lt} \left(\frac{\sin x - x\cos x}{x\sin x}\right) \quad \left| \frac{0}{0} \textit{form} \right.$$

$$\Rightarrow l = \underset{x \to 0}{Lt} \frac{\cos x - 1.\cos x + x.\sin x}{1.\sin x + x.\cos x},$$

$$\Rightarrow e = \underset{x \to 0}{Lt} \frac{x\sin x}{\sin x + x\cos x} \quad \left| \frac{0}{0} \textit{form} \right.$$

$$\Rightarrow L = \underset{x \to 0}{Lt} \frac{\sin x + x.\cos x}{\cos x + \cos x - x.\sin x}, \Rightarrow 1 = \underset{x \to 0}{Lt} \frac{\sin x + x \cos}{2\cos x - x \sin x}$$

$$\Rightarrow L = \frac{(0)+0(1)}{2(1)-(0)\,(0)} = \frac{0}{2}, (\because \sin 0^\circ = 0, \quad \cos 0^\circ = 1) \quad \Rightarrow l = 0$$

Example 6: Prove that $\underset{x \to 1}{Lt} \left[\frac{2}{x^2-1} - \frac{1}{x-1}\right] = \frac{-1}{2}$

Prof : Let $l = \underset{x \to 1}{Lt} \left[\frac{2}{x^2-1} - \frac{1}{x-1}\right]$ | $\propto - \propto$ *form* as $\frac{1}{0} = \propto$

$$\Rightarrow l = \underset{x \to 1}{Lt} \left[\frac{2-(x+1)}{x^2-1}\right]$$

$$\Rightarrow l = \underset{x \to 1}{Lt} \left[\frac{1-x}{-(1-x^2)}\right], \Rightarrow l = \underset{x \to 1}{Lt} \frac{-1}{1+x}$$

Hence $l = \frac{-1}{1+x}$, $\Rightarrow L = \frac{-1}{2}$

Example 7 : If limit, $\underset{x \to 1}{Lt} \frac{\text{Sin}2x + a \sin x}{x^3}$ be finite then find the values of a and the limit.

Solution : Let $l = \underset{x \to 0}{Lt} \frac{\text{Sin}2x + a \sin x}{x^3}$ | $\frac{0}{0}$ form

So by L - Hospital rule we get

$$l = \underset{x \to 0}{Lim} \frac{2\cos 2x + a \cos x}{3x^2}$$

Since deno. $= 3x^2 = 0$

when x = 0 and the limit is finite so nuem. = 2 cos0° + a cos0° = 0.

$\Rightarrow 2\times1 + a\times1=0$, Þ a = –2

(Again using L - Hospital rule)

Now $L = \lim_{x \to 0} \frac{-4\ \sin 2x - a\ \sin x}{6x}$ ....(i)

Again deno = 6x = 6× 0 = 0 at x =0

and nuem = –4 sin 2× 0° –a sin 0° = –4×0 –a×0 = 0.

So again apply L - Hospital rule in (i), we get

$L= \lim_{x \to 0} \frac{-8\ \cos 2x - a\cos x}{6}$

$\Rightarrow L = \frac{-8\cos 0^\circ - (-2)\cos 0^\circ}{6} = \frac{-8\times1+2\times1}{6} = \frac{-6}{6} = -1$

$\Rightarrow$ L = –1 and a = –2.

Example 8 : Prove that

Sol. Let $l= \underset{x \to \infty}{\text{Lt}} (1+x)^{1/x}$ $\left| \infty^\circ \right.$ form.

$\therefore \log l = \underset{x \to \alpha}{\text{Lt}} \frac{1}{x}\log(1+x)$

$\Rightarrow \log l = \underset{x \to \alpha}{\text{Lt}} \frac{\log(1+x)}{x}$ $\left| \frac{0}{0} \right.$ form

Using L-Hospital rule we get.

$\Rightarrow \log l = \underset{x \to \infty}{\text{Lt}} \left[\frac{\left(\frac{1}{1+x}\right)}{1.x^{1-1}} \right], \Rightarrow \log l = \underset{x \to \infty}{\text{Lt}} \left(\frac{1}{1+x}\right)$

$$\Rightarrow \log l = \frac{1}{1+\propto}, \Rightarrow \log l = \frac{1}{\propto}, \ \Rightarrow \log l = \frac{1}{\propto} = 0$$

Since $\frac{1}{\propto} = 0, \ \therefore \ \log l = 0, \ \Rightarrow l = e^{0} = 1$

Hence value of the given limit is 1

Example : Prove that $\underset{x \to 0}{\text{Lim}} \left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}$

Solution : Let $l = \underset{x \to 0}{\text{Lim}} \left(\frac{\tan x}{x}\right)^{1/x^2}$, then $\log l = \underset{x \to 0}{\text{Lt}} \ \frac{1}{x^2} . \log\left(\frac{\tan x}{x}\right)$

$$\Rightarrow \log l = \underset{x \to 0}{\text{Lt}} \left[\frac{log\left(\frac{tanx}{x}\right)}{x2}\right] \text{ which is in } \frac{0}{0} \text{ form.}$$

$$\therefore \ \log l \ = \underset{x \to 0}{\text{Lt}} \left[\left(\frac{x}{\tan \ x}\right) . \frac{x \sec^2 x - 1. \tan x}{x2} . \frac{1}{2x}\right]$$

$$\Rightarrow \ \log l = \underset{x \to 0}{\text{Lt}} \ (1) . \frac{x \sec^2 x - \tan x}{2x^3} \qquad \left| \frac{0}{0} \text{form}\right.$$

$$\Rightarrow \ \log l = \underset{x \to 0}{\text{Lt}} \ \frac{1. \sec^2 x + 2x \sec^2 x \tan x - \sec^2 x}{6x^2}$$

$$\Rightarrow \ \log l = \underset{x \to 0}{\text{Lt}} \ \frac{2x \sec^2 x \tan x}{6x^2} = \frac{1}{3} \ \underset{x \to 0}{Lt} \sec^2 x \left(\frac{\tan x}{x}\right)$$

$$\Rightarrow \log l = \frac{1}{3}(1)^2.(1), \Rightarrow \log l = \frac{1}{3}$$

Hence. L = $e^{1/3}$. Since Sec 0°=1, $\frac{\tan x}{x} \to 1$, as $x \to 0$

Activity 1 Evaluate the $\left(\frac{\sin x}{x}\right)^{1/x^2}$ as $x \to \infty$.

Activity 2 Find the value of $\underset{x \to 0}{\text{Lt}} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right]$

Activity 3 Find the value of $\underset{x \to 1}{\text{Lt}} \left[\frac{x}{x-1} - \frac{1}{\log x}\right]$

Exercises : Solve the following limits

i) $\underset{x \to \infty}{\text{Lt}} \frac{3x+4}{\sqrt{2x^2+5}}$

ii) $\underset{x \to 0}{\text{Lt}} \left[\frac{a^x - 1 - x\log a}{x^2}\right]$

iii) Find the values of a and b so that

$\underset{x \to 0}{\text{Lt}} \frac{x(1 - a\cos x) + b\sin x}{x^3}$ exists and equals to $\frac{1}{3}$.

iv) $\underset{x \to 0}{\text{Lt}} \frac{3^x - 2^x}{\sqrt{x}}$

v) $\underset{x \to 0}{\text{Lt}} \frac{e^x \sin x - x - x^2}{x^3}$

vi) $\underset{x \to 0}{Lt} \; \dfrac{2 \cos x - 2 + x^2}{\chi^4}$

χ

vii) $\underset{x \to 0}{Lt} \; \dfrac{x + x - \log(1 + x)}{\chi 2}$

viii) $\underset{x \to 0}{Lt} \; \left(\dfrac{x}{x - 1} - \dfrac{1}{\log x} \right)$

ix) $\underset{x \to \infty}{Lt} \; 2x. \sin \dfrac{a}{2x}$

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CALCULUS OF VECTOR VALUED FUNCTIONS

3.1 Introduction

Throughout this unit it is presumed that students are familiar with vector calculus already studied in their previous classes.

3.2 Objectives

To study the differentiation and integration of vectors.

Some Points to Remember

(i) If $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$ then $|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$ is called modulus or magnitude of vector $\vec{a}$.

(ii) Let $\vec{a}$ and $\vec{b}$ be any two vectors then the dot product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}.\vec{b}$ and is defined as

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$,

where θ is the angle between $\vec{a}$ and $\vec{b}$.

Also $\hat{i}.\hat{i} = 1 = \hat{j}.\hat{j} = \hat{k}.\hat{k}$, $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{i}.\hat{k} = 0$

(iii) Cross product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ and is defined as

$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\hat{\theta}\hat{\eta}$,

θ is the angle between $\vec{a}$ and $\vec{b}$, $\hat{\eta}$ is unit vector $\perp$ to both $\vec{a}$

and $\vec{b}$. Also $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$, $\hat{i}\times\hat{j}=\hat{k}, \hat{j}\times\hat{k}=\hat{i}$ and $\hat{k}\, x\, \hat{i}=\hat{j}$,

(iv) Scalar tripple product of vectors $\vec{a},\vec{b},\vec{c}$ is $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \vec{a}\,.\left(\vec{b}\times\vec{c}\right)$

Also $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \left[\vec{c}\ \vec{a}\ \vec{b}\right] = \left[\vec{b}\ \ \vec{c}\ \ \vec{a}\right]$;

$\left[\vec{a}\ \vec{a}\ \vec{b}\right]=0$

(v) Vector tripple product $=\vec{a}\times\left(\vec{b}\times\vec{c}\right)$.

3.3 Differentiation of Vectors Functions of Single Scalar Variable

If to each value of scalar variable 't' in some interval [*a, b*], there corresponds, by any law whatsoever, a value of variable vector $\vec{F}$, then $\vec{F} = \vec{F}(t)$ is a single valued vector function of scalar variable '*t*'.

Decomposition of a Vector Functions

Any vector function $\vec{F}(t)$ can be decomposed as a linear combination of $\hat{i},\hat{j},\hat{k}$, So we write

$$\vec{F}(t)=f_1(t)\hat{i}+f_2(t)\hat{j}+f_3(t)\hat{k}$$

where $f_1(t), f_2(t), f_3(t)$ are scalar function of *t* and are called components of $\vec{F}(t)$.

Example : $\vec{F}(t) = a\cos\ t\ \hat{i}+b\ \sin t\ \hat{j}+0\hat{k}$, which is of the form

$$\vec{F}(t)=f_1(t)\hat{i}+f_2(t)\hat{j}+f_3(t)\hat{k}.$$

Limit of a Vector Function

Definition 3.1 A vector function $\vec{F}(t)$ is said to tend to a vector *l* when *t* tends *c,* if for any $\varepsilon>0$ (however small), there corresponds a positive number δ such that

$$|\vec{F}(t)-l|<\varepsilon \text{ wherever } 0<|t-c|<\delta$$

and we write

$$\lim_{t \to c} \vec{F}(t) = l$$

Continuity

Definition 3.2 A vector function $\vec{F}(t)$ is said to be a continuous at c, if

$$\lim_{t \to c} \vec{F}(t) = \vec{F}(c)$$

Also $\vec{F}(t)$ is said to be a continuous function, if it is continuous for every value of the interval of definition of the function.

Differentiability

Definition 3.3. A vector function $\vec{F}(t)$ is said to be differentiable at $t = c$ if

$$\lim_{h \to c} \frac{F(c + h - f(c))}{h} \text{ exists}$$

and the limit is called derivative of $\vec{F}(t)$ at c and is denoted by $\vec{F}'(t)$.

We also write

$$\vec{F}'(t) = \lim_{t \to c} \frac{\vec{F}(t) - \vec{F}(c)}{t - c}$$

A function is said to be differentiable or derivable if it is derivable for every value of the interval of definition of the function.

Theorem 3.1 $\vec{F}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ is derivative function if and only if $f_1(t)$, $f_2(t)$, $f_3(t)$ are also derivable and

$$\frac{d\vec{F}}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$$

Proof : Since $\dfrac{\vec{F}(t+h) - \vec{F}(t)}{h} = \dfrac{f_1(t+h)\hat{i} + f_2(t+h)\hat{j} + f_3(t+h)\hat{k} - [f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}]}{h}$

$$= \frac{f_1(t+h)-f_1(t)}{h}\hat{i} + \frac{f_2(t+h)-f_2(t)}{h}\hat{j} + \frac{f_3(t+h)-f_1(t)}{h}\hat{k} \qquad ...(1)$$

Taking the limit $h \to 0$ on both side of *eq*. (1), we get

$$\lim_{h\to 0}\frac{\vec{F}(t+h)-\vec{F}(t)}{h} = \lim_{h\to 0}\frac{f_1(t+h)-f_1(t)}{h}\hat{i}$$

$$+ \lim_{h\to 0}\frac{f_2(t+h)-f_2(t)}{h}\hat{j} + \lim_{h\to 0}\frac{f_3(t+h)-f_3(t)}{h}\hat{k}$$

$$\frac{d\vec{F}}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$$

Definition 3.4. If $\vec{r}$ is the position vector of a moving point P then we define velocity $\vec{v}$ and acceleration $\vec{a}$ of moving point P as under

$$\vec{v} = \frac{d\vec{r}}{dt}$$

and $$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}.$$

EXAMPLE 1. If a particle moves along a curve $x=4\cos t$, $y = 4\sin t$, $z = 6t$, then find the velocity and acceleration at $t = 0$. Also find magnitudes of velocity and acceleration at any time t.

Sol. We have $x = 4\cos t$, $y = 4\sin t$, $z = 6t$

Let $\vec{r}$ be the position vector of the particle at any time t, then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

i.e. $$\vec{r} = 4\cos t\ \hat{i} + 4\sin t\ \hat{j} + 6t\hat{k}$$

$\therefore$ Velocity $(\vec{v}) = \dfrac{d\vec{r}}{dt} = -4\sin t\,\hat{i} + 4\cos t\ \hat{j} + 6\hat{k}$

and acceleration $(\vec{a}) = \dfrac{d^2\vec{r}}{dt^2} = -\ 4\cos t\ \hat{i} - 4\sin t\,\hat{j}$

At $t = 0$, we have $\vec{v} = \left.\dfrac{d\vec{r}}{dt}\right|_{t=0} = 4\hat{j} + 6\hat{k}$

and $\left.\vec{a}\right|_{t=0} = \left.\dfrac{d^2\vec{r}}{dt^2}\right|_{t=0} = -\ 4$

their magnitudes,

$$|\vec{v}| = \sqrt{16\sin^2 t + 16\cos^2 t + 36} = \sqrt{16+36} = \sqrt{52} = 2\sqrt{13}$$

and $|\vec{a}| = \sqrt{16\cos^2 t + 16\sin^2 t} = \sqrt{16} = 4$

EXAMPLE 2. Let $\vec{u} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$

and $\vec{v} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$, find $\dfrac{d}{dt}(\vec{u}.\vec{v})$ at $t = 2$.

Sol. Here $\vec{u} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$ and $\vec{v} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$.

$\therefore$ $\vec{u}.\vec{v} = [t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}].[(2t-3)\hat{i} + \hat{j} - t\hat{k}]$

$$= t^2(2t-3) + (-t)(1) + (2t+1)(-t)$$

$$= 2t^3 - 3t^2 - t - 2t^2 - t = 2t^3 - 5t^2 - 2t$$

and hence $\dfrac{d}{dt}(\vec{u}.\vec{v}) = \dfrac{d}{dt}(2t^3 - 5t^2 - 2t) = 6t^2 - 10t - 2$

At $t = 2$, we get

$$\frac{d}{dt}(\vec{u}.\vec{v}) = 6(2)^2 - 10(2) - 2 = 24 - 20 - 2 = 2.$$

THEOREM 3.2 The vector function $\vec{f}(t)$ is of constant magnitude iff

$$\vec{f}(t).\frac{d\vec{f}}{dt} = \vec{0}.$$

PROOF: First we assume that the vector function $\vec{f}(t)$ has a constant magnitude, i.e.,

$$|\vec{f}| = \text{constant}$$

$\Rightarrow \qquad |\vec{f}|^2 = \text{constant}$

$\Rightarrow \qquad \vec{f}.\vec{f} = \text{constant}$

$\Rightarrow \qquad \frac{d}{dt}(\vec{f}.\vec{f}) = 0$

$\Rightarrow \qquad \vec{f}.\frac{d\vec{f}}{dt} + \frac{d\vec{f}}{dt}.\hat{f} = 0$

$\Rightarrow \qquad 2\vec{f}.\frac{d\vec{f}}{dt} = 0 \quad \Rightarrow \quad \hat{f}.\frac{d\vec{f}}{dt} = 0$

Conversely, we assume that

$\vec{f}.\frac{d\vec{f}}{dt} = 0 \quad \Rightarrow \quad 2\vec{f}.\frac{d\vec{f}}{dt} = 0 \quad \Rightarrow \quad \vec{f}.\frac{d\vec{f}}{dt} + \frac{d\vec{f}}{dt}.\vec{f} = 0$

$\Rightarrow \qquad \frac{d}{dt}(\vec{f}.\vec{f}) = 0$

$\Rightarrow \qquad \vec{f}.\vec{f} = \text{constant}$

$\Rightarrow \qquad |\vec{f}|^2 = \text{constant}$

$\Rightarrow \qquad |\vec{f}| = \text{constant}$

i.e. $\qquad \vec{f}(t)$ is of constant magnitude.

Note. Above theorem can also be stated as :

The necessary and sufficient condition for the vector function $\vec{f}(t)$ to have constant magnitude is $\vec{f}.\frac{d\vec{f}}{dt} = 0$.

THEOREM 3.3 The vector function $\vec{f}(t)$ has constant direction if and only if $\hat{f} \times \frac{d\hat{f}}{dt} = 0$.

PROOF: First, let us take $\vec{f} = f\vec{\mathrm{F}}$ where $\vec{\mathrm{F}}$ is a unit vector in the direction of $\vec{f}$.

$$\therefore \qquad \frac{d\vec{f}}{dt} = f\frac{d\vec{F}}{dt} + \frac{df}{dt}\vec{F}$$

and hence $\vec{f} \times \frac{d\hat{f}}{dt} = (f\vec{F}) \times \left(f\frac{d\vec{F}}{dt} + \frac{df}{dt}\vec{F} \right)$

$$= f^2\vec{F} \times \frac{d\vec{F}}{dt} + f\frac{df}{dt}\vec{F} \times \vec{F}$$

$$= f^2\vec{F} \times \frac{d\vec{F}}{dt} + \vec{0}$$

$$\therefore \qquad \vec{f} \times \frac{d\vec{f}}{dt} + \delta^2\vec{F} \times \frac{d\vec{F}}{dt} \qquad \text{...(3.1)}$$

Now we assume that $\vec{f}$ has constant direction

$\Rightarrow \qquad \vec{F}$ is a constant vector

$$\Rightarrow \qquad \frac{d\vec{F}}{dt} = 0$$

so from eq. (3.1) $\vec{f} \times \frac{d\vec{f}}{dt} = \vec{0}$

Conversely, we assume that

$\vec{f} \times \frac{d\vec{f}}{dt} = \vec{0}$ so from (3.1), we have

$$\delta^2\vec{F} \times \frac{d\vec{F}}{dt} = \vec{0} \qquad \Rightarrow \qquad \vec{F} \times \frac{d\vec{F}}{dt} = 0 \qquad \text{...(3.2)}$$

Since $\vec{F}$ is of constant magnitude

$$\therefore \qquad \vec{F} \times \frac{d\vec{F}}{dt} = 0 \qquad \text{...(3.3)}$$

From (3.2) and (3.3), we have

$$\frac{d\vec{F}}{dt} = 0$$

$\therefore$ $\vec{F}$ is a constant vector both in magnitude and direction.

$\Rightarrow$ $\vec{F}$ has a constant direction

EXERCISE 3.1

Q. 1 A particle moving along the curve x = e^t, y = 2 cos 3t, z = 2 sin 3t. Determine the velocity and acceleration at t = 0.

Q. 2. If A = $t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$ and $\vec{B} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$ then find $\frac{d}{dt}(\vec{A} \,.\, \vec{B})$, $\frac{d}{dt}(\vec{A} \,.\, \vec{B})$, at t = 1.

Q. 3. If $\vec{A} = t^2\hat{i} - t\hat{j} + (3t+1)\hat{k}$, $\vec{B} = t\hat{i} + \sin t\hat{j} + \cos t\hat{k}$ then find

(i) $\frac{d}{dt}(\vec{A}.\vec{B})$ (ii) $\frac{d}{dt}(\vec{A}\times\vec{B})$

(iii) $\left|\frac{d\vec{A}}{dt}\right|$ (iv) $\frac{d}{dt}(\vec{A}.\vec{A})$

Q. 4. If $\vec{A} = \vec{a}e^{\alpha t} + \vec{b}e^{+\alpha t}$ then show that $\frac{d^2\vec{A}}{dt^2} - \alpha^2\vec{A} = 0$

Q. 5. If $\hat{r}$ denote unit vector, prove $\hat{r} \times \frac{d\hat{r}}{dt} = \hat{r} \times \frac{d\hat{r}}{dt}$ where $r = \vec{r}\ \hat{r}$

Q. 6. If $\vec{A} = 3\hat{i} - 6t^2\hat{j} + 4t\hat{k}$, then find $\frac{d^2\vec{A}}{dt^2}$.

....

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3.4 Partial Derivatives of Vectors

If $\vec{F}$ is a vector function depending on more than one variables say *x, y, z* we write

$$\vec{F} = \vec{F}(x, y, z);$$

then the paritial derivative of $\vec{F}$ w.r.t *x* is defined as

$$\frac{\partial \vec{F}}{\partial x} = \lim_{\delta \to 0} \frac{\vec{F}(x+\delta x, y, z) - \vec{F}(x, y, z)}{\delta x} \text{ if this limit exists}$$

Similarly $$\frac{\partial \vec{F}}{\partial y} = \lim_{\delta y \to 0} \frac{\vec{F}(x, y+\delta y, z) - \vec{F}(x, y, z)}{\delta y}$$

and $$\frac{\partial \vec{F}}{\partial z} = \lim_{\delta z \to 0} \frac{\vec{F}(x, y, z+\delta z) - \vec{F}(x, y, z)}{\delta z}$$

are the partial derivatives of $\vec{F}$ w.r.to *y* and *z* respectively if these limit exists.

In case

$$\vec{F}(x, y, z) = F_1(x, y, z)\,\hat{u} + F_2(x, y, z)\hat{j} + F_3(x, y, z)\hat{k}$$

then $$\frac{\partial \vec{F}}{\partial x} = \frac{\partial \vec{F}_1}{\partial x}\hat{i} + \frac{\partial \vec{F}_2}{\partial x}\hat{j} = \frac{\partial \vec{F}}{\partial \text{x}}\hat{k}$$

$$\frac{\partial \vec{F}}{\partial y} = \frac{\partial F_1}{\partial z}\hat{i} + \frac{\partial F_2}{\partial z}\hat{j} + \frac{\partial F_3}{\partial y}\hat{k}$$

and $$\frac{\partial \vec{F}}{\partial z} = \frac{\partial F_1}{\partial z}\hat{i} + \frac{\partial F_2}{\partial z}\hat{j} + \frac{\partial F_3}{\partial z}\hat{k}$$

Partial Derivatives of Higher Order

Higher order partial derivatives can be defined as

$$\frac{\partial^2 \vec{F}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{F}}{\partial x}\right), \quad \frac{\partial^2 \vec{F}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{F}}{\partial y}\right),$$

$$\frac{\partial^2 \vec{F}}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \vec{F}}{\partial z}\right), \quad \frac{\partial^2 \vec{F}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{F}}{\partial y}\right),$$

In general, $\dfrac{\partial^2 \vec{F}}{\partial x \partial y} = \dfrac{\partial^2 \vec{F}}{\partial y \partial x}$

Total Differential

Definition 3.5 The total differential of $\vec{F} = \vec{F}(x, y)$ is given by

$$d\vec{F} = \frac{\partial \vec{F}}{\partial x}dx + \frac{\partial \vec{F}}{\partial x}dy$$

If $\vec{A}$ and $\vec{B}$ be two vector functions of variable x and y then

$$\frac{\partial}{\partial x}\left(\vec{A}.\vec{B}\right) = \vec{A}.\frac{\partial \vec{B}}{\partial x} + \frac{\partial \vec{A}}{\partial x}.\vec{B}$$

$$\frac{\partial}{\partial x}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{\partial \vec{B}}{\partial x} + \frac{\partial \vec{A}}{\partial x} \times \vec{B}$$

Limit of Vector function of two variables

Definition 3.6 A vector function $\vec{F}$ (u, v) is said to tend to a limit 'l' when u tend to c and v tend to d, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\vec{F}(u,v) - l \,| < \varepsilon \text{ wherever } 0 < |u - c| \leq \delta, \; 0 < |u - d| \leq \delta$$

Definition 3.7 A vector function $\vec{F}(u, v)$ is said to be continuous for $u = c,\ v = d$ if

$$\lim_{(u,v)\to(c,d)} \vec{F}(u, v) = \vec{F}(c, d)$$

It can be easily shown that

$$\vec{F}(u, v) = f_1(u, v)\hat{i} + f_2(u, v)\hat{j} + f_3(u, v)\hat{k}$$

is continuous if and only if

$f_1(u, v), f_2(u, v), f_3(u, v)$ is continuous.

EXAMPLE 1. If $\vec{A} = xyz\hat{i} + xz^2\hat{j} - y^3\hat{k}$

and $\vec{B} = x^3\hat{i} - xyz\hat{j} + x^2z\hat{k}$

then find the value of $\dfrac{\partial^2\vec{A}}{\partial y^2} \times \dfrac{\partial^2\vec{B}}{\partial x^2}$ at the point (1, 1, 0)

Sol. We have $\dfrac{\partial\vec{A}}{\partial y} = xz\hat{i} - 3y^2\hat{k}$

$$\therefore \quad \frac{\partial^2\vec{A}}{\partial y^2} = -6y\hat{k}$$

Now $$\frac{\partial\vec{B}}{\partial x} = 3x^2\hat{i} - yz\hat{j} + 2xz\hat{k}$$

and $$\frac{\partial^2\vec{B}}{\partial x^2} = 6x\hat{i} + 2z\hat{k}$$

$$\therefore \quad \frac{\partial^2\vec{A}}{\partial y^2} \times \frac{\partial^2\vec{B}}{\partial y^2} = (-\ 6y\hat{k}) \times (6x\hat{i} + 2z\hat{k})$$

$$= (-6y)\ (6x)\ (\hat{k} \times \hat{i}) + (-6y)(2z)(\hat{k} + \hat{k}).$$

$= -\ 36xy\,\hat{j} + 0 = \ 36\hat{j}$ at the point (1, 1, 0)

EXAMPLE 2. If $\vec{A} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$ and $\vec{B} = 2x\hat{i} + y\hat{j} - x^2\hat{k}$

then find $\dfrac{\partial^2}{\partial x \partial y}\ (\vec{A}\times\vec{B})$ at the point.

Sol. Here $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2yz & -zxz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$

$$= i\begin{vmatrix} -2xz^3 & xz^2 \\ y & -x^2 \end{vmatrix} - \hat{j}\begin{vmatrix} x^2yz & xz^2 \\ 2z & -x^2 \end{vmatrix} + \hat{k}\ \begin{vmatrix} x^2yz & -2xz^3 \\ 2z & -y \end{vmatrix}$$

$$= \ \hat{i}(2x^3z^3 - xyz^2) - \hat{j}(-x^4yz - 2xz^3) + \hat{k}\,(x^2y^2z + 4xz^4)$$

or $\vec{A} \times \vec{B} = (2x^3z^3 - xyz^2)\hat{i} + (x^4yz + 2xz^3)\,\hat{j}\ + (x^2y^2z + 4xz^4)\hat{k}$ ...(*)

Differentiating equation (*) partially w.r. to *y,* we get

$$\frac{\partial}{\partial y}(\vec{A} \times \vec{B}) = -xz^2\hat{i} + x^4z\hat{j} + 2x^2yz\hat{k} \qquad \text{...(**)}$$

Again differentiating (**) partially w.r. to *x*, we get

$$\frac{\partial^2}{\partial x \partial y}(\vec{A} \times \vec{B}) = -z^2\hat{i} + 4x^3z\hat{j} + 4xyz\hat{k}$$

$$\therefore \qquad \frac{\partial^2}{\partial x \partial y}(\vec{A} \times \vec{B}) = -4\hat{i} - 8\,\hat{j} + 0\hat{k} = -4\left(\hat{i} + 2\,\hat{j}\right)$$

EXERCISE

Q. 1. Find $\left[\frac{\partial \vec{r}}{\partial u} \ \frac{\partial \vec{r}}{\partial v} \ \frac{\partial^2 \vec{r}}{\partial u^2}\right]$ and $\left[\frac{\partial \vec{r}}{\partial u} \ \frac{\partial \vec{r}}{\partial v} \ \frac{\partial^2 \vec{r}}{\partial u \partial v}\right]$

where $\vec{r} = \frac{1}{2}a(u+v)\hat{i} + \frac{1}{2}b(u-v)\hat{j} + \frac{1}{2}uv\hat{k}$

Q. 2. If $\phi(x, y, z) = xy^2 z$ and $\vec{A} = xz\hat{i} - xy\hat{j} + yz\hat{k}$,

find $\frac{\partial^3(\phi\vec{A})}{\partial x^2 \partial z}$ at (2, – 1, 1).

Q. 3. If $\vec{A} = (2x^2 y - x^4)\hat{i} + (e^{xy} - y\sin x)\hat{j} + x^2 \cos y\hat{k}$

Verify $\frac{\partial^2 \vec{A}}{\partial x \partial y} = \frac{\partial^2 \vec{A}}{\partial y \partial x}$.

Q. 4. If $\vec{F} = xyz\hat{i} + xz^2\hat{j} - y^2\hat{k}$ and $\vec{G} = x^2\hat{i} - xyz\hat{j} + x^2 z\hat{k}$
then show that

$\frac{\partial^2 \vec{F}^2}{\partial y^2} \times \frac{\partial^2 \vec{G}}{\partial x^2} = 0$ at (1, 1, 0) equal to $-4\hat{j}$.

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3.5 DIRECTIONAL DERIVATIVE AND GRADIENT

Directional Derivative (i) If $\phi(x, y, z)$ is a scalar point function, then the directional derivative along the direction of coordinate axes is defined as

$$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \text{ respectively.}$$

(ii) If $\vec{F}(x, y, z) = \vec{F}_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is a vector point function then the directional derivative along the +ve direction of coordinate axes is defined as :

$$\frac{\partial\vec{F}}{\partial x}, \frac{\partial\vec{F}}{\partial y}, \frac{\partial\vec{F}}{\partial z} \text{ respectively,}$$

where $$\frac{\partial\vec{F}}{\partial x} = \frac{\partial\vec{F}_1}{\partial x}\hat{i} + \frac{\partial\vec{F}_2}{\partial z}\hat{j} + \frac{\partial\vec{F}_3}{\partial x}\hat{k}$$

The vector operator ∇ (del) is defined as :

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\hat{l} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

Definition 3.5 (The Gradient of a scalar function)

Let $\phi(x, y, z)$ be a scalar point function defined and differentiable at each point (*x, y, z*). Then the gradient of ϕ is denoted by $\nabla\phi$ or grad ϕ and is defined as :

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial y}\hat{k}\right)\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}.$$

EXAMPLE. If $f = 3x^2y - y^3z^2$, then grad f at the point (1, – 2, – 1)

Sol. We have $f = 3x^2y - y^3z^2$

$\therefore$ $\frac{\partial f}{\partial x} = 6xy$; $\frac{\partial f}{\partial y} = 3x^2 - 3y^2z^2$ and $\frac{\partial f}{\partial z} = -2y^3z$

$\therefore$ grad $f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = 6xy\hat{i} + (3x^2 - 3y^2z^2)\hat{j} -$

$2y^3z\,\hat{k}$

and hence grad $f\big|_r$ at the point (1, – 2, – 1) is

grad $f = -12\hat{i} - 9\hat{j} - 16\hat{k}$.

THEOREM 3.5 If ϕ and ψ are scalar function of *x, y, z,* then

(i) grad $(\phi \pm \psi)$ = grad $\phi \pm$ grad ψ

(ii) grad $(\phi\psi)$ = ϕ grad ψ + ψ grad ϕ

(iii) grad $\left(\frac{\phi}{\psi}\right)$ = $\frac{\psi \text{ grad } \phi \text{ grad } \psi}{\psi^2}, \psi \neq 0$

Proof : (i) grad $(\phi+\psi)$ = $\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(\phi+\psi)$

$$= \hat{i}\frac{\partial}{\partial x}(\phi \pm \psi) + \hat{j}\frac{\partial}{\partial y}(\phi \pm \psi) + \hat{k}\frac{\partial}{\partial z}(\phi \pm \psi)$$

$$= \hat{i}\left[\frac{\partial \phi}{\partial x} \pm \frac{\partial \psi}{\partial x}\right] + \hat{j}\left[\frac{\partial \phi}{\partial y} \pm \frac{\partial \psi}{\partial y}\right] + \hat{k}\left[\frac{\partial \phi}{\partial z} \pm \frac{\partial \psi}{\partial z}\right]$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} \pm \hat{j}\frac{\partial\phi}{\partial y} \pm \hat{k}\frac{\partial\phi}{\partial z}\right) + \left(\hat{i}\frac{\partial\psi}{\partial x} \pm \hat{j}\frac{\partial\psi}{\partial y} \pm \hat{k}\frac{\partial\psi}{\partial z}\right)$$

$$= \text{grad } \phi \pm \text{grad } \psi .$$

(ii) and (iii) Exercise for the students.

EXAMPLE $\nabla\phi$ i.e, grad ϕ where ϕ is given below :

$$\phi = r^n = (x^2 + y^2 + z^2)^{n/2}$$

Sol. Here $\phi = r^n = (x^2 + y^2 + z^2)^{n/2}$

$$\therefore \quad \frac{\partial \phi}{\partial x} = \frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1}$$

$$\frac{\partial}{\partial x}(x^2 + y^2 + z^2) = \frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n-2}{2}} \times 2x$$

$$= nr^{n-2}x$$

Similarly $\frac{\partial \phi}{\partial y} = nr^{n-2}y$, $\frac{\partial \phi}{\partial z} = nr^{n-2}z$

$$\therefore \quad \text{grad } \phi = \sum_{\hat{i}} \hat{i}\frac{\partial \phi}{\partial x} = nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k}) = nr^{n-2}.\vec{r}.$$

EXAMPlE Find the directional derviative at (1, 2, – 1) of

$f(x, y, z) = 2x^3y - 3y^2z$

in the direction of the vector $2\vec{e}_1 - 3\vec{e}_2 + 6\vec{e}_3$ *i.e.*, $2\hat{i} - 3\hat{j} + 6\hat{k}$.

Sol. Here $f(x, y, z) = 2x^3y - 3y^2z$

$$\therefore \quad \text{grad } f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}(6x^2y) + \hat{j}(2x^3 - 6yz) + \hat{k}(-3y^2)$$

$$= 12\hat{i} + 14\hat{j} - 12\hat{k} \text{ at } (1,\ 2,\ -1)$$

$\therefore$ Directional derivative = $\hat{a}$ grad f

where $\hat{a} = \frac{\hat{a}}{|\vec{a}|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{49}}$

$$= \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$$

Hence the required directional derivative

$$= \frac{1}{7}(2\hat{i} - 3\hat{j} + 6\hat{k}).2(6\hat{i} + 7\hat{j} - 6\hat{k})$$

$$= \frac{2}{7}(12 - 21 - 36) = \frac{2}{7} \times -45 = -\frac{90}{7}$$

EXERCISES

Q. 1. If $\vec{F} = \left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y} \right)\hat{i} + \left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z} \right) + \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} \right)\hat{k}$ prove that

(i) $\vec{F} = \vec{r} \times \nabla f$ (ii) : = 0 (iii) $\vec{F}.\nabla f = 0$

Q. 2. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then prove that

(i) $\vec{a}.\nabla\vec{r} = 3\vec{a}$ (ii) grad $(\vec{a}.\vec{r}) = \vec{a}$

Q. 3. Find the directional derviative of $\phi = xyz$ at (1, 2, 3) in the direction of $\hat{i}$.

Q. 4. Find the maximum value of directional derivative of $\phi = 2x^2 + 3y^2 + 5z^2$ at the point (1, 1, – 4).

Q. 5. In what direction from the point (2, 1, – 1) is directional derivative of $\phi = x^2yz^3$ maximum and what is its magnitude ?

DIVERGENCE AND CURL OF A VECTOR

Definition (The Divergence). If $\vec{F}$ is a given point vector function which is continuously differentiable, then the divergence of $\vec{F}$ is written as $\nabla.\vec{F}$ or div $\vec{F}$, defined by

$$\nabla.\vec{F} = \text{div } \vec{F} = \hat{i}\frac{\partial \vec{F}}{\partial x} + \hat{j}\frac{\partial \vec{F}}{\partial y} + \hat{k}\frac{\partial \vec{F}}{\partial z} = \sum_{\hat{i}} \frac{\partial \vec{F}}{\partial x}$$

In the above summation $\hat{i}$ is to be replaced by $\hat{j}$ and $\hat{k}$ and x to be replaced by y and z respectively.

Note that if div $\vec{F} = 0$ then $\vec{F}$ is called solenoidal vector.

Definition (The Curl of a Vector) If $\vec{F}$ is a given vector point function which continuously differentiable, then the Curl of $\vec{F}$ is written as Curl $\vec{F}$ or $\nabla \times \vec{F}$ and is defined by

$$\text{Curl } \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{F} = \sum_{\hat{i}} \hat{i} \times \frac{\partial \vec{F}}{\partial x}$$

Now that if Curl $\vec{F} = 0$ then $\vec{F}$ is irrotational vector.

In short we define the divergence and Curl of

$$\vec{F} = \vec{F_1}\hat{i} + \vec{F}_2\hat{j} + \vec{F}_3\hat{k} \text{ as under}$$

$$\text{div}\vec{F} = \nabla.\vec{F} = \sum \hat{i}\frac{\partial \vec{F}}{\partial x} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{Curl } \vec{F} = \nabla.\vec{F} = \sum \hat{i}\frac{\partial \vec{F}}{\partial x} = \begin{vmatrix} i & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

The Laplacian Operator ∇^2

The Laplacian operator is denoted by ∇^2 and defined as :

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

EXAMPLE 1. Prove that div (grad ϕ) = Ñ$^2 f$

Sol : div (grad$f$) $= \nabla.(\nabla \phi) = \sum_{\hat{i}} i \frac{\partial}{\partial x}(\nabla \phi)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right)(\hat{i}.\hat{i}) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right)(\hat{j}.\hat{j}) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)(\hat{k}.\hat{k})$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi \quad \because \quad \hat{i}.j = j.k = k.i = 0$$

and $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

EXAMPLE 2. Show that Curl (Curl $\vec{F}$) = grad div $\vec{F} - \nabla^2 \vec{F}$ proof is left for the students.

EXAMPLE 3. Find the Curl of the vector function

$\vec{F} = y(x+z)\hat{i} + z(x+y)\hat{j} + x(y+z)\hat{k}$ and hence find the value of Curl (Curl $\vec{F}$)

Sol.

$$\text{Curl F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y(x+z) & z(x+y) & x(y+z) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}x(y+z)-\frac{\partial}{\partial z}z(x+y)\right]-\hat{j}\left[\frac{\partial}{\partial x}x(y+z)-\frac{\partial}{\partial z}y(x+z)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}z(x+y)-\frac{\partial}{\partial y}y(x+z)\right]$$

$$=\hat{i}(x-x-y)-\hat{j}(y+z-y)+\hat{k}(z-x-z)$$

$$\text{Curl } \vec{F} = -iy-\hat{j}z-\hat{k}x=-y\hat{i}-z\hat{j}-x\hat{k}$$

$$\therefore \text{ Curl (Curl } \vec{F}\text{)} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & -z & -x \end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}(-x)-\frac{\partial}{\partial x}(-z)\right]-\hat{j}\left[\frac{\partial}{\partial x}(-x)-\frac{\partial}{\partial z}(-y)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(-z)-\frac{\partial}{\partial y}(-y)\right]$$

$$=\hat{i}+\hat{j}+\hat{k}$$

EXAMPLE If $\phi = x^3+y^3+z^2-3xyz$, then find div grad ϕ, Curl grad ϕ.

Sol. grad $\phi = \nabla\phi$

$\therefore$ div (grad ϕ) = $\nabla.(\nabla\phi) = (\nabla^2\phi) = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$

$$=\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}+\left(\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}+\left(\frac{\partial\phi}{\partial z}\right)$$

$$=\frac{\partial}{\partial x}(3x^2-3yz)+\frac{\partial}{\partial y}(3y^2-3xz)+\frac{\partial}{\partial z}+(3z^2-3xy)$$

$$= 6x + 6y + 6z$$

Now Curl (grad ϕ) $= \nabla \times \nabla \phi = 0$.

EXERCISES

Q. 1. If $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$, then prove that Curl (Curl $\vec{F}$) = 0

Q. 2. If $\vec{F} = xy^2\hat{i} - 2y^2z^2\hat{j} + xyz^2\hat{k}$, then find div $\vec{F}$ at (1, – 1, 1).

Q. 3. If $\vec{A} = x^2\hat{i} + y\hat{j} + z^2\hat{k}$, $\vec{B} = yz\hat{i} + xz\hat{j} + xy\hat{k}$

then find $\frac{\partial^2}{\partial x \partial y}(\vec{A} \times \vec{B}$ at (1, – 1, 0)

Q. 4. If $\vec{v} = e^{-\lambda x}(c_1 \cos \lambda y + c_2 \sin \lambda y)$ where c_1 and c_2 are constant vectors, then find

$$\frac{\partial^2 \vec{v}}{\partial x^2} \times \frac{\partial^2 \vec{v}}{\partial y^2}.$$

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Line Integrals

$$\int_{p_1}^{p_2} \vec{r}.d\vec{r} = \int_c \vec{r}.d\vec{r} = \int_c (F_1 dx + F_2 dy + F_3 dz)$$

is called a line integral

If C is a closed curve, then the integral around C is denoted by

$$\oint_c \vec{F}.d\vec{r} = \oint_c (F_1 dx + F_2 dy + F_3 dz)$$

it represends the work done by the force $\vec{F}$.

In general, any integral which is to be evaluated along a curve is called a line integral.

EXERCISE 1. Find the total work done in moving a particle in a force field is given by

$$\vec{F} = = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$$

along the curve $x = t^2 + 1$, $y = 2t^2$, $z=t^3$ from t=1 to t= 2

Sol. Total work done = $\int_c \vec{F}.d\vec{r} = \int_c 3xy\ dx - 5z\ dy + 10xdz.$

$$= \int_{t=1}^{2} 3(t^2+1)2t^2 d(t^2+1) - 5t^3 d(2t^2) + 10(t^2+1)d(t^3)$$

$$= \int_1^2 (6t^4 + 6t^2)2t\ dt - 5t^3.4t\ dt + 10(t^2+1)3t^2 dt$$

$$= \int_1^2 \left[(12t^5 + 12t^3) - 20t^4 + 30t^4 + 30t^2\right]dt$$

$$= \int_1^2 \left[(12t^5 + 10t^4 + 12t^3 + 30t^2\right]dt$$

$$= \left|\frac{12t^6}{6} + \frac{10t^5}{5} + \frac{12t^4}{4} + \frac{30t^3}{3}\right|_1^2$$

$$= \left|2t^6 + 2t^5 + 3t^4 + 10t^3\right|_1^2$$

$$= 2(2^6 - 1^6) + 2(2^5 - 1^5) + 3(2^4 - 1^4) + 10(2^3 - 1^3)$$

$$= 2\ (63) + 2\ (31) + 3\ (15) + 10\ (7)$$

$= 126 + 62 + 45 + 70 = 303$ **Ans.**

EXERCISE 2. If $\phi = 2xyz^2, \mathrm{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and c is the curve $x = t^2, y = 2t, z = t^3$ from t = 0 to t = 1, then evaluate the following line integrals :

(a) $\int_c \phi d\vec{r}$ (b) $\int_c \vec{\mathrm{F}} \times d\vec{r}$

Sol. (a) Along C, $\phi = 2xyz^2 = 2t^2(2t)(t^3)^2 = 4t^9$,

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$

and $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k} = (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})dt$, then

$$\int_c \phi d\vec{r} = \int_{t=0}^1 4t^9(2t\ \hat{i} + 2\hat{j} + 3t^2\hat{k})dt = \int_0^1 (8t^{10}\hat{i} + 8t^9\hat{j} + 12t^{11}\hat{k})dt$$

$$= \left.\frac{8t^{11}}{11}\hat{i} + \frac{8t^{10}}{10}\hat{j} + \frac{12t^{12}}{12}\hat{k}\right|_0^1 = \frac{8}{11}\hat{i} + \frac{8}{10}\hat{j} + \hat{k}$$

(b) Along 'C', $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k} = 2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}$

then $\vec{F} \times d\hat{r} = (2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}) \times (2t\hat{j} + 2\hat{j} + 3t^2\hat{k})dt$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} -t^3 & t^4 \\ 2 & 3t^2 \end{vmatrix} - \hat{j} \begin{vmatrix} 2t^3 & t^4 \\ 2t & 3t^2 \end{vmatrix} + \hat{k} \begin{vmatrix} 2t^3 & -t^3 \\ 2t & 2 \end{vmatrix}$$

$$= (-3t^5 - 2t^4)\hat{i} - \hat{j}(6t^5 - 2t^5) + \hat{k}(4t^3 + 2t^4)$$

$$= (-3t^5 - 2t^4)\hat{i} - 4t^5\hat{j} + (4t^3 + 2t^4)\hat{k}$$

$\therefore$ $$\int \vec{F} \times d\vec{r} = \hat{i}\int_0^1 (-3t^5 - 2t^4)dt + \hat{j}\int_0^1 (4t^5)dt + k\int_0^1 (4t^3 + 2t^4)dt$$

$$= \hat{i}\left[-\frac{3t^6}{6} - \frac{2t^5}{5}\right]_0^1 + \hat{j}\left|-\frac{4t^6}{6}\right|_0^1 + \hat{k}\left|\frac{4t^4}{4} + \frac{2t^5}{5}\right|_0^1$$

$$= \hat{i}\left[-\frac{1}{2} - \frac{2}{5}\right] + \hat{j}\left[-\frac{2}{3}\right] + \hat{k}\left[1 + \frac{2}{5}\right]$$

$$= -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k} \text{ Ans.}$$

EXERCISE 3. If $\vec{A}(t) = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ and $B(t) = 2t^2\hat{i} + 6t\hat{k}$ then find values of

(a) $\int_0^2 \vec{A}.\vec{B}\, dt$ (b) $\int_0^2 \vec{A} \times \vec{B}\, dt$

Sol. (a) $\vec{A}.\vec{B} = [t\hat{i} - t^2\hat{j} + (t-1)\hat{k}].\,[2t^2\hat{i} + 6t\hat{k}]$

$$= (t)\ (2t^2) - t^2.(0) + (t-1)\ (6t)$$

$$= 2t^3 + 6t^2 - 6t$$

$$\therefore \quad \int_0^2 \vec{A}.\vec{B}\ dt = \int_0^2 (2t^3 + 6t^2 - 6t)dt = \left| \frac{2t^4}{4} + \frac{6t^3}{3} - \frac{6t^2}{2} \right|_0^2$$

$$= \frac{1}{2}t^4 + 2t^3 - 3t^2 \Big|_0^2 = \frac{1}{2}[2^4 - 0] + 2[2^3 - 0] - 3[2^2 - 0]$$

$$= \frac{1}{2} \times 2^4 + 2^4 - 3 \times 4$$

= 8 + 16 – 12 = 12.

(b) $$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ t & -t^2 & t-1 \\ 2t^2 & 0 & 6t \end{vmatrix} = \hat{i}(-6t^3 - 0) - \hat{j}(6t^2 - 2t^3 + 2t^2) + 2t^4 k$$

$$\therefore \quad \int_0^2 \vec{A} \times \vec{B} dt = \hat{i}\int_0^2 -6t^3 dt - \hat{j}\int (8t^2 - 2t^3)dt + k^n \int 2t^4 dt$$

$$= -\frac{6t^4}{4}\Big|_0^2 - \hat{j}\left| \frac{-2t^4}{4} + \frac{8t^3}{3} \right|_0^2 + \hat{k}\left| \frac{2t^5}{5} \right|_0^2$$

$$= -3 \times 8\hat{i} - \hat{j}\left(-8 + \frac{64}{3}\right) + \hat{k}\frac{64}{5}$$

$$= -24\hat{i} - \frac{40}{3}\hat{j} + \frac{64}{5}\hat{k} \text{ Ans.}$$

EXERCISES

Q. 1. Evaluate $\oint_c \vec{F}.d\vec{r}$, where $\vec{F} = x_2x_3\vec{e}_1 + x_3x_1e_2 + x_1x_2\vec{e}_3$ and C is the curve $\vec{r} = \vec{e}_1 t + \vec{e}_2 t^2 + e_3 t^3$ varying from $t = -1$ to $t = 1$.

Q. 2. Evaluate $\int \vec{F}.d\vec{r}$ where $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ in the arc of the curve $\vec{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$ from $t = 0$ to $t = 2\pi$.

Q. 3. If $\vec{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$ then evaluate $\int \vec{F}.d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the following paths.

$x = t,\ y = t^2,\ z = t^3.$

Q. 4. Line integral $\int^{p_2} \vec{F}.\vec{dr}$ is independent of the path joining any two points P_1 and P_2 in a given region if and only if

$\oint_c \vec{F}.d\vec{r} = 0$ for all closed paths in the region.

Q. 5. Find work done in moving a particle in the force field

$\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along

(a) The line joining (0, 0, 0) to (2, 1, 3)

(b) The curve $x = 2t^2$, $y = t$, $z = 4t^2 - t$ from $t = 0$ to $t = 1$.

Q. 6. Find $\int_c \vec{A}.\vec{dr}$ where $\int_c \vec{A} = x^2\hat{i} - xy\,\hat{j}$ from point (1, 1) to (9, 3) to parabola $y^2 = x$.

....

Mohammad Rasul Choudhary

Rajouri

POLAR CO-ORDINATES

(1) Introduction

(2) Relationship between Cartesian & Polar Co-ordinates

- (i) Working Rule
- (ii) Solved Examples
- (iii) Exercise

(3) Angle between Radius Vector and Tangent

- (i) Solved Examples
- (ii) Angle of intersection between two curves
- (iii) Solved Examples
- (iv) Exercise

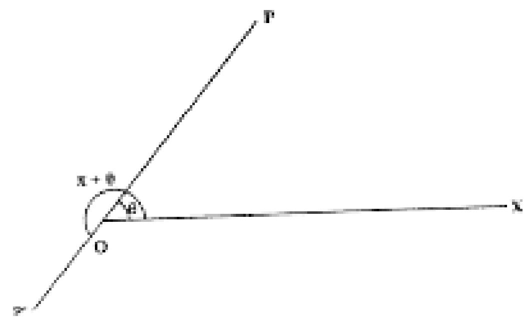
INTRODUCTION

The students are familiar with Cartesian System of co-ordinates. Besides this system, the position of a point P in a plane can be indicated by (i) its distance γ from a fixed point O and (ii) the inclination θ of the line OP with a fixed line OX in the plane. γ and θ are called the Polar co-ordinates of P. Here

(a) O is called pole

(b) OX is called initial line

(c) r is called radius vector



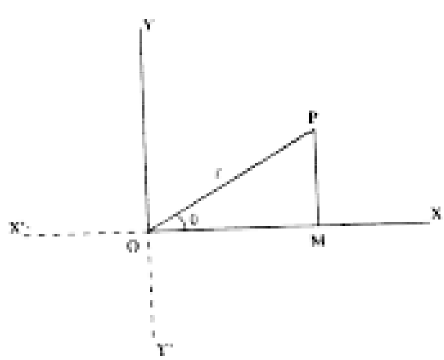
(d) θ is the vectorial angle of P

r is considered to be +ve if it is measured from the pole O along the line bounding the vectorial angle and negative when measured in the opposite direction. And θ is considered to be +ve if it is measured in the anti-clockwise sense.

Remark :- If r and θ are given then there is only one point whose co-ordinates are (r, θ). But if the point P be given, its co-ordinate may be $(\pm r, \theta)$, $(\pm r, \theta + 2\pi)$, $(\pm r, \theta \pm 4\pi)$ .... etc.

RELATIONSHIP BETWEEN CARTESIAN AND POLAR CO-ORDINATES

Let P (x, y) be the Cartesian Co-ordinates of the point P in a plane and P(r, θ) be its polar co-ordinates. Join OP. Draw PM ⊥ X'OX. Then OMP is a rt. triangle.



$$\frac{OM}{OP} = \cos\theta \text{ and } \frac{PM}{OP} = \sin\theta$$

$\Rightarrow$ $OM = OP \cos\theta$ $\qquad$ $PM = OP \sin\theta$

$\Rightarrow$ $x = r\cos\theta$ .... (1) and $y = r\sin\theta$ ...... (2)

With the help of relation (1) and (2) the Polar Co-ordinates are transformed into Cartesians and vice versa.

Working rule :

(i) To change a cartesian equations to Polar equations,

put $x = r\cos\theta$ and $y = r\sin\theta$ and simplify the result.

(ii) To change a Polar equation into caretsian equation,

put $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$ and clear the fractions after putting $r = \sqrt{x^2 + y^2}$ where required.

Example 1. Transform the following equations to polar co-ordinates :

(i) $x^2 + y^2 = 36$ $\qquad$ (ii) $x^2 - y^2 = 9$

(iii) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ $\qquad$ (iv) $x^2 + y^2 = 2ax$

(v) $x^2 - y^2 = 2ay$ $\qquad$ (vi) $x^3 = y^2(2a - x)$

Solution : (i) Put $x = r\cos\theta$ and $y = r\sin\theta$

Then $x^2 + y^2 = 36$ becomes

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 36$$

$$\Rightarrow \quad r^2(\cos^2\theta + \sin^2\theta) = 36$$

$$\Rightarrow \quad r^2.1 = 36$$

$$\Rightarrow \qquad \boxed{\mathbf{r^2 = 36}}$$

(ii) Put $x = r\cos\theta$ and $y = r\sin\theta$

Then given equation $x^2 - y^2 = 9$ becomes

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 9$$

or $$r^2(\cos^2\theta - \sin^2\theta) = 9$$

$$\boxed{\mathbf{r^2 .\cos 2\theta = 9}}$$

(iii) Put $x = r\cos\theta$ and $y = r\sin\theta$

then given equation $(x^2+y^2)^2 = a^2(x^2-y^2)$ becomes

$$[(r\cos\theta)^2+(r\sin\theta)^2]^2 = a^2[(r\cos\theta)^2-(r\sin\theta)^2]$$

$$\Rightarrow \quad [(r^2\cos^2\theta + r^2\sin^2\theta] = a^2[r^2\cos^2\theta - r^2\sin^2\theta]$$

$$\Rightarrow \quad r^4[\cos^2\theta + \sin^2\theta] = a^2.r^2[\cos^2\theta - \sin^2\theta]$$

$$\Rightarrow \quad r^2.(1)^2 = a^2.\cos 2\theta$$

$$\Rightarrow \quad \boxed{\mathbf{r^2 = a^2\cos 2\theta}}$$

(iv) Put $x = r\cos\theta$ and $y = r\sin\theta$

then given equation $x^2 + y^2 = 2ax$ becomes

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 2a.\ r\cos\theta$$

$$r^2[\cos^2\theta + \sin^2\theta] = 2ar.\cos\theta$$

$$r^2.1 = 2ar\cos\theta$$

$$\Rightarrow \quad \boxed{\mathbf{r = 2a\cos\theta}}$$

(v) Put $x = r\cos\theta$ and $y = r\sin$

then given equation $x^2-y^2 = 2ay$ becomes

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 2a.r\sin\theta$$

$$\Rightarrow \quad r^2[\cos^2\theta - \sin^2\theta] = 2ar\sin\theta$$

$$\Rightarrow \quad \boxed{\mathbf{r \cos 2\theta = 2a \sin \theta}}$$

(vi) Put $x = r\cos\theta$ and $y = r\sin\theta$

then given equation $x^3 = y^2 (2a - x)$ becomes

$$(r\cos\theta)^3 = (r\sin\theta)^2 (2a - r\cos\theta)$$

$$\Rightarrow \quad r^3 \cos^3\theta = r^2 \cdot \sin^2\theta\,(2a - r\cos\theta)$$

$$\Rightarrow \quad r\frac{\cos^3\theta}{\sin^3\theta} = 2a - r\cos\theta$$

$$\Rightarrow \quad r\frac{\cos^3\theta}{\sin^2\theta} + r\cos\theta = 2a$$

$$\Rightarrow \quad r\cos^3\theta + r\cos\theta \,.\, \sin^2\theta = 2a\sin^2\theta$$

$$\Rightarrow \quad r\cos\theta\,(\cos^2\theta + \sin^2\theta) = 2a\sin^2\theta$$

$$\Rightarrow \quad r\cos\theta \,.\, 1 = 2a\sin^2\theta$$

$$\Rightarrow \quad \boxed{\mathbf{r \cos\theta = 2a \sin^2\theta}}$$

Example 2. Transform the following Polar curves in cartesian form

(i) $r = a\cos\theta$ (ii) $r = \sin\theta + \cos\theta$

(iii) $\theta = \tan^{-1}(m)$ (iv) $r^2\cos 2\theta = a^2$

(v) $r^2 \sin 2\theta = 2k$ (vi) $\sqrt{r}\cos\frac{\theta}{2} = \sqrt{a}$

Sol. : (i) Put $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$ and $r = \sqrt{x^2 + y^2}$

then given equation $r = a\cos\theta$ becomes

$$r = a.\frac{x}{r}$$

$$\Rightarrow \quad r^2 = ax$$

$$\Rightarrow \quad \left(\sqrt{x^2+y^2}\right)^2 = ax$$

$$\Rightarrow \quad \boxed{\mathbf{x^2 + y^2 = ax}}$$

(ii) Put $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$ and $r = \sqrt{x^2+y^2}$

then given equation $r = \sin\theta + \cos\theta$ becomes

$$\sqrt{x^2+y^2} = \frac{y}{r} + \frac{x}{r}$$

$$\Rightarrow \quad \sqrt{x^2+y^2} = \frac{y+x}{r}$$

$$\Rightarrow \quad \sqrt{x^2+y^2} = \frac{x+y}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \quad \boxed{\mathbf{x^2 + y^2 = x + y}}$$

(iii) $\theta = \tan^{-1}(m)$ $\Rightarrow \tan\theta = m$

$$\Rightarrow \quad \frac{\sin\theta}{\cos\theta} = m$$

$$\Rightarrow \quad \sin\theta = m\cos\theta \qquad (*)$$

Now put $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$

then (*) becomes

$$\frac{y}{r} = m.\frac{x}{r}$$

$$\Rightarrow \qquad \boxed{\mathbf{y = mx}}$$

(iv) $r^2 \cos 2\theta = a^2$ becomes

$$r^2 [\cos^2\theta - \sin^2\theta] = a^2$$

$$\Rightarrow \quad r^2\cos^2\theta - r^2\sin^2\theta = a^2$$

$$\Rightarrow \quad (r\cos\theta)^2 - (r\sin\theta)^2 = a^2$$

Now put $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$

Then we get $\left(r.\frac{x}{r}\right)^2 - \left(r.\frac{y}{r}\right)^2 = a^2$

$$\Rightarrow \qquad \boxed{\mathbf{x^2 - y^2 = a^2}}$$

(v) $r^2 \sin 2\theta = 2k$ becomes

$$r^2 . 2\sin\theta\cos\theta = 2k$$

$$\Rightarrow \quad r^2\sin\theta\cos\theta = k$$

Now put $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$

Then we get

$$r^2.\frac{y}{r}.\frac{x}{r} = k$$

$\Rightarrow$ $$\boxed{\mathbf{xy = k}}$$

(vi) $\sqrt{r}\ \cos\ \frac{\theta}{2} = \sqrt{a}$

sq. on both sides, we get

$$r \cos^2 \frac{\theta}{2} = a$$

$\Rightarrow$ $r\left[\frac{1+\cos 2.\theta/2}{2}\right] = a$

$\Rightarrow$ $r\ [1 + \cos\ \theta] = 2a$

$\Rightarrow$ $r + r \cos\ \theta = 2a$

$\Rightarrow$ $r \cos\ \theta = 2a - r$

Now put $\cos\ \theta = \frac{x}{r}$

Then we get

$$r.\frac{x}{r} = 2a - r$$

$\Rightarrow$ $x = 2a - r$

$\Rightarrow$ $r = 2a - x$

$\Rightarrow$ $\sqrt{x^2 + y^2} = 2a - x$

$$\left(\because r = \sqrt{x^2 + y^2}\right)$$

sq. on both sides we get

$$x^2 + y^2 = 4a^2 - 4ax + x^2$$

$$\Rightarrow \quad y^2 = 4a^2 - 4ax$$

$$\Rightarrow \quad \boxed{\mathbf{y^2 = 4a(a - x)}}$$

ASSIGNMENT QUESTIONS

Q 1. Transform the following Cartesian curves into Polar co-cordinates :

(a) $x^2 + y^2 = x + y$

(b) $x^{2/3} + y^{2/3} = a^{2/3}$

(c) $x^2 - y^2 = (x + y)^2$

Q 2. Transform the following Polar curves into Cartesian co-ordinates :

(a) $r = a(1 + \cos\theta)$

(b) $r^2 = a^2 \cos 2\theta$

(c) $r^2 = a^2 \sin 2\theta$

(d) $\frac{2a}{r} = 1 - \cos\theta$

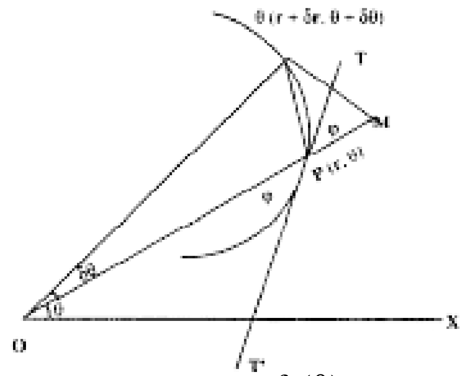
ANGLE BETWEEN RADIUS VECTOR AND TANGENT

Find the angle between radius vector and tangent at any point of a given curve.

OR

Prove that $\tan\varphi = r\frac{d\theta}{dr}$; where the symbols have their usual meaning.

Proof. Let P (r, θ) and Q (r + δr, θ + δθ) be any two neighbouring points on the curve.



$r = f(\theta)$

Let TPT' be the Tangent to the given curve at P. Let φ be the angle between radius vector and Tangent. It is required to find φ.

Draw QM perpendicular to OP (produced if necessary). It is clear that when Q → P δθ → 0, the secant PQ → the Tangent PT and ∠QPM → φ.

$$\therefore \quad \varphi = \underset{\delta\theta \to 0}{\text{lt}} \angle QPM$$

$$\text{or} \quad \tan\varphi = \underset{\delta\theta \to 0}{\text{lt}} \tan \angle QPM \qquad (*)$$

Now in rt. ΔOMQ

$$\frac{OM}{OQ} = \cos\delta\theta$$

$$\Rightarrow \quad OM = OQ \cos\delta\theta$$

$$\Rightarrow \quad OM = (r + \delta r)\cos\delta\theta$$

$$\& \qquad \frac{OM}{OQ} = \sin \delta\theta$$

$$\Rightarrow \qquad QM = OQ \sin \delta\theta$$

$$\Rightarrow \qquad QM = (r + \delta\theta) \sin \delta\theta$$

$$\text{So} \qquad PM = OM - OP$$

$$= (r + \delta r) \cos \delta\theta - r$$

$$= \delta r \cos \delta\theta - r + r \cos \delta\theta$$

$$= \delta r \cos \delta\theta - r (1 - \cos \delta\theta)$$

$$= \delta r \cos \delta\theta - 2 r \sin^2 \frac{\delta\theta}{2}$$

This (*) becomes

$$\tan \varphi = \underset{\delta\theta \to 0}{lt} \frac{QM}{PM}$$

$$= \underset{\delta\theta \to 0}{lt} \frac{(r+\delta r)\sin \delta\theta}{\delta r\cos \delta\theta - 2r \sin^2 \delta\theta/2}$$

$$= \underset{\delta\theta \to 0}{lt} \frac{(r+\delta r)\dfrac{\sin \delta\theta}{\delta\theta}}{\dfrac{\delta r}{\delta\theta}\cos \delta\theta - r \dfrac{\sin \delta\theta/2}{\delta\theta/2}\sin\delta\theta/2}$$

$$= \frac{(r+0).1}{\dfrac{dr}{d\theta}.1 - r.1.0} \left(\because \underset{\delta\theta \to 0}{lt} \frac{\sin\delta\theta}{\delta\theta} = 1 \;\&\; \underset{\delta\theta \to 0}{lt} \cos\delta\theta = 1\right)$$

$$= \frac{r}{\frac{dr}{d\theta}}$$

$$= r\frac{d\theta}{dr}$$

Thus $\boxed{\tan\varphi = r\frac{d\theta}{dr}}$

Note : If we suppose φ to be less than π numerically and define it to be the angle between the positive direction of the radius vector and that direction of the tangent in which θ increases, then it is clear that above formula holds true whether $r\frac{d\theta}{dr}$ be +ve or –ve.

If $r\frac{d\theta}{dr}$ is –ve, it implies that the value of φ is greater than $\frac{\pi}{2}$.

Example 1. Show that in the equiangular spiral $r = ae^{\theta\cot\alpha}$, the angle between radius vector and tangent is constant.

Sol. : Given equation of the curve is

$$r = ae^{\theta\cot\alpha}$$

Diff. w.r. to θ, we get

$$\frac{d\theta}{dr} = ae^{\theta\cot\alpha}\frac{d}{d\theta}(\theta\cot\alpha)$$

$$= ae^{\theta\cot\alpha}.\cot\alpha$$

$$= a \cot \alpha . e^{\theta \cot \alpha}$$

$$\therefore \qquad \frac{d\theta}{dr} = \frac{1}{a \cot \alpha . e^{\theta \cot \alpha}}$$

$$\Rightarrow \qquad r\frac{d\theta}{dr} = \frac{r}{a \cot \alpha . e^{\theta \cot \alpha}}$$

$$= \frac{ae^{\theta \cot \alpha}}{a \cot \alpha . e^{\theta \cot \alpha}}$$

$$\Rightarrow \qquad \tan \varphi = \frac{1}{\cot \alpha}$$

$$= \tan \alpha$$

$$\Rightarrow \qquad \tan \varphi = \tan \alpha$$

$$\Rightarrow \qquad \boxed{\varphi = \alpha} \quad ; \text{ which is constant.}$$

Example 2. Find the angle between radius vector and tangent at any point for the following curves :

(i) $r = a\ (1 - \cos\ \theta)$ (ii) $r = a\ (1+\cos\ \theta)$

(iii) $r = a\ (1 + \sin\ \theta)$ (iv) $r = a\theta$

(v) $r^m = a^m \cos\ m\ \theta$ (vi) $r^m = b^m \sin m\theta$

Sol. (i) Given equation of the curve is

$$r = a\ (1 - \cos\ \theta)$$

Diff. w.r. to θ, we get

$$\frac{dr}{d\theta} = a\ [0-(-\sin\ \theta)]$$

$$= a \sin\theta$$

r

$$\Rightarrow \quad \frac{d\theta}{dr} = \frac{1}{a\sin\theta}$$

$$\Rightarrow \quad \frac{r d\theta}{dr} = \frac{r}{a\sin\theta}$$

$$\Rightarrow \quad \tan\varphi = \frac{a(1-\cos\theta)}{a\sin\theta}$$

$$= \frac{\cancel{a}.\cancel{2}\sin^2\theta/2}{\cancel{a}.\cancel{2}\sin\theta/2.\cos\theta/2}$$

$$= \tan\frac{\theta}{2}$$

$$\Rightarrow \quad \boxed{\tan\varphi = \tan\frac{\theta}{2}}$$

$$\Rightarrow \quad \varphi = \frac{\theta}{2}$$

(ii) Given equation of the curve is

$$r = a\,(1 + \cos\theta)$$

Diff. w.r. to θ, we get

$$\frac{dr}{d\theta} = a\,[0 - \sin\theta]$$

$$= -a\sin\theta$$

$$\Rightarrow \quad r\frac{d\theta}{dr} = \frac{-r}{a\sin\theta}$$

$$= \frac{-\not{a}(1+\cos\theta)}{\not{a}\sin\theta}$$

$$\Rightarrow \quad \tan\varphi = -\frac{\not{2}\cos^2\theta/2}{\not{2}\sin\theta/2.\cos\theta/2}$$

$$= -\cot\frac{\theta}{2}$$

$$\Rightarrow \quad \tan\varphi = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\Rightarrow \quad \boxed{\varphi = \frac{\pi}{2}+\frac{\theta}{2}}$$

(iii) $r = a(1+\sin\theta)$

Diff. w.r. to θ, we get

$$r \quad \frac{dr}{d\theta} = a(0+\cos\theta)$$

$$= a\cos\theta$$

$$\Rightarrow \quad \frac{rd\theta}{dr} = \frac{r}{a\cos\theta}$$

$$= \frac{\not{a}(1+\sin\theta)}{\not{a}\cos\theta}$$

$$\Rightarrow \qquad \tan\varphi = \frac{1+\sin\theta}{\cos\theta}$$

$$= \frac{1+\cos(\pi/2-\theta)}{\sin(\pi/2-\theta)}$$

$$= \frac{2\cos^2(\pi/4-\theta/2)}{2\sin(\pi/4-\theta/2).\cot(\pi/4-\theta/2)}$$

$$= \cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\tan\varphi = \cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$= \tan\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$\Rightarrow \qquad \varphi = \frac{\pi}{2}-\frac{\pi}{4}+\frac{\theta}{2}$$

$$\boxed{\varphi = \frac{\pi}{4}+\frac{\theta}{2}}$$

(iv) $r = a\theta$

Diff. w.r. to θ, we get

$$\frac{dr}{d\theta} = a$$

$$\frac{rd\theta}{dr} = \frac{r}{a}$$

$$= \frac{a\theta}{a}$$

$$= \theta$$

$$\Rightarrow \boxed{\tan\varphi = \theta}$$

$$\Rightarrow \varphi = \tan^{-1}(\theta)$$

(v) $$r^m = a^m \cos m\,\theta$$

Taking log on both sides, we get

$$\log r^m = \log a^m \cos m\,\theta$$

$$\Rightarrow m \log r = \log a^m + \log \cos m\,\theta$$

Diff. w. r. to θ, we get

$$m.\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos m\theta}\frac{d}{d\theta}(\cos m\,\theta)$$

$$\Rightarrow m.\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\cos m\theta}(-\sin m\theta)\frac{d}{d\theta}(m\theta)$$

$$\Rightarrow m.\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin m\theta}{\cos m\theta}.m$$

$$\Rightarrow \frac{rd\theta}{dr} = -\frac{\cos m\theta}{\sin m\theta}$$

$$= -\cot m\theta$$

$$\Rightarrow \qquad \tan\varphi = \tan\left(\frac{\pi}{2}+m\theta\right)$$

$$\Rightarrow \qquad \boxed{\varphi = \frac{\pi}{2} + m\theta}$$

(vi) $$r^m = b^m \sin m$$

Taking log on both sides, we get

$$\log r^m = \log b^m \sin m\theta$$

$$\Rightarrow \qquad m \log r = \log b^m + \log \sin m\theta$$

Diff. w. r. to θ, we get

$$m.\frac{1}{r}\frac{dr}{d\theta}(r) = 0 + \frac{1}{\sin m\theta}\frac{d}{d\theta}(\sin m\theta)$$

$$\Rightarrow \qquad m.\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\sin m\theta} \cos m\theta.m$$

$$\Rightarrow \qquad r.\frac{d\theta}{dr} = \frac{\sin m\theta}{\cos m\theta}$$

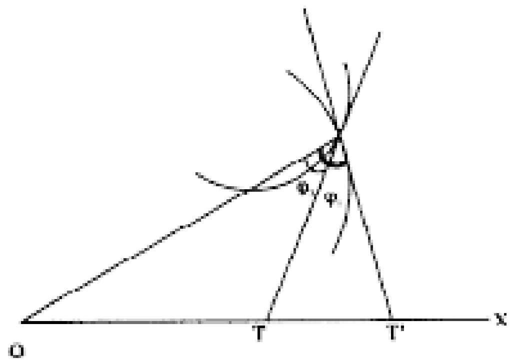
$$= \tan m\theta$$

$$\Rightarrow \qquad \tan\varphi = \tan m\theta$$

$$\Rightarrow \qquad \boxed{\varphi = m\theta}$$

ANGLE OF INTERSECTION OF TWO CURVES

If two curves whose polar equations are given. Let the two curves intersect at P and let φ_1 and φ_2 be the angles which the tangents of two curves make with radius vector respectively. Then obviously



the angle of intersection of two curves at P is $\varphi_1 - \varphi_2$ or $\varphi_2 - \varphi_1$.

Note : If $\tan \varphi_1 = n_1$ and $\tan \varphi_2 = n_2$ then the angle of intersection between the two curves is evidently $\tan^{-1}\left(\frac{n_1 - n_2}{1 + n_1 n_2}\right)$

Note. Two curves cut orthogonally if $\varphi_1 - \varphi_2 = \frac{\pi}{2}$

or $\tan \varphi_1 . \tan \varphi_2 = -1$

i.e. $n_1 . n_2 = -1$

Example 3. Find the angle of intersection for following pairs of curves.

(i) $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$

(ii) $r = a\theta$ and $r = \frac{a}{\theta}$

(iii) $r = \sin\theta + \cos\theta$ and $r = 2\cos\theta$

(iv) $r = a\cos\theta$ and $r = a(1 - \cos\theta)$

(v) $r^2 = a^2 \cos 2\theta$ and $r = a(1 + \cos\theta)$

(vi) $r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$

Sol. : (i) The equations of the curves are

$$r = a(1 + \cos\theta) \quad \text{........ (1)}$$

and $$r = a(1 - \cos\theta) \quad \text{........ (2)}$$

From (1) and (2), we get

$$r = r$$

$$\Rightarrow \quad a(1 + \cos\theta) = a(1 - \cos\theta)$$

$$\Rightarrow \quad 1 + \cos\theta = 1 - \cos\theta$$

$$\Rightarrow \quad \cos\theta = -\cos\theta$$

$$\Rightarrow \quad \cos\theta + \cos\theta = 0$$

$$\Rightarrow \quad 2\cos\theta = 0$$

$$\Rightarrow \quad \cos\theta = 0$$

$$= \cos\frac{\pi}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}$$

$\therefore$ point of intersection of two curves is $\left(\pi, \frac{\pi}{2}\right)$

For Curve (1)

$$r = a(1 + \cos\theta)$$

Diff. w. r. to θ, we get

$$\frac{d\theta}{dr} = a(0 - \sin\theta)$$

$$= -a\sin\theta$$

$$\therefore \qquad r\frac{dr}{d\theta} = \frac{a(1+\cos\theta)}{-a\ \sin\theta}$$

$$= \frac{-2\cos^2\theta/2}{2\ \sin\theta/2\ \cos\theta/2}$$

$$\Rightarrow \qquad \tan\ \varphi_1 = -\ \cot\ \frac{\theta}{2}$$

$$= \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\Rightarrow \qquad \varphi_1 = \frac{\pi}{2}+\frac{\theta}{2}$$

For Curve 2

$$r = a\ (1 - \cos\ \theta)$$

Diff. w. r. to θ, we get

$$\frac{dr}{d\theta} = a\ (0 + \sin\ \theta)$$

$$= a\ \sin\ \theta$$

$$\therefore \qquad \frac{d\theta}{dr} = \frac{a(1-\cos\theta)}{a\ \sin\theta}$$

$$= \frac{2\sin^2\theta/2}{2\sin\theta/2\ \cos\theta/2}$$

$$\tan \varphi_2 = \tan \frac{\theta}{2}$$

$$\Rightarrow \qquad \varphi_2 = \frac{\theta}{2}$$

If α denotes the angle of intersection, then

$$\alpha = |\varphi_1 - \varphi_2|$$

$$= \left|\frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2}\right|$$

$$= \frac{\pi}{2}$$

Note : These two curves cut orthogonally.

(ii) The equations of the curves are

$$r = a\theta \qquad \text{...... (1)}$$

$$\& \qquad r\theta = a \qquad \text{...... (2)}$$

From (1) & (2) we get

$$a\theta = \frac{a}{\theta}$$

$$\Rightarrow \qquad \theta^2 = 1$$

$$\Rightarrow \qquad \theta = \pm 1$$

For Curve (1)

$$r = a\theta$$

$$\frac{dr}{d\theta} = a$$

So $$r\frac{d\theta}{dr} = \frac{a\theta}{a} = \theta$$

$\Rightarrow$ $$\tan \varphi_1 = \theta = 1$$

$$= \tan\frac{\pi}{4} \qquad (\because \theta = 1)$$

For Curve (2)

$$r = \frac{a}{\theta}$$

Diff. w.r. to θ, we get

$$\frac{dr}{d\theta} = \frac{-a}{\theta^2}$$

So $$r\frac{d\theta}{dr} = \frac{-\theta^2}{a}.\frac{a}{\theta}$$

$\Rightarrow$ $$\tan \varphi_2 = -\theta$$

$$= -1$$

$$= -\tan\frac{\pi}{4} \qquad (\because \theta = 1)$$

$\Rightarrow$ $$\tan \varphi_2 = \tan\left(\pi - \frac{\pi}{2}\right)$$

$$\Rightarrow \qquad \varphi_2 = \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

If α denotes the angle of intersection. Then

$$\alpha = |\varphi_1 - \varphi_2|$$

$$= \left|\frac{\pi}{4} - 3\frac{\pi}{4}\right|$$

$$= \left|\frac{-2\pi}{4}\right|$$

$$= \left|-\frac{\pi}{2}\right|$$

$$= \frac{\pi}{2}$$

Note : These curves also cut orthogonally.

(iii) The equations of the curves are

$$r = \sin\theta + \cos\theta \qquad \text{....... (1)}$$

and $$r = 2\cos\theta \qquad \text{....... (2)}$$

From (1) & (2), we get

$$\sin\theta + \cos\theta = 2\cos\theta$$

$$\sin\theta = \cos\theta$$

$$\Rightarrow \quad \tan\theta = 1$$

$$= \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

$\therefore$ Point of intersection of two curves is $\left(\frac{2}{\sqrt{2}} - \frac{\pi}{4}\right)$

For Curve (1)

$$r = \sin\theta + \cos\theta$$

Diff. w. r. to θ, we get

$$\frac{dr}{d\theta} = \cos\theta - \sin\theta$$

$$\therefore \quad r\frac{d\theta}{dr} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$$

$$\Rightarrow \quad (\tan\varphi_1)_{\theta = \pi/4} = \infty$$

$$= \tan\frac{\pi}{2}$$

$$\Rightarrow \quad \varphi_1 = \frac{\pi}{2}$$

For Curve (2)

$$r = 2\cos\theta$$

Diff. w.r. to θ, we get

$$\frac{dr}{d\theta} = -2 \sin\theta$$

$$\therefore \qquad r\frac{d\theta}{dr} = \frac{-2\cos\theta}{2\sin\theta}$$

$$= -\cot\theta$$

$$= \tan\left(\frac{\pi}{2}+\theta\right)$$

$$\tan\varphi_2 = \tan\left(\frac{\pi}{2}-\theta\right)$$

$$\Rightarrow \qquad \varphi_2 = \frac{\pi}{2}+\theta$$

i.e.

$$\varphi_2 = \frac{\pi}{2}+\frac{\pi}{4} = \frac{3\pi}{4}$$

If α denotes the angle of intersection, then

$$\alpha = |\varphi_1 - \varphi_2|$$

$$= \left|\frac{\pi}{2} - \frac{3\pi}{4}\right|$$

$$= \frac{\pi}{4}$$

i.e. $\boxed{\alpha = \frac{\pi}{4}}$

(iv) Exercise for Students

(v) The equations of the curves are

$$r^2 = a^2 \cos 2\theta \quad \text{....... (1)}$$

and $$r = a\,(1+\cos\theta) \quad \text{....... (2)}$$

From (1) and (2), we get

$$a^2 \cos 2\theta = a^2 (1 + \cos\theta)^2$$

$$\cos^2\theta - \sin^2\theta = 1 + \cos^2\theta + 2\cos\theta$$

$$-(1-\cos^2\theta) = 1 + 2\cos\theta$$

$$-1 + \cos^2\theta - 1 - 2\cos\theta = 0$$

$$\Rightarrow \quad \cos^2\theta - 2\cos\theta - 2 = 0$$

$$\therefore \quad \cos\theta = \frac{2 \pm \sqrt{4 - 4.1.(-2)}}{2.1}$$

$$= \frac{2 \pm \sqrt{12}}{2}$$

$$= \frac{2 \pm 2\sqrt{3}}{2}$$

$$= 1 \pm \sqrt{3}$$

Since maximum value of cos θ is 1.

$$\therefore \quad \cos\theta = 1+\sqrt{3} \text{ is not possible}$$

So $$\cos\theta = 1-\sqrt{3}$$

$$\Rightarrow \quad 1 - \cos\theta = \sqrt{3}$$

$$\Rightarrow \quad 2\sin^2\frac{\theta}{2} = \sqrt{3}$$

$$\Rightarrow \quad \sin^2\frac{\theta}{2} = \frac{\sqrt{3}}{2}$$

$$= \left(\frac{3}{4}\right)^{1/2}$$

$$\therefore \quad \sin\frac{\theta}{2} = \left(\frac{3}{4}\right)^{1/4}$$

$$\Rightarrow \quad \theta = 2\sin^{-1}\left(\frac{3}{4}\right)^{1/4} \qquad (*)$$

For Curve (1)

$$r^2 = a^2\cos 2\theta$$

Taking log on both sides, we get

$$\log r^2 = \log a\ 2\cos 2\theta$$

$$\Rightarrow \quad 2\log r = \log a^2 + \log\cos 2\theta$$

Diff. w. r. to θ, we get

$$2.\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos 2\theta}\frac{d}{d\theta}(\cos 2\theta)$$

$$2.\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin 2\theta}{\cos 2\theta}.2$$

$$\Rightarrow \quad r\frac{d\theta}{dr} = -\cot 2\theta$$

$$\Rightarrow \quad \tan \varphi_1 = \tan\left(\frac{\pi}{2}+2\theta\right)$$

$$\Rightarrow \quad \varphi_1 = \frac{\pi}{2}+2\theta$$

For Curve (2)

$$r = a(1 + \cos\theta)$$

Diff. w.r. to θ, we get

$$\frac{dr}{d\theta} = a(0 - \sin\theta)$$

$$= -a\sin\theta$$

$$r\frac{d\theta}{dr} = \frac{-a(1+\cos\theta)}{a\sin\theta}$$

$$= \frac{-2\cos^2\theta/2}{2\sin\theta/2\ \cos\theta/2}$$

$$\tan\varphi_2 = -\cot\frac{\theta}{2}$$

$$= \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\Rightarrow \qquad \varphi_2 = \frac{\pi}{2}+\frac{\theta}{2}$$

If α denotes the angle of intersection, then

$$\alpha = |\varphi_1 - \varphi_2|$$

$$= \left|\frac{\pi}{2} + 2\theta - \frac{\pi}{2} - \frac{\theta}{2}\right|$$

$$= \frac{3\theta}{2}$$

$$= \frac{3}{2}.2\sin^{-1}\left(\frac{3}{4}\right)^{1/4} \qquad \text{(From *)}$$

$$\Rightarrow \qquad \boxed{\alpha = 3\ \sin^{-1}\left(\frac{3}{4}\right)^{1/4}}$$

(vi) The equations of the curves are

$$r^2 = \frac{4}{\sin 2\theta}$$

$$= 4\ \text{cosec}\ 2\theta \qquad \text{........ (1)}$$

and

$$r^2 = 16\ \sin\ 2\theta \qquad \text{........ (2)}$$

From (1) & (2), we get

$$4 \,.\, \text{cosec } 2\theta = 16 \sin\theta$$

$$\Rightarrow \quad \frac{1}{\sin 2\theta} = 4 \sin 2\theta$$

$$\Rightarrow \quad \sin^2 2\theta = \frac{1}{4}$$

$$\Rightarrow \quad \sin 2\theta = \frac{1}{2}$$

$$= \sin \frac{\pi}{6}$$

$$\Rightarrow \quad 2\theta = \frac{\pi}{6}$$

$$\Rightarrow \quad \theta = \frac{\pi}{12}$$

$\therefore$ Point of intersection is $\left(2\sqrt{2}, \frac{\pi}{12}\right)$

For Curve (1)

$$r^2 = 4 \text{ cosec } 2\theta$$

Taking log on both sides, we get

$$\log r^2 = \log 4 \text{ cosec } 2\theta$$

$$\Rightarrow \quad 2 \log r = \log 4 + \log \text{cosec } 2\theta$$

Diff. w.r. to θ, we get

$$2.\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\operatorname{cosec} 2\theta}(-\operatorname{cosec}2\theta \ \cot 2\theta).2$$

$$\Rightarrow \qquad r\frac{d\theta}{dr} = -\tan 2\theta$$

$$= \tan(\pi - 2\theta)$$

$$\Rightarrow \qquad \tan\varphi_1 = \tan(\pi - 2\theta)$$

$$\Rightarrow \qquad \varphi_1 = \pi - 2\theta$$

$$= \pi - 2.\frac{\pi}{12}$$

$$= \frac{5\pi}{6}$$

For Curve (2)

$$r^2 = 16 \sin 2\theta$$

Taking log on both sides, we get

$$\log r^2 = \log 16 \sin 2\theta$$

$$2 \log r = \log 16 + \log \sin 2\theta$$

Diff. w. r. to θ, we get

$$2.\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sin 2\theta}.\cos 2\theta.2$$

$$\Rightarrow \qquad r\frac{d\theta}{dr} = \tan 2\theta$$

$$\tan \varphi_2 = \tan 2\theta$$

$$\Rightarrow \quad \varphi_2 = 2\theta$$

$$= 2.\frac{\pi}{12}$$

$$= \frac{\pi}{6}$$

If α denotes the angle of intersection, then

$$\alpha = |\varphi_1 - \varphi_2|$$

$$= \left| 5\frac{\pi}{6} - \frac{\pi}{6} \right|$$

$$= \frac{2\pi}{3}$$

$$\boxed{\alpha = \frac{2\pi}{3}}$$

Example 4. Show that following curves cut orthogonally.

(i) $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$

(ii) $r = a(1 + \sin\theta)$ and $r = a(1 - \sin\theta)$

(iii) $r = \dfrac{a}{1+\cos\theta}$ and $r = \dfrac{b}{1+\cos\theta}$

(iv) $r^n = a^2 \cos n\theta$ and $r^n = b^n \sin n\theta$

Sol. : Part (i) and (ii) are already solved.

(iii) The equation of the (1) curve is

$$r = \frac{a}{1+\cos\theta}$$

$$= \frac{a}{2\cos^2\theta/2}$$

$$= \frac{a}{2}\sec^2\frac{\theta}{2}$$

Diff. w.r. to θ, we get

$$\frac{dr}{d\theta} = \frac{a}{2}.2\sec\frac{\theta}{2}.\left(\sec\frac{\theta}{2}\tan\frac{\theta}{2}\right).\frac{1}{2}$$

$$\therefore \qquad r.\frac{d\theta}{dr} = \frac{a/2\ \sec^2\theta/2}{a/2\ \sec^2\theta/2.\tan\theta/2}$$

$$= \cot\frac{\theta}{2}$$

$$\tan\varphi_1 = \cot\frac{\theta}{2}$$

$$= \tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

$$\Rightarrow \qquad \varphi_1 = \frac{\pi}{2} - \frac{\theta}{2}$$

The equation of the 2nd curve is

$$r = \frac{b}{1 - \cos\theta}$$

$$= \frac{b}{2\sin^2 \theta/2}$$

$$= \frac{b}{2} \operatorname{cosec}^2 \frac{\theta}{2}$$

Diff. w. r. to θ, we get

$$\frac{dr}{d\theta} = \frac{b}{2} 2 . \operatorname{cosec} \frac{\theta}{2} \left(-\operatorname{cosec} \frac{\theta}{2} \cot \frac{\theta}{2} \right) . \frac{1}{2}$$

$$\therefore \qquad r\frac{d\theta}{dr} = -\tan \frac{\theta}{2}$$

$$\Rightarrow \qquad \tan \varphi_2 = \tan \left(\pi - \frac{\theta}{2} \right)$$

$$\Rightarrow \qquad \varphi_2 = \pi - \frac{\theta}{2}$$

If α denotes the angle of intersection, then

$$\alpha = \left| \varphi_1 - \varphi_2 \right|$$

$$= \left| \frac{\pi}{2} - \frac{\theta}{2} - \pi + \frac{\theta}{2} \right|$$

$$\Rightarrow \alpha = \frac{\pi}{2}$$

Hence curves cut orthogonally.

(iv) The equation of the 1st curve is

$$r^n = a^n \cos n\theta$$

Taking log on both sides, we get

$$\log r^n = \log a^n \cos n\theta$$

$$\Rightarrow \quad n \log r = \log a^n + \log \cos n\theta$$

Diff. w. r. to θ, we get

$$n.\frac{1}{r}.\frac{d\theta}{dr} = 0 + \frac{1}{\cos n\theta}.(-\sin \theta).n$$

$$\Rightarrow \quad r\frac{d\theta}{dr} = -\cot n\theta$$

$$\Rightarrow \quad \tan \varphi_1 = \tan\left(\frac{\pi}{2} + n\theta\right)$$

$$\Rightarrow \quad \varphi_1 = \frac{\pi}{2} + n\theta$$

The equation of 2nd curve is

$$r^n = b^n \sin n\theta$$

Taking log on both sides, we get

$$\log r^n = \log b^n \sin n\theta$$

$$\Rightarrow \quad n \log r = \log b^n + \log \sin n$$

Diff. w. r. to θ, we get

$$n \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{\sin n\theta} (\cos n\theta) \cdot n$$

$$\Rightarrow \quad r\frac{d\theta}{dr} = \tan n\theta$$

$$\Rightarrow \quad \tan \varphi_2 = \tan n\theta$$

$$\Rightarrow \quad \varphi_2 = n\theta$$

If α denotes the angle of intersection,

$$\text{Then } \alpha = \left| \varphi_1 - \varphi_2 \right|$$

$$= \left| \frac{\pi}{2} + n\theta - n\theta \right|$$

$$\Rightarrow \quad \alpha = \frac{\pi}{2}$$

Thus curves cut orthogonally.

EXERCISE

1. Show that in the Parabola $\frac{2a}{r} = 1 - \cos \theta$, $\varphi = \pi - \frac{1}{2}\theta$.

2. Show that at any point P(r, θ) of the curve

$$\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right);$$

(i) $\cos\varphi = \dfrac{a}{r}$ (ii) $\tan\varphi = \dfrac{\sqrt{r^2 - a^2}}{a}$

3. Find the angle of intersection of the curves

$r = 6\cos\theta$ and $r = 2(1+\cos\theta)$

4. Show that the circle $r = b$ cuts the curve $r^2 = a^2\cos 2\theta + b^2$ at an angle $\tan^{-1}\left(\dfrac{a^2}{b^2}\right)$.

5. For the curve $\dfrac{l}{r} = 1 + e\cos\theta$ prove that

$$\varphi = \tan^{-1}\left(\frac{1+e\cos\theta}{e\sin\theta}\right).$$

6. For the curve $r^n = a^n \sec(n\theta + \alpha)$

Show that $\varphi = \dfrac{\pi}{2} - (n\theta + \alpha)$.

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GRAPHING TECHNIQUES IN POLAR CO-ORDINATES

(1) Objectives of Graphs

- (i) Procedures
- (ii) Solved examples
- (iii) Exercise

(2) Area in Polar Co-ordinates

- (i) Introduction
- (ii) Formulae for finding Polar Co-ordinates
- (iii) Solved Examples
- (iv) Exercise

OBJECTIVES :

The objective of curve tracing is to determine the approximate shape of the curve without plotting a large number of points.

We shall find that the equations of the curves which we shall be required to trace are invariably such that they can be solved for *r*. There will be certain cases where the cartesian equation of the curve cannot be solved either for y or for x but can be solved for *r* when transformed into polar co-ordinates. As stated in the beginning we shall not try to find the shape of the curve by plotting a large number of points but find the approximate shape of the curves with the help of rules.

PROCEDURE OF TRACING THE POLAR CURVES

To trace a polar curve we consider the following points :

I. Symmetry : Determine if the curve has any symmetry by applying the following rules :

(a) About the initial line : If the equation of the curve remains unchanged when θ is changed into $-\theta$, then the curve is symmetrical about the initial line.

(b) About the Pole : If the equation of the curve remains unchanged when r is changed into $-r$ or θ into $\pi - \theta$, then the curve is symmetrical about pole.

(c) About the line $\theta = \frac{\pi}{2}$:

(i) If the equation of the curve remains unchanged when θ is changed into $\pi - \theta$, then the curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

(ii) If the equation of the curve remains unchanged when θ is changed into $-\theta$ and r to $-r$, then the curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

(d) About the line $\theta = \frac{\pi}{4}$: If the equation of the curve remains unchanged when θ is changed into $\frac{\pi}{2} - \theta$, then the curve is symmetrical about the line $\theta = \frac{\pi}{4}$.

(e) ***About the line*** $\boldsymbol{\theta = \frac{3\pi}{4}}$: If the equation of the curve remains unchanged when θ is changed into $\frac{3\pi}{4} - \theta$, then the curve is symmetrical about the line $\theta = \frac{3\pi}{4}$.

II. Pole : (i) Find whether the equation of the curve passes through the pole or not. This can be done by putting r = 0 in the equation and then finding some real value of θ. If it is not possible to find a real value of θ for which r = 0, then curve does not pass through the pole (origin).

(ii) Find the tangents at the pole. Putting $r = 0$, the real values of θ gives the tangents at the pole.

(iii) Find the points where the curves meets the initial line and the line $\theta = \frac{\pi}{2}$.

III. Value of φ : Find tan φ and hence φ. Then find the point where φ =0 or $\frac{\pi}{2}$.

IV. Asymptotes : If $r \to \infty$ as $\theta \to \theta$, then there is an asymptote find it by the following method.

(a) Write down the given equation as $\frac{1}{r} = f(\theta)$

(b) Equate $f(\theta)$ to zero and solve for θ.

(c) Find $f'(\theta)$ and calculate at $\theta = \theta_1, \theta_2, ...$ where $\theta_1, \theta_2, ...$ are values of $f(\theta)$ when equated to zero.

(d) Asymptotes are $r \sin(\theta - \theta_1) = \dfrac{1}{f'(\theta_1)}$

$$r \sin(\theta - \theta_2) = \frac{1}{f'(\theta_2)}, \ ....$$

V. Special points : Find some points on the curve for convenient values of θ.

VI. Region : Solve the givenequation for r or θ. Find the region where the curve does not lie. This can be done by the following manner :

(a) No part of the curve lies between $\theta = \alpha$ and $\theta = \beta$ if for $\alpha < \theta < \beta$; r is imaginary.

(b) If the greatest numerical value of r be α, then whole curve lies within the circle $r = \alpha$. And if the least numerical value of r is β, then the whole curve lies outside the circle $r = \beta$.

(c) Finally trace the variations of r when θ varies in the interval $(0, \infty)$ and $(-\infty, 0)$ and making values of θ for which $r = 0$ or attains a maximum or minimum values. Plot the points so obtained.

VII. Conversion into Cartesians : Transform the equation to Cartesian Co-ordinates whenever required.

Example : Trace the following curves :

(a) $r = a(1 + \cos\theta)$

(b) $r = a(1 - \cos\theta)$

(c) $r = a(1 + \sin\theta)$

(d) $r = a(1 - \sin\theta)$

(e) $r = a\cos 2\theta$

(f) $r = a\sin 2\theta$

(g) $r = a\cos 3\theta$

(h) $r = a\sin 3\theta$

(i) $r = a + b\cos\theta$

(a) $r = a(1+\cos\theta)$

(1) **Symmetry :** Since the equation of the curve remains unchanged when θ is changed into $-\theta$.

$\therefore$ the curve is symmetrical about the initial line.

(2) (i) Pole : When $r = 0$

$$\text{then } a(1+\cos\theta) = 0$$

$$\Rightarrow \cos\theta = -1$$

$$= \cos\pi$$

$$\Rightarrow \theta = \pi$$

$\therefore$ the curve passes through pole and tangent at the pole is $\theta = \pi$.

(ii) The curve cuts the initial line at $(2a, 0)$ ($\theta = 0$) and the lines

$$\theta = \pm\frac{\pi}{2} \text{ at } \left(a, \frac{\pi}{2}\right) \text{ and } \left(a, -\frac{\pi}{2}\right).$$

3. (Value of φ) :

Here $$\varphi = \frac{\pi}{2} + \frac{\theta}{2}$$

$$\therefore \qquad \varphi = \frac{\pi}{2}$$

when $\qquad \theta = 0$

& $\qquad r = 2a$

$\therefore$ at (2a, 0), the tangent is $\perp$ to initial line.

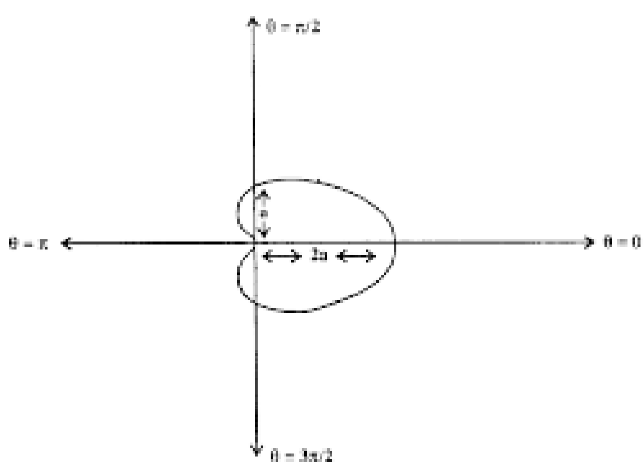
4. **(Asymptote) :** There is no asymptote to the curve because the value of *r* does not tend to infinity for any finite value of θ.

5. **(Special Points) :** We have

θ	0°	30°	60°	90°	120°	150°	180°
r	2a	1.87a	1.5a	a	0.5a	0.13a	0

6. **(Region)** : The maximum value of *r* is 2a and minimum value of *r* is 0 $\therefore$ entire curve lies within the circle r = 2a and outside the circle $r = 0$.

Hence the shape of the curve is as shown below :



(b)r = a (1–cos θ)

(1) Since the equation of the curve remains unchanged when θ is changed into $-\theta$

$\therefore$ the curve is symmetrical about the initial line.

(2) When

$$\begin{aligned} r &= 0 \\ \Rightarrow \quad a\,(1-\cos\,\theta) &= 0 \\ \Rightarrow \quad \cos\,\theta &= 1 \\ &= \cos\,0 \\ \Rightarrow \quad \theta &= 0 \end{aligned}$$

$\therefore$ the curve passes through the pole and tangent at the pole is $\theta = 0°$.

(3) The curve meets the initial line at $(0, 0°)$ and the lines $\theta = \pm\frac{\pi}{2}$ at $\left(a, \frac{\pi}{2}\right)$ and $\left(a, -\frac{\pi}{2}\right)$ respectively.

(4) Also $\varphi = \frac{\theta}{2}$

Now $\varphi = 0$ when $\theta = 0$ and $\varphi = \frac{\pi}{2}$ when $\theta = \pi$

$\therefore$ at $(2a, \pi)$ the tangent is $\perp$ to the line $\theta = \pi$.

(5) There is no asymptote to the curve because the value of r does not tend to infinity for any finite value of θ.

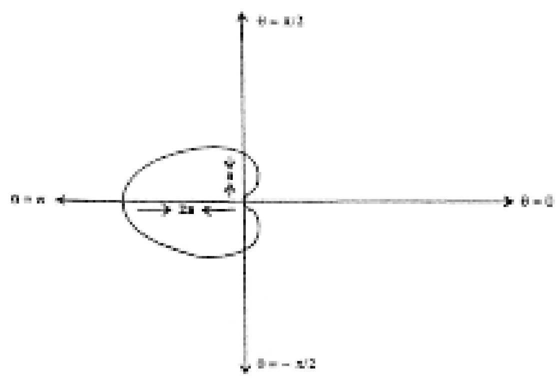
(6) Special points are :

θ	0°	30°	60°	90°	120°	150°	180°
r	0	0.13a	.5a	a	1.5a	1.87a	2a

(7) The maximum value of r is 2a

∴ whole curve lies inside the circle r = 2a. And minimum value of g is 0 ∴ entire lies outside the circle r = 0.

Hence the shape of the curve is given below :



(c)r = a (1 + sin θ)

(1) Since the equation of the curve remains unchanged when θ is changed in π–θ. Therefore the curve is symmetrical about the line

$$\theta = \frac{\pi}{2}.$$

3

(2) When r = 0

$$\Rightarrow \quad 1 + \sin\theta = 0$$

$$\Rightarrow \quad \sin\theta = -1$$

$$= \sin\frac{3\pi}{2}$$

$$\Rightarrow \quad \theta = \frac{3\pi}{2}$$

∴ The curve passes through pole and tangent at the pole is $\theta = \frac{3\pi}{2}$.

(3) The curve meets the initial line at (a, 0); the line $\theta = \frac{\pi}{2}$ at $\left(2a, \frac{\pi}{2}\right)$ and the line $\theta = -\frac{\pi}{2}$ at $\left(0, -\frac{\pi}{2}\right)$.

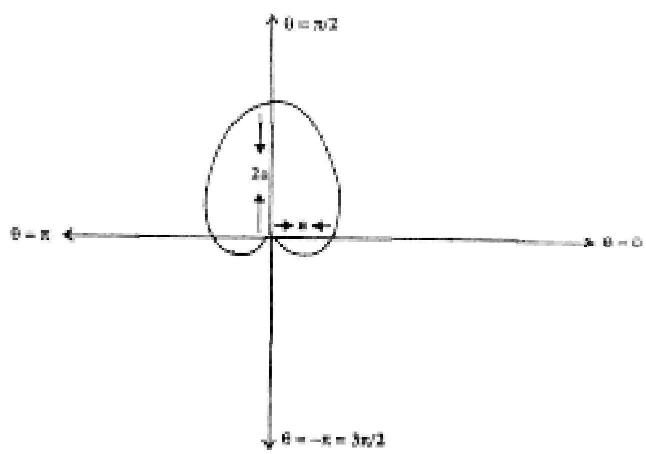
(4) There is no asymptote to the curve because the value of r does not tend to infinity for finite value of θ.

(5) Special points :

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°
r	a	1.5a	1.87a	2a	1.87a	1.5a	a	0.5a	0.13a	0

(6) The maximum value of r is 2a ∴ entire curves lies within the circle $r = 2a$ and the minimum value of r is 0 ∴ entire curve outside the circle $r = 0$.

Hence the shape of the curve is as shown :



(d) $r = a\,(1 - \sin\theta)$

(1) Since the equation of the curve remains unchanged when θ is changed into $\pi-\theta$.

$\therefore$ The curve is symmetrical about the line $\theta = \dfrac{\pi}{2}$.

(2) When $r = 0$

$$\Rightarrow \quad a\,(1 - \sin\theta) = 0$$

$$\Rightarrow \quad \sin\theta = 1$$

$$= \sin\frac{\pi}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}$$

$\therefore$ The curve passes through the pole and tangent at the pole is $\theta = \dfrac{\pi}{2}$.

(3) The curve meets the initial line at (a, 0); the line $\theta = \dfrac{\pi}{2}$ at $\left(0, \dfrac{\pi}{2}\right)$ and the line $\theta = -\dfrac{\pi}{2}$ at $\left(2a, -\dfrac{\pi}{2}\right)$

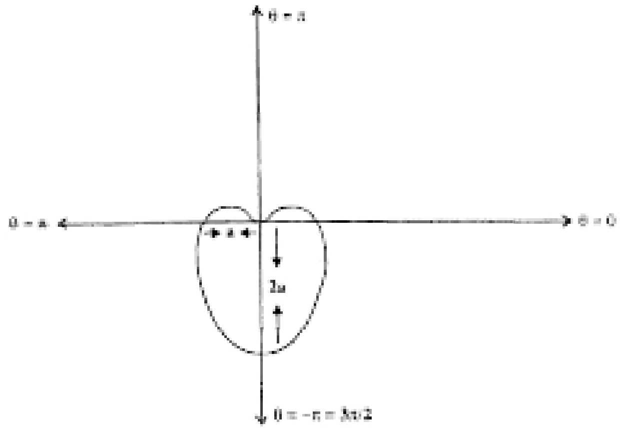
(4) There is no asymptote to the curve because the value of r does not tend to infinity for finite value of θ.

(5) Special point

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°
r	a	.5a	.13a	0	.5a	.13a	a	1.5a	1.87a	2a

(6) The maximum value of r is 2a, therefore whole curve lies inside the circle r=2.

Hence the shape of the curve is as shown below :



(e) Given = n of curve r = a cos 2θ

(1) Since the = n the curve remains unchanged when θ is changed into –θ and θ is changed into π–θ therefore the curve is symmetrical about the initial line and the line $\theta = \frac{\pi}{2}$.

(2) When θ = 0

$$\Rightarrow \quad \cos 2\theta = 0 = \cos\frac{\pi}{2}$$

$$\therefore \quad 2\theta = \pm\frac{\pi}{2}, \pm 3\frac{\pi}{2}, \pm 5\frac{\pi}{2}, \pm 7\frac{\pi}{2}, \ldots\ldots\ldots$$

$$\Rightarrow \quad \theta = \pm\frac{\pi}{4}, \pm 3\frac{\pi}{4}, \pm 5\frac{\pi}{4}, \pm 7\frac{\pi}{4}, \ldots\ldots\ldots$$

∴ the curves passes through origin and tangent at the origin are

$$0 = \pm\frac{\pi}{4}, \pm 3\frac{\pi}{4}, \pm 5\frac{\pi}{4}, \pm 7\frac{\pi}{4}, ...$$

(3) The curve meets the initial line at (a, θ); the line $\theta = \frac{\pi}{2}$, at $\left(-a, \frac{\pi}{2}\right)$.

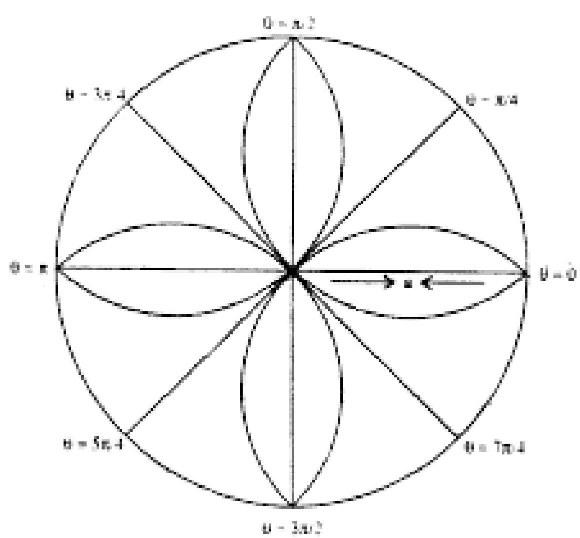
(4) There is no asymptote to the curve because the value of *r* does not tend to infinity for any finite value of θ.

(5) Special points are ;

0	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
r	a	·87a	·5a	0	–·5a	–·87a	–a	–·87a	–·5a	0	1·5a	·87a	a

(6)The maximum value of *r* is a ∴ whole curve lies inside the circle r = a.

Hence the shape of the curve is as shown below :



(f) Given = n of the curve is r = a sin 20.

(1) Since the = n of the curve remains unchanged when θ is changed into $\frac{\pi}{2}-\theta$ and $3\frac{\pi}{2}-\theta$. $\therefore$ the curve is symmetrical about the line $\theta = \frac{\pi}{4}$ and $\theta = -\frac{\pi}{4}$

(2) When r = 0

$\Rightarrow$ Sin $2\theta = 0$

$\Rightarrow$ $20 = 0, \pi, 2\pi, 3\pi, 4\pi,$ ....

$\therefore$ $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi,$ ....

$\therefore$ The curve passes through pole and tangent at the pole are

$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi,$ ....

(3) The curve meets the initial line at (0, 0), the line $0 = \frac{\pi}{2}$ at (0, 0).

(4) There is no asymptote to the curve because the value of r does not tend to infinity for any finite value of θ.

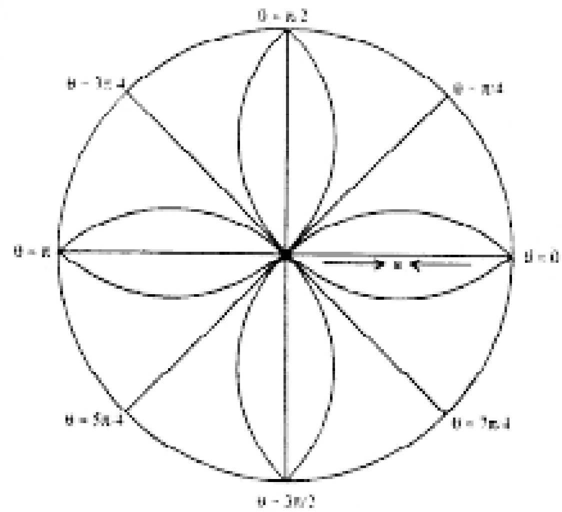
(5) The maximum value of r is a

$\therefore$ whole curves lies inside the circle r = a.

(6) Special points

θ	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°
r	0	·5a	·87a	a	.87a	·5a	0	–·5a	–·87	·a

Hence the shape of the curve is shown below :



(g) The given equation of the curve r = a cos 3θ.

(1) Since the equation of the curve remains unchanged when θ is changed into –θ ∴ the curve is symmetrical about the initial line.

(2) When r = 0

$\Rightarrow \quad a \cos 3\theta = 0$

$\Rightarrow \quad \cos 3\theta = 0 = \cos\frac{\pi}{2}$

$\Rightarrow \quad 3\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots\ldots\ldots\ldots$

$\Rightarrow \quad \theta = \pm\frac{\pi}{6}, \pm\frac{3\pi}{6}, \pm\frac{5\pi}{6}, \pm\frac{7\pi}{6}, \ldots\ldots\ldots\ldots$

∴ The curve passes through pole and tangent at pole are

$$\theta = \pm\frac{\pi}{6}, \pm\frac{3\pi}{6}, \pm\frac{5\pi}{6}, \pm\frac{7\pi}{6}, \text{............}$$

(3) The curve meet the initial line at (a, 0), the line $\theta = \frac{\pi}{2}$, at $\left(-a, \frac{\pi}{2}\right)$ and the line $\theta = -\frac{\pi}{2}$ at $\left(-a, \frac{\pi}{2}\right)$

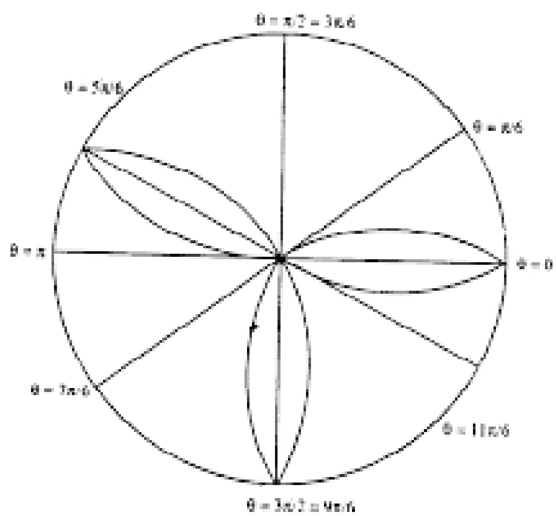
(4) There is no asymptotes to the curve because the value of r does not tend to infinity for any finite value of θ.

(5) The maximum value of r is a ∴ whole curve lies inside the circle r = a.

(6) Special points :

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
r	a	·87a	·5a	0	–·5a	–·87a	–a	–·87a	–·5a	0

Hence the shape of the curve is given below :



(h) The given equation of the curve r = a sin 3θ.

(1) Since the equation of the curve remains unchanged when θ is changed π–θ.

$\therefore$ the curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

(2) When r = 0

$$\Rightarrow \quad a \sin 3\theta = 0$$

$$\Rightarrow \quad \sin 3\theta = 0 = \sin 0$$

$$\Rightarrow \quad 3\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, \ldots\ldots\ldots\ldots$$

$$\Rightarrow \quad \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots\ldots\ldots$$

$\therefore$ The curve passes through pole and tangent at the pole are

$$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots\ldots\ldots$$

(3) The curve meets the initial line at (0, 0); the line $0 = \frac{\pi}{2}$ at

$$\left(a, \frac{\pi}{2}\right)$$

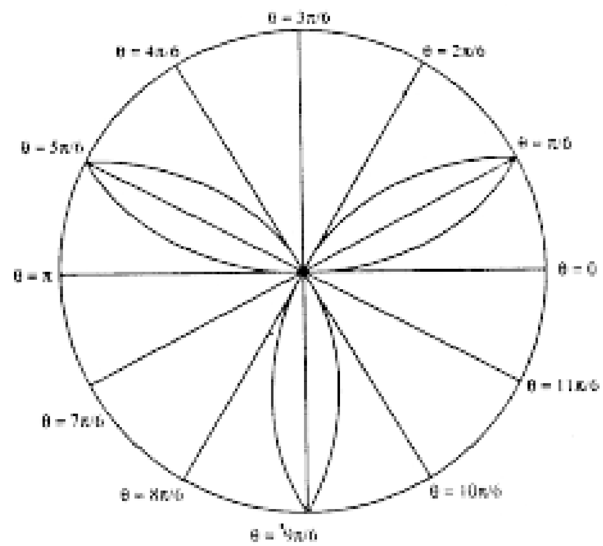
(4) There is no asymptote to the curve because the value of r does not tend to infinity for any finite value of θ.

(5) The maximum value of r is a $\therefore$ whole curve lies inside the circle r = a.

(6) Special points :

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
r	0	·5a	·87a	a	·87a	·5a	0	–·5a	–·87a	–a

Hence the shape is



(i) The given equation of the curve $r = a + b \cos \theta$.

(This curve is called Limacon of Pascal).

(1) Since the equation of the curve remains unchanged when θ is changed into $-\theta$ $\therefore$ the curve is symmetrical about the pole.

(2) When $r = 0$

$$\Rightarrow \quad a + b \cos \theta = 0$$

$$\Rightarrow \quad \cos \theta = \left(\frac{-a}{b}\right)$$

$$\Rightarrow \qquad \theta = \cos^{-1}\left(\frac{-a}{b}\right)$$

Now three cases arises

Case (a) : When a = b, then Limacon of Pascal becomes a Cardiod of the type $r = a(1 + \cos\theta)$ Which has already discussed in part (a).

Case (b) : When a < b, then the curve passes through the pole and tangent at the pole is $\theta = \cos^{-1}\left(\frac{-a}{b}\right)$

(3) The curve cuts the initial line at (a + b, 0), the line $\theta = \frac{\pi}{2}$ at $\left(a, \frac{\pi}{2}\right)$ and the line $\theta = \frac{-\pi}{2}$ at $\left(a, \frac{-\pi}{2}\right)$.

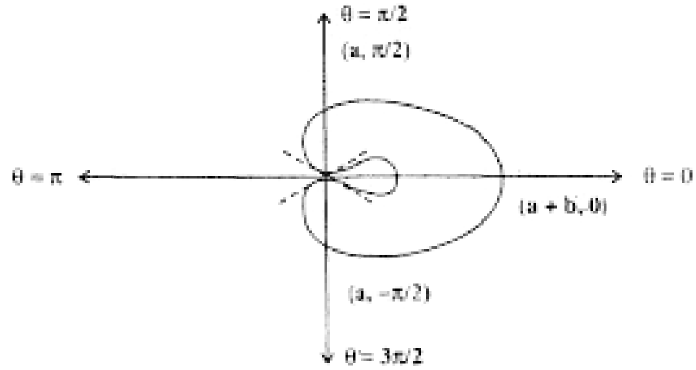
(4) There is no asymptote to the curve because the value of r does not tend to infinity for any finite value of θ.

(5) The maximum value of r is a+b ∴ whole curve lies within the circle r = a+b. And the minimum value of r is a – b ∴ the whole curve lies outside the circle r = a – b.

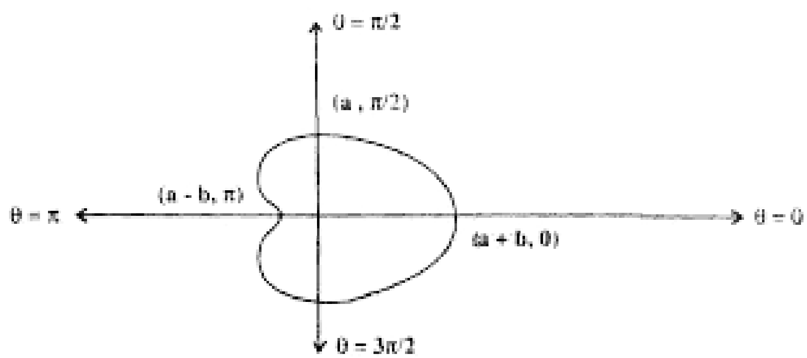
(6) Special points :

θ	0°	30°	60°	90°	120°	150°	180°
r	a+b	a+·87a	a+·5b	a	a–·5b	a–·87b	a–b

Hence the shape is



Case (c) : When a > b then the curve never passes through the pole and consequently its shape is



EXERCISE

Trace the following curves.

(1) $r = 2 + 3 \cos \theta$

(2) $r = 2 (1 + \cos \theta)$

(3) $r = 6 \sin 3\theta$

(4) r = 2a cos θ

(5) r = 2a sin θ

(6) r = 4 – 3 cos θ

Note : (i) The curve r = a (1 ± cos θ) and r = a (1 ± sin θ) are called cardiod

(ii) r = a sin 2θ

r = a cos 2θ (4 leared rose)

(iii) r = a sin 3θ

r = a cos 3θ (three leared rose)

(iv) r = a + b cosθ, a, b > 0 (Limacon of Pascal)

AREA IN POLAR CO-ORDINATES

INTRODUCTION

The process of finding the area bounded by any portion of a plane cure is called Quadrature.

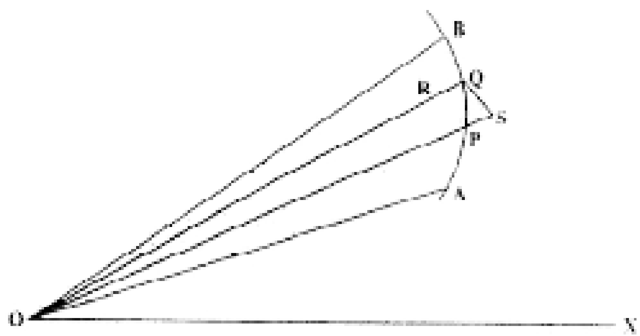
Area formula for Polar Co-ordinates

Art (1) : If r = f(θ) be the equation the curve then the area of the sector enclosed by the curve and two radii vector θ = α and θ = β is

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta.$$

Proof : Let AB be the curve r = f (θ); OA and OB are the radii vectors θ = α and θ = β respectively. Take any point P (r, θ) on the curve. Let Q (r + δr, θ + δθ) be another point on the curve. Close to P.

With O as centre and radii OP, OQ respectively, draw the circular arcs PR and QS. Then PR = rδθ and QS = (r + δr) δθ



$$\therefore \text{ Sectorial area OPR} = \frac{1}{2} r \cdot r\delta\theta$$

$$= \frac{1}{2} r^2 \delta\theta$$

$$\text{and Sectorial area OSQ} = \frac{1}{2}(r + \delta r) \cdot (r + \delta r)\ \delta\theta$$

$$= \frac{1}{2}\ (r + \delta r)^2 .\ \delta\theta$$

If S and S + δS denotes the areas OAP and OAQ respectively, then

δS = Area OAQ–Area OAP

= Area OPQ

Now the area OPQ (= δS) lies between the areas OPR and OSQ so that

$$\frac{1}{2}\ r^2 \delta\theta < \delta S < \frac{1}{2}(r+\delta r)^2 \cdot \delta\theta$$

i.e. $$\frac{r^2}{2} < \frac{\delta S}{\cdot\delta\theta} < \frac{\left(r + \delta r\right)^2}{2} \qquad \text{.......... (1)}$$

In the limit when $\delta\theta \to 0$, $\delta\gamma$ also $\to 0$ and consequently (1) becomes

$$\frac{dS}{d\theta} = \frac{r^2}{2}$$

$$= \frac{\left[f(\theta)\right]^2}{2} \qquad \text{............ (2)}$$

Hence from relation (2) it is dovious that

$$\int_\alpha^\beta \frac{1}{2} r^2 d\theta = \int_\alpha^\beta \frac{dS}{d\theta} \cdot d\theta$$

$$= \int_\alpha^\beta dS$$

$$= \left| S \right|_\alpha^\beta$$

= (Value of S when $\theta = \beta$) – (Value of S when $\theta = \alpha$)

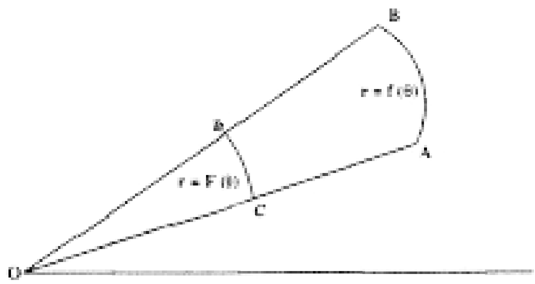
= Area OAB

Hence Area OAB = $\frac{1}{2}\int_\alpha^\beta r^2 d\theta$

Art No. 2 Prove that the area bounded by the curves $r = f(\theta)$, $r = F(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ is

$$\frac{1}{2}\int_\alpha^\beta \left(r_1^2 - r_2^2\right) d\theta.$$

Proof : Let AB, CD be the curves $r = f(\theta)$, $r = F(\theta)$, and OCA, ODB and the radii vectors $\theta = \alpha$, $\theta = \beta$. Then



Area CABD $=$ Area OAB – Area OCD

$$\text{Area CABD} = \frac{1}{2}\int_{\alpha}^{\beta}\left[f(\theta)^2\right]d\theta - \int_{\alpha}^{\beta}\left[f(\theta)^2\right]d\theta$$

$$= \frac{1}{2}\int_{\alpha}^{\beta}\left\{\left[f(\theta)\right]^2 - \left[F(\theta)\right]^2\right\}d\theta$$

$$= \frac{1}{2}\int_{\alpha}^{\beta}\left(r_1^2 - r_2^2\right)d\theta;$$

where r_1 [= f(θ)] and r_2 [=f(θ)].

Note : Determination of the limits of integration.

(1) If the curve is symmetrical about the initial line only then the required integral may be evaluated from 0 to π and whole result is multiplied by 2.

(2) If the curve is symmetrical about both initial line and the line $\theta = \frac{\pi}{2}$, then integral must be evaluated from 0 to $\frac{\pi}{2}$ and the whole result is multiplied by 4.

(3) In case of a loop, the limits of integration for finding its area are two successive value of θ which make r = 0.

Example : Find the areas of the cardiods

(a) $r = a\,(1 + \cos\theta)$

(b) $r = a\,(1 - \cos\theta)$

(c) $r = a\,(1 + \sin\theta)$

(d) $r = a\,(1 - \sin\theta)$

Sol : Students are advised to first trace the curve as done in previous chapter. Then proceed further as follows :

(a) Since the curve is symmetrical about initial line only

$$\therefore \quad \text{Area} = 2\int_0^{\pi} \frac{1}{2} r^2 d\theta$$

$$= 2 \cdot \frac{1}{2} \int_0^{\pi} a^2 (1 + \cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^{\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= a^2 \int_0^{\pi} \left(1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= a^2 \int_0^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= a^2 \left| \frac{3}{2}\theta + 2\sin\theta + \frac{1}{2}\frac{\sin 2\theta}{2} \right|_0^{\pi}$$

$$= a^2\left[\left(\frac{3}{2}\pi + 0 + 0\right) - (0 + 0 + 0)\right]$$

$$= \frac{3\pi a^2}{2} \textbf{\textit{ Ans.}}$$

(b) .... Trace first ....

Since the curve is symmetrical about the initial line only.

$$\therefore \text{ Area } = 2\int_0^{\pi} \frac{1}{2} r^2 d\theta$$

$$= \int_0^{\pi} a^2 (1 - \cos\theta)^2 d\theta$$

$$= a^2 \int_0^{\pi} (1 - 2\cos\theta + \cos^2\theta) d\theta$$

$$= a^2 \int_0^{\pi} \left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$\left(\because \cos^2\theta = \frac{1 + \cos 2\theta}{2}\right)$$

$$= a^2 \left|\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right|_0^{\pi}$$

$$= a^2\left[\left(\frac{3}{2}\pi - 0 + 0\right) - (0 + 0 + 0)\right]$$

$$= \frac{3\pi a^2}{2} \textbf{\textit{ Ans.}}$$

(c) .... Trace first ....

Since the curve is symmertical about the line $\theta = \frac{\pi}{2}$

$$\text{Area} = 2\int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2 d\theta = \int_{-\pi/2}^{\pi/2} r^2\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} a^2 (1+\sin\theta)^2\, d\theta$$

$$= a^2 \int_{-\pi/2}^{\pi/2} \left(1+2\sin\theta+\sin^2\theta\right) d\theta$$

$$= a^2 \int_{-\pi/2}^{\pi/2} \left(1+2\sin\theta+\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= a^2 \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2}+2\sin\theta-\frac{1}{2}\cos 2\theta\right) d\theta$$

$$= a^2 \left|\left(\frac{3}{2}\theta+2(-\cos\theta)-\frac{1}{2}\frac{\sin 2\theta}{2}\right)\right|_{-\pi/2}^{\pi/2}$$

$$= a^2 \left[\left(\frac{3}{2}\cdot\frac{\pi}{2}-0-0\right)-\left(\frac{3\pi}{2}\cdot\frac{1}{2}-0-0\right)\right]$$

$$= a^2\left[\frac{3\pi}{4}+\frac{3\pi}{4}\right]$$

$$= a^2.\frac{6\pi}{4} = 3a^2\frac{\pi}{2} \qquad \textit{\textbf{Ans.}}$$

(d) Solve yourself as above.

Example : Obtain the area of a loop of the following curves. Also find the total area of the curves.

(i) r = a sin 3θ

(ii) r = a cos 3θ

(iii) r = a sin 2θ

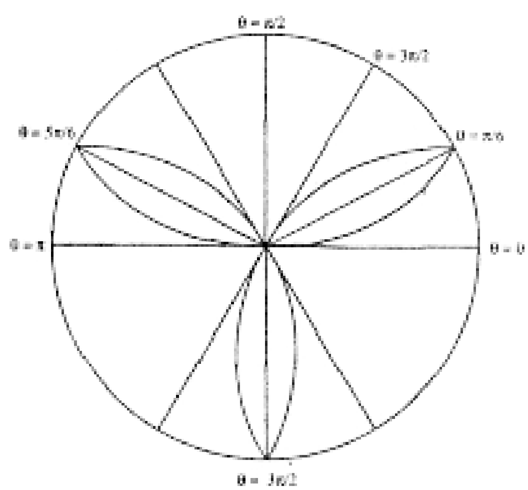
(iv) r = a sin 3θ

Sol. : (i) The given =n of the curve is

r = a sin 3θ

.... Trace it .... (As done in previous chapter)

For a loop of the curve



$$r = 0$$

$$\Rightarrow \quad a \sin 3\theta = 0$$

$$\Rightarrow \quad 3\theta = 0$$

$$\Rightarrow \quad \theta = 0, \ \frac{\pi}{3}$$

So, Area of a loop $= \int_0^{\pi/3} \frac{1}{2} r^2 d\theta$

$$= \frac{1}{2} \int_0^{\pi/3} a^2 \sin^2 3\theta d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/3} \frac{1-\cos 6\theta}{2} d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/3} (1-\cos 6\theta) d\theta$$

$$= \frac{a^2}{4} \left| \theta - \frac{\sin 6\theta}{6} \right|_0^{\pi/3}$$

$$= \frac{a^2}{4} \left[\left(\frac{\pi}{3} - \frac{1}{6} ,0 \right) - (0-0) \right]$$

$$= \frac{\pi a^2}{12}$$

Since the whole curve lies within the circle r=a and there are three equal loops.

$\therefore$ whole are $= 3.\dfrac{\pi a^2}{12} = \dfrac{\pi a^2}{4}$

(ii) Hint .... Trace the curve ....

Then for limits of integration put

$$r = 0$$

$$\Rightarrow \quad a \cos 3\theta = 0$$

$$\Rightarrow \quad \cos 3\theta = 0$$

$$\Rightarrow \quad 3\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\ldots..$$

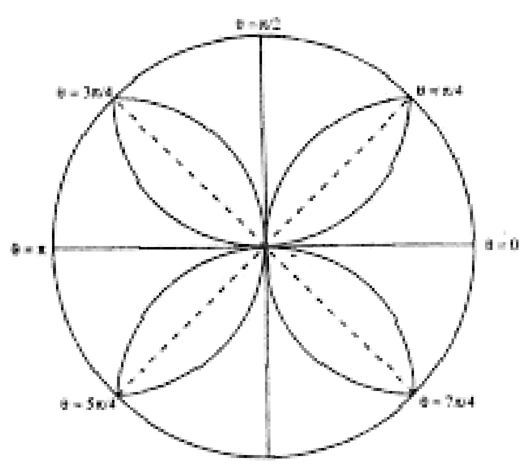
$$\Rightarrow \quad \theta = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \ldots\ldots...$$

and solve as above.

(iii) Given equation of the curve is

$$r = a \sin 2\theta$$

.... Trace the curve ....



For a loop of the curve

$$\gamma = 0$$

$$\Rightarrow \quad a \sin 3\theta = 0$$

$$\Rightarrow \quad 2\theta = 0, \pi, \ldots\ldots..$$

$$\Rightarrow \quad \theta = 0, \frac{\pi}{2}, \ldots\ldots..$$

So area of a loop of the curve

$$= \frac{1}{2}\int_0^{\pi/2} r^2 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} a^2 \sin^2 2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1-\cos 4\theta}{2} d\theta$$

Area of a loop $= \dfrac{a^2}{4}\displaystyle\int_0^{\pi/2} (1 - \cos 4\theta) d\theta$

$$= \frac{a^2}{4}\left| \theta - \frac{\sin 4\theta}{4} \right|_0^{\pi/2}$$

$$= \frac{a^2}{4}\int_0^{\pi/2} \left(\frac{\pi}{2} - 0\right) - (0 - 0)$$

$$= \frac{\pi a^2}{8}$$

Since the curve lies within the circle r = a and there are four equal loops.

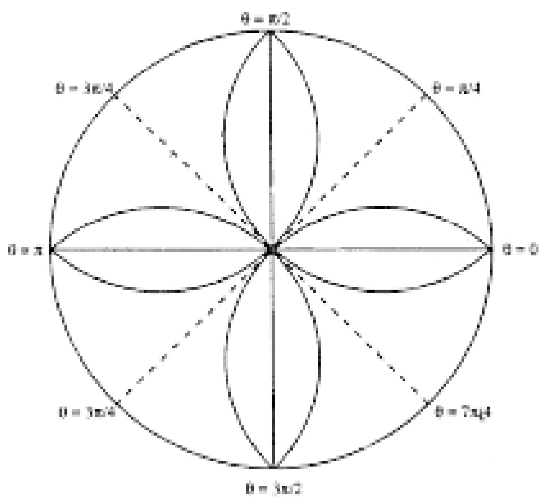
$$\therefore \quad \text{Whole area} = 4.\frac{\pi a^2}{8}$$

$$= \frac{\pi a^2}{2}$$

(vi) Given =n of the curve

$$r = a \cos 2\theta$$

.... Trace the curve ....



For a loop $r = 0$

$\Rightarrow \quad a \cos 2\theta = 0$

$\Rightarrow \quad \cos 2\theta = 0$

$\Rightarrow \quad 2\theta = \pm\frac{\pi}{2}$

$$\Rightarrow \qquad \theta = \pm\frac{\pi}{4}$$

Area of a loop $= = \frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2 d\theta$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} a^2 \sin^2 2\theta \, d\theta$$

$$= \frac{1}{2}a^2\int_{\pi/4}^{\pi/4} \frac{1+\cos 4\theta}{2} d\theta$$

$$= \frac{a^2}{4}\left| \theta + \frac{\sin 4\theta}{4} \right|_{-\pi/4}^{\pi/4}$$

$$= \frac{a^2}{4}\left[\left(\frac{\pi}{3}+0\right)-\left(\frac{-\pi}{4}-0\right)\right]$$

$$= \frac{a^2}{4}\left[\frac{\pi}{4}+\frac{\pi}{4}\right]$$

$$= \frac{\pi a^2}{8}$$

Since the whole curve lies within the circle r = a and there are four equal loops.

$\therefore$ whole area of the curve $= 4.\frac{\pi a^2}{8} = \frac{\pi a^2}{2}$

Example : Find the whole area of the curve

$$r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$$

Sol. : Given =n of the curve is

$$r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$$

Since the equation of the curve remains unchanged when θ is changed into –θ. ∴ the curve is symmetrical about the initial line.

Also it remains unchanged when θ is changed into π – θ, ∴ curve is also symmetrical about the line $\theta = \frac{\pi}{2}$.

For the curve in the first quadrant, θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \quad \text{Required area} = \frac{1}{2}\int_0^{\pi/2} \frac{1}{2} r^2 d\theta$$

$$= 2\int_0^{\pi/2} \left(a^2\cos^2\theta + b^2\sin^2\theta\right) d\theta$$

$$= 2\left[a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} + b^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2}(a^2 + b^2)$$

EXERCISE

Find the area of the following curves :

(a) $r^2 = a^2 \sin 2\theta$ (*Ans.* a^2)

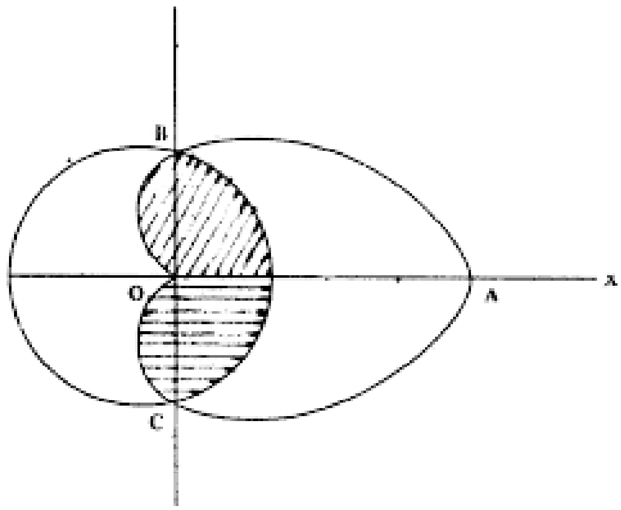
(b) $r^2 = a^2 \cos 2\theta$ (*Ans.* a^2)

(c) $r = 2 \cos$ (*Ans.* πa^2)

(d) $r = a + b \cos 2\theta$ $(a<b)$ $\left(Ans.\ \pi \left(a^2 + \frac{b^2}{2} \right) \right)$

(e) Find the area common to the curves $r = a$ and $r = a(1+\cos\theta)$.

Sol. (e) : The = n of the curves are



$r = a$ and $r = a(1+\cos\theta)$.

Now $r = a$ is a circle and $r = a(1+\cos\theta)$ is a cardiod. Hence common area is given by

$$\text{Required area} = 2\left[\text{Area of circle from 0 to } \pi + \text{Area of cardiod from } \frac{\pi}{2} \text{ to } \pi\right]$$

$$= 2\int_0^{\pi/2} \frac{1}{2} a^2 d\theta + 2\int_{\pi/2}^{\pi} \frac{a^2}{2}(1+\cos\theta)^2 d\theta$$

$$= a^2 \int_0^{\pi/2} d\theta + a^2 \int_{\pi/2}^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta \right) d\theta$$

$$= a^2 \left|\theta\right|_0^{\pi/2} + a^2 \left| \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta \right|_{\pi/2}^{\pi}$$

$$= a^2 \left[\left(\frac{\pi}{2} - 0 \right) + \left(\frac{3}{2}\pi + 0 + 0 \right) - \left(\frac{3}{2}\frac{\pi}{2} + 2 + 0 \right) \right]$$

$$= a^2 \left[\frac{\pi}{2} + \frac{3\pi}{2} - \frac{3\pi}{4} - 2 \right]$$

$$= a^2 \left(\frac{2\pi + 6\pi - 3\pi}{4} - 2 \right)$$

$$= a^2 \left(\frac{5\pi}{4} - 2 \right)$$

Mohammad Rasul Choudhary
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APPLICATIONS OF INTEGRATION

1. Introduction & Definition
2. List of formulae for integration
3. Reduction Formula
   i) Introduction & Definition
   ii) Reduction formulae and their applications
   iii) Examples & Exercise

Review of integrals

INTRODUCTION :

Integral calculas is the outcome of the attempts made by mathematicians to find some general method of finding the area of the plane regions bounded by given curved lines. To find such an area it was found necessary to divide the region into a large number of very small elements and then to find some method of evaluating the limit of the sum of the areas of all these elements when each element is infinitely small and their number infinitely increased.

In fact, the name 'integral calculas' has its origin in this process of summation. Later it was found that the process of integration can also be viewed as the inverse operation of differentiation.

It may be of interest to the students to know that, although we are venturing to study integration as the inverse operation of differentation, it was integral calculas which was discovered first.

Besides the area enclosed by plane curves, integral calculus is also applied to other important problmes such as finding the legnth of curves, surface, volumes etc.

Def. : If a function f(x) is given, then any function F(x) whose derivative is equal to f(x) is called anti-derivative or integral or a primitive of f(x).

We use the symbol $\int f(x)dx = F(x)$ and it is read as integral of f(x) with respect to x.

For example $\int \cos x dx = \sin x \left(\because \frac{d}{dx}(\sin x) = \cos x \right)$

and $\int x^3 dx = \frac{x^4}{4}$ because $\frac{d}{dx}\left(\frac{x^4}{4}\right) = x^3$.

Note : Since $\frac{d}{dx}$ (F(x) + c) = $\frac{d}{dx}$(F(x) = f(x)

$\therefore \int f(x) = F(x) + C$;

where C is called constant of integration

Before the discussion of integration and its various application the following formulae are to be remembered.

List of Formulae

1. $\int a dx = ax + c$

2. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$; provided $n \neq -1$

3. $\int \frac{1}{x} dx = \log x + c$

4. $\int (ax+b)^n \, dx = \frac{(\text{x}x+b)^{n+1}}{n+1} \cdot \frac{1}{a}$ + c; provided n ≠ – 1

5. $\int \frac{1}{a\text{x}+b} d\text{x}$ = log (ax+b)+c

6. $\int f(x)^n.f'(x) \, dx = \frac{j\text{X}x)^{n+1}}{n+1}$ + c ; provided n ≠ – 1

7. $\int \frac{f'(\text{x})}{f(\text{x})} d\text{x}$ = log f(x) + c

8. $\int e^{mx} dx = \frac{e^{mx}}{m}$ + c

9. $\int a^x dx = \frac{a^{\text{x}}}{\log a}$ + c

10. $\int a^{px+q}.dx = \frac{a^{p\text{x}+q}}{p \log a}$ + c

11. $\int \sin (ax+b) \, dx = \frac{-\cos(\text{xx}+b)}{a} + c$

12. $\int \cos (ax+b) \, dx = \frac{\sin(a\text{x}+b)}{a} + c$

13. $\int \tan (ax+b) \, dx = \frac{\log \sec(a\text{x}+b)}{a} + c$

14. $\int$ cot (ax+b) dx = $\dfrac{\log \sin (a\text{x}+b)}{a}+c$

15. $\int$ sec (ax+b)dx = $\dfrac{1}{a}$ log (sec (ax+b) + tan (ax+b) + c

16. $\int$ cosec (ax+b)dx = $\dfrac{1}{a}$ log [cosec (ax+b] – cot (ax+b)] + c

17. $\int$ $\sec^2$ (ax+b) dx = $\dfrac{1}{\text{a}}$ tan (ax+b)+c

18. $\int$ cosec^2 (ax + b) dx = $\dfrac{-1}{\text{a}}$ cot (ax+b) + c

19. $\int$ sec (ax + b) tan (ax + b) dx = $\dfrac{1}{\text{a}}$ sec(ax+b) + c

20. $\int$ cosec (ax + b) cor (ax + b) dx = $\dfrac{-1}{\text{a}}$ cosec (ax+b) + c

21. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{1}\left(\dfrac{x}{a}\right)+c$ or $-\cos^{1}\left(\dfrac{x}{a}\right)+c$ x

22. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)+ c$ or $\dfrac{-1}{a}\cot^{-1}\left(\dfrac{x}{a}\right)+ c$

23. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left(\dfrac{x}{a}\right)+c$ or $\dfrac{-1}{a}\text{cosec}^{-1}\left(\dfrac{x}{a}\right)+c$

24. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \log\left(\frac{x + \sqrt{a^2 + x^2}}{a}\right) + c$

25. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right) + c$

26. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a + x}{a - x} + c$

27. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x - a}{x + a} + c$

28. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \sin{-1}\left(\frac{x}{a}\right) + c$

29. $\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \log\left(\frac{x + \sqrt{a^2 + x^2}}{a}\right) + c$

30. $\int \sqrt{x^2 - a^2 dx} = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) + c$

31. $\int e^{ax} \sin bx = \frac{e^{ax}\left[a \sin bx - b \cos bx\right]}{a^2 + b^2} + c$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin\left[bx - \tan^{-1}\left(\frac{b}{a}\right)\right] + c$$

32. $\int e^{ax} \cos bx = \dfrac{e^{ax}\left[a\cos bx + b\sin bx\right]}{a^2+b^2} + c$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left[bx - \tan^{-1}\left(\frac{b}{a}\right)\right] + c$$

Note : The students once again are advised to memorise the above mentioned formula before proceeding futher.

Illustration 1. $\int \dfrac{x^5+x^3+3x^2+2x+5}{\sqrt{x}}dx$

Sol. : $\int \dfrac{x^5+x^3+3x^2+2x+5}{x^{1/2}}dx$

$$= \int\left(\frac{x^5}{x^{1/2}} + \frac{x^3}{x^{1/2}} + \frac{3x^2}{x^{1/2}} + \frac{2x}{x^{1/2}} + \frac{5}{x^{1/2}}\right)dx$$

$$= \int\left(x^{9/2} + x^{5/2} + x^{3/2} + 2x^{1/2} + 5x^{-1/2}\right)dx$$

$$= \frac{x^{9/2+1}}{9/2+1} + \frac{x^{5/2+1}}{5/2+1} + \frac{3x^{3/2+1}}{3/2+1} + \frac{2x^{1/2+1}}{1/2+1} + \frac{5x^{1/2+1}}{-1/2+1} + c$$

$$= \frac{2}{11}x^{11/2} + \frac{2}{7}x^{1/2} + \frac{6}{5}x^{5/2} + \frac{4}{3}x^{3/2} + 10\sqrt{x} + c$$

(ii) $\int \sqrt{1+\sin 2x}\, dx$

$$= \int \sqrt{1 + 2\sin x \cos x}\ dx$$

$$= \int \sqrt{\sin^2 x + \cos^2 + 2\sin x \cos x}\ dx$$

$$= \int \sqrt{\left(\sin x + \cos x\right)^2}\ dx$$

$$= \int \left(\cos x + \sin x\right) dx$$

$$= \sin x - \cos x + c$$

(iii) $$\int \frac{x^2 \tan^{-1}\left(x^3\right)}{1 + x^6} dx = \int \frac{\tan^{-1}\left(x^3\right) \cdot x^2 dx}{1 + \left(x^3\right)^2}$$

Now put $x^3 = t$

Diff.

$$3x^2\ dx = dt$$

$$\therefore \quad x^2\ dx = \frac{dt}{3}$$

$$\therefore \quad \int \frac{x^2 \tan^{-1}\left(x^3\right)}{1 + x^6} dx = \int \frac{\tan^{-1}\left(t\right) \cdot \frac{dt}{3}}{1 + t^2}$$

$$= \frac{1}{3} \int \frac{\tan^{-1}\left(t\right)}{1 + t^2} dt$$

Put $\tan^{-1}(t) = u$

Diff.

$$\frac{1}{1+t^2}dt. = du$$

$$\therefore \quad \int \frac{x^2 \tan^{-1}(x^3)}{1+x^6}\,dx = \frac{1}{3}\int u\,du$$

$$= \frac{1}{3}\ \frac{u^2}{2} + c$$

$$= \frac{1}{6}\ (\tan^{-1}(x^3))^2 + c$$

Alt. : Put $\tan^{-1}(x^3) = t$

Diff.

$$\frac{1}{1+x^6}.\ 3x^2\ dx = dt$$

$$\therefore \quad \frac{x^2}{1+x^6}dx = \frac{dt}{3}$$

$$\therefore \quad \int \frac{x^2 \tan^{-1}(x^3)}{1+x^6}\,dx = \int t.\ \frac{dt}{3}$$

$$= \frac{t^2}{3.2} + c$$

$$= \frac{1}{6}[\tan^{-1}(x^3)]^2 + c$$

REDUCTION FORMULAE

INTRODUCTION

Many functions occur whose integrals are not reducible immediately to one or other of the standard forms, and whose integrals are not obtainable directly. In some cases however may be linearly connected by some algebraic formula with the integrals of another expressions, which is either immediately integrable or relatively easier to integrate than the origin function.

Def. : An algebraic relation which connects an integral linearly with another integral in which the integrand is of the same type, but is of a lower degree or order or relatively easier to integrate is called Reduction Formula.

Usually a reduction formula has to be used separately to compute the integral of the given function. This method of integration is called integration by successive reduction.

Note : Reduction formulae are generally obtained by the method of integration by parts.

Reduction Formula for $\int \sin^n x dx$ and $\int \cos^n xd$; where n is a +ve integer.

Sol. : $\int \sin^n x dx = \int \sin^{n-1}x . \sin x dx$

$$= \int \sin^{n-1}x (-\cos x) - \int -\cos x . (n-1) \sin^{n-2}x \frac{d}{dx}(\sin x)$$

$$\int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x . \cos x . \cos x \, dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x . \cos^2 x \, dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} (1-\sin^2 x) \, dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx - (n-1) \int \sin^2 x dx$$

Transporting the last integral to the left side, we get

$$\int \sin^n x dx + (n-1)\int \sin^n x dx \quad = \; -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x dx$$

$$(1+n-1)\int \sin^n x dx \quad = \; -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x dx$$

$$\Rightarrow \quad \boxed{\int \sin^n x dx \; = \; \frac{-\cos x \sin^{n-1}x}{n} + \frac{n-1}{n}\int \sin^{n-2}x dx}$$

which is required reduction Formula for $\int \sin^n x dx$.

$$\int \cos^n x dx \qquad = \; \int \cos^{n-1}x \,.\, \cos x \, dx$$

$$= \; -\cos^{n-1} x \,.\, \sin x - \int \sin x \,.\, (n-1) \cos x^{n-2} \frac{d}{dx}(\cos x)\, dx$$

$$= \quad \sin x \cos^{n-1}x - (n-1) \int \sin x \,.\, \cos^{n-2}(-\sin x)\, dx$$

$$= \quad \sin x \cos^{n-1}x + (n-1) \int \cos^{n-2}(1-\cos^2 x)\, dx$$

$$= \quad \sin x \cos^{n-1}x + (n-1) \int \cos^{n-2}x \, dx - (n-1) \int \cos^n x \, dx$$

Transposing the last integral on L.H.S., we get

$$\int \cos^n x \, dx + (n-1) \int \cos^n x \, dx = \sin x \cos^{n-1} x \, dx + (n-1) \int \cos^n x \, dx$$

$$\Rightarrow \qquad (1+n-1)\int \cos^n x \, dx \;\; = \;\int \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \, dx$$

$$\Rightarrow \quad \boxed{\int \cos^n x \, dx \; = \; \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x \, dx}$$

which is required reduction formula for $\int \cos^n x \, dx$.

Reduction formula for $\int_0^{\pi/2} \sin^n x\, dx$ and $\int_0^{\pi/2} \cos^n x\, dx$. (These are called wall's Formulae)

Sol. : Let $S_n = \int_0^{\pi/2} \sin^n x\, dx$

Then by reduction formula for $\int \sin^n x\, dx$ we have

$$S_n = \left| \frac{-\cos x \sin^{n-1} x}{n} \right|_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$= (0-0) + \frac{n-1}{n}\left[\left| \frac{-\cos x \sin^{n-3} x}{n-2} \right|_0^{\pi/2} + \frac{n-3}{n-2} \int_0^{\pi/2} \sin^{n-4} x\, dx \right]$$

$$= \frac{n-1}{n}\left[(0+0)\frac{n-3}{n-2}. S_{n-4} \right]$$

$$= \frac{(n-1)(n-3)}{n(n-2)} S_{n-4}$$

Continuing this process, we get

$$S_n = \begin{cases} \dfrac{(n-1)(n-3).........4.2}{n(n-2)..............3.1}. S_1 \text{ ; if n is odd} \\ \dfrac{(n-1)(n-3).........3.1}{n(n-2)..............4.2}. S_0 \text{ ; if n is even} \end{cases}$$

$$\text{Where } S_1 = \int_0^{\pi/2} \sin x \, dx$$

$$= -\left|\cos x\right|_0^{\pi/2}$$

$$= -\left(\cos\frac{\pi}{2} - \cos 0\right)$$

$$= -(0 - 1)$$

$$= 1$$

$$\& \qquad S_0 = \int_0^{\pi/2} \sin^0 x \, dx$$

$$= \int_0^{\pi/2} 1 . \, dx$$

$$= \left|x\right|_0^{\pi/2}$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

$$\text{Thus } S_n = \begin{cases} \dfrac{(n-1)(n-3).........4.2}{n(n-2)..............3.1} & \text{if n is odd} \\ \dfrac{(n-1)(n-3).........3.1}{n(n-2)..............4.2} . \dfrac{\pi}{2} & \text{if n is even} \end{cases}$$

Illustration : (1) Evaluate $\int \sin^6 x\, dx$

$$\int \sin^6 x\, dx = \frac{-\cos x \sin^5 x}{6} + \frac{5}{6} \int \sin^4 x\, dx$$

$$= \frac{-\cos x \sin^5 x}{6} + \frac{5}{6} \left[\frac{-\cos x \sin^3 x}{6} + \frac{3}{4} \int \sin^2 x\, dx \right]$$

$$= \frac{-\cos x \sin^5 x}{6} - \frac{5}{24} \cos x \sin^3 x + \frac{15}{24} \left[\frac{-\cos x \sin x}{6} + \frac{1}{2} \int \sin x\, dx \right]$$

$$= \frac{-\cos x \sin^5 x}{6} - \frac{5}{2} \cos x \sin^3 x - \frac{15}{48} \cos x \sin x + \frac{15}{48} \int 1 \,.\, dx$$

$$= \frac{-\cos x \sin^6 x}{7} - \frac{5}{2} \cos x \sin^3 x - \frac{15}{48} \cos x \sin x + \frac{15}{48} x + c$$

(2) Evaluate $\int \cos^7 x\, dx$

$$\int \cos^7 x\, dx = \frac{\sin x \cos^6 x}{7} + \frac{6}{7} \int \cos^5 x dx$$

$$= \frac{\sin x \cos^6 x}{7} + \frac{6}{7} \left[\frac{\sin x \cos^4 x}{5} + \frac{4}{5} \int \cos^3 x\, dx \right]$$

$$= \frac{\sin x \cos^6 x}{7} + \frac{6}{35} \sin x \cos^4 x + \frac{24}{35} \left[\frac{\sin x \cos^2 x}{3} + \frac{2}{3} \int \cos^2 x\, dx \right]$$

$$= \frac{\sin x \cos^6 x}{7} + \frac{6}{35} \sin x \cos^4 x + \frac{24}{105} \sin x \cos^2 x + \frac{48}{105} \sin x + c$$

EXERCISE

1. Integrate with respect to x

(i) $\sin^4 x$ (ii) $\sin^5 x$ (iii) $\sin^7 x$

(iv) $\cos^6 x$ (v) $\cos^9 x$ (vi) $\cos^8 x$

(vii) $\sin^8 x$

Reduction formula for $\int_0^{\pi/2} \cos^n x \, dx$

Let $C_n = \int_0^{\pi/2} \cos^n x \, dx.$

Then by reduction formula for $\int \cos^n x dx$, we get

$$C_n = \left|\frac{\sin x \cos^{n-1} x}{n}\right|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x dx$$

$$= (0-0) + \frac{n-1}{n}\left[\left|\frac{\sin x \cos^{n-3} x}{n-2}\right|_0^{\pi/2} + \frac{n-3}{n-2}\int_0^{\pi/2} \cos^{n-4} x dx\right]$$

$$= \frac{n-1}{n}\left[(0-0) + \frac{n-3}{n-2} C_{n-4}\right]$$

$$= \frac{(n-1)(n-3)}{n\,(n-2)} C_{n-4}$$

Continuing this process, we get

$$C_n = \begin{cases} \dfrac{(n-1)(n-3).........4.2}{n(n-2).............3.1}.\ C_1 \ ; \text{if n is odd.} \\ \dfrac{(n-1)(n-3).........3.1}{n(n-2).............4.2}.\ C_0 \ ; \text{if n is even.} \end{cases}$$

Where $C_1 = \int_0^{\pi/2} \cos^n x dx$

$= \left| \sin x \right|_0^{\pi/2}$

$= \sin \dfrac{\pi}{2} - \sin 0$

$= 1 - 0 \qquad = 1$

& $\quad C_0 = \int_0^{\pi/2} \cos^0 x dx$

$= \int_0^{\pi/2} 1.\, dx$

$= \left| x \right|_0^{\pi/2}$

$= \dfrac{\pi}{2} - 0$

$= \dfrac{\pi}{2}$

Thus

$$C_n = \begin{cases} \dfrac{(n-1)(n-3)\ldots\ldots\ldots 4.2}{n(n-2)\ldots\ldots\ldots\ldots. 3.1} & \text{if n is odd.} \\ \dfrac{(n-1)(n-3)\ldots\ldots\ldots 3.1}{n(n-2)\ldots\ldots\ldots\ldots. 4.2} \cdot \dfrac{\pi}{2} & \text{if n is even.} \end{cases}$$

Illustration : Evaluate (i) $\int_0^{\pi/2} \sin^8 x dx$ and (ii) $\int_0^{\pi/2} \cos^9 x dx$

(i) $\int_0^{\pi/2} \sin^8 x dx = \dfrac{7.5.\not{3}.1}{8.\not{6}.4.2} \cdot \dfrac{\pi}{2} = \dfrac{35\pi}{256}$

(ii) $\int_0^{\pi/2} \cos^9 x dx = \dfrac{8.\not{6}.4.2}{9.7.5.\not{3}.1} = \dfrac{128}{315}$

EXERCISE

Evaluate the following definite integrals

(1) $\int_0^{\pi/2} \cos^4\theta d\theta$

(2) $\int_0^{\pi/2} \cos^6\theta d\theta$

(3) $\int_0^{\pi/4} \cos^2 2\theta d\theta$

(4) $\int_0^{\pi/4} \sin^4 2\theta d\theta$

(5) $\int_0^{a} \dfrac{x^4}{a^2 - x^2} dx$

(6) $\int_0^{\infty} \dfrac{dx}{(a^2 + x^2)^4}$

Reduction Formula For $\int \tan^n x dx$.

$$\int \tan^n x dx = \int \tan^{n-2} x.\tan^2 x dx$$

$$= \int \tan^{n-2} x.(\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x.\sec^2 x dx - \int \tan^{n-2} x dx$$

$$= \int \tan^{n-2} x.d(\tan x)\, dx - \int \tan^{n-2} x dx$$

$$= \frac{\tan^{n-2+1} x}{n-2+1} - \int \tan^{n-2} x dx$$

$$\boxed{\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx}$$

This is required reduction formula for $\int \tan^n x dx$.

Reduction formula for $\int \cot^n x dx$

$$\int \cot^n x dx = \int \cot^{n-2} x dx\, .\, \cot^2 x\, dx$$

$$= \int \cot^{n-2} x\, .\, (\mathrm{cosec}^2 x - 1)\, dx$$

$$= \int \cot^{n-2} x\, \mathrm{cosec}^2 x\, dx - \int \cot^{n-2} x dx$$

$$= \frac{-\cot x^{n-2+1}}{n-1} - \int \cot^{n-2} x\, dx$$

$$\boxed{\int \cot^n x dx = \frac{-\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx}$$

This is required reduction formula for $\int \cot^n x dx$.

Reduction formula for $\int \sec^n x dx$

$$\int \sec^n x dx \quad = \quad \int \sec^{n-2} x \,.\, \sec^2 x\, dx$$

$$= \quad \sec^{n-2} x \,.\, \tan x - \int \tan x \,.\, (n-2) \sec^{n-3} x \frac{d}{dx}(\sec x)\, dx$$

$$= \quad \sec^{n-2} x \,.\, \tan x - \int \tan x \,.\, (n-2) \sec^{n-3} x \frac{d}{dx}(\sec x)\, dx$$

$$= \quad \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x\, dx$$

$$= \quad \sec^{n-2} x \,.\, \tan x - (n-2) \int \sec^{n-2} x \,.\, (\sec^2 x - 1)\, dx$$

$$=$$

$$\sec^{n-2} x \,.\, \tan x - (n-2) \int \sec^n x\, dx + (n-2)\, dx \int \sec^{n-2} x\, dx$$

$$= \quad \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x\, dx - (n-2) \int \sec^n x\, dx.$$

Transposing the last integral on L.H.S., we get

$$\int \sec^n x\, dx + (n-2) \int \sec^n x\, dx = \tan x \sec^{n-2} x + (n-2) \int \sec^{n-2} x\, dx$$

$$\Rightarrow \qquad (1+n-2) \int \sec^n x\, dx = \tan x \sec^{n-2} x + (n-2) \int \sec^{n-2} x\, dx.$$

$$\Rightarrow \qquad (n-1) \int \sec^n x\, dx = \tan x \sec^{n-2} x + (n-2) \int \sec^{n-2} x\, dx$$

$$\Rightarrow \qquad \boxed{\int \sec^n x\, dx = \frac{\tan x \,.\, \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx.}$$

This is the required reduction formula for $\int \sec^n x dx$.

Reduction formula for $\int \mathbf{cosec^n x dx}$

$$\int \text{cosec}^n x\, dx = \int \text{cosec}^{n-2} x \,.\, \text{cosec}^2 x dx$$

$$= \int \text{cosec}^{n-2} x \,.\, (-\cot x) - \int -\cot x \,.\, (n-2)\, \text{cosec}^{n-3} x \,\frac{d}{dx}(\text{cosec } x)$$

$$= -\cot x \,.\, \text{cosec}^{n-2} x + (n-2) \int \cot x \,.\, \text{cosec}^{n-3} x \,.\, (-\text{cosec } x \cot x)\, dx$$

$$= -\cot x \,.\, \text{cosec}^{n-2} x - (n-2) \int \text{cosec}^{n-2} x \,.\, \cot^2 x\, dx$$

$$= -\cot x \,.\, \text{cosec}^{n-2} x - (n-2) \int \text{cosec}^{n-2} x\, (\text{cosec}^2 x - 1)\, dx$$

$$= -\cot x \,.\, \text{cosec}^{n-2} x - (n-2) \int \text{cosec}^n x dx + (n-2) \int \text{cosec}^{n-2} x\, dx$$

$$= -\cot x\, \text{cosec}^{n-2} x + (n-2) \int \text{cosec}^{n-2} x dx + (n-2) \int \text{cosec}^n x\, dx$$

Transposing the last integral on L.H.S., we get

$$\int \text{cosec}^n\, x dx + (n-2) \int \text{cosec}^n x\, dx = -\cot x\, \text{cosec}^{n-2} x + (n-2) \int \text{cosec}^{n-2} x$$

$$\Rightarrow \quad (1+n-2) \int \text{cosec}^n x = -\cot x\, \text{cosec}^{n-2}\, x + (n-2) \int \text{cosec}^{n-2} x\, dx$$

$$\Rightarrow \quad \int \text{cosec}^n x\, dx = \frac{-\cot x\, \text{cosec}^{n-2}\, x}{n-1} + \frac{n-2}{n-1} \int \text{cosec}^{n-2} x\, dx$$

This is the required reduction formula for $\int \text{cosec}^n x$.

Examples : (Application of Reduction Formulae)

(i) Evaluate $\int \tan^5 x \, dx$

Sol. : $\int \tan^5 x \, dx = \frac{\tan^4 x}{4} \int \tan^3 x \, dx$

$$= \frac{1}{4} \tan^4 x - \left[\frac{\tan^2 x}{2} - \int \tan^2 x dx \right]$$

$$= \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \log \sec x + c$$

(ii) Evaluate $\int \tan^6 x \, dx$

Sol. : $\int \tan^6 x \, dx = \frac{\tan^5 x}{5} \int \tan^4 x \, dx$

$$= \frac{1}{5} \tan^5 x - \left[\frac{\tan^3 x}{3} - \int \tan^2 x dx \right]$$

$$= \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + \left[\frac{\tan x}{1} - \int \tan^0 x dx \right]$$

$$= \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + \tan x - \int 1.dx$$

$$= \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + \tan x - x + c$$

(iii) Evaluate $\int \cot^6 x \, dx$

Sol. : $\int \cot^6 x \, dx \quad = \quad -\frac{\cot^5 x}{5} - \int \cot^4 x \, dx$

$$= -\frac{1}{5}\cot^5 x - \left[\frac{-\cot^3 x}{3} - \int \cot^2 x dx\right]$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x + \left[\frac{-\cot x}{1} - \int \cot^0 x dx\right]$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - \int 1.dx$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - x + c$$

(iv) Evaluate $\int \cot^7 x \, dx$

Sol. : $\int \cot^7 x \, dx \quad = \quad -\frac{\cot^6 x}{5} - \int \cot^5 x \, dx$

$$= -\frac{1}{6}\cot^6 x - \left[\frac{-\cot^4 x}{4} - \int \cot^3 x dx\right]$$

$$= -\frac{1}{6}\cot^6 x + \frac{1}{4}\cot^4 x + \left[\frac{-\cot^2 x}{2} - \int \cot x dx\right]$$

$$\int \cot^7 x\, dx = -\frac{1}{6}\cot^6 x + \frac{1}{4}\cot^4 x - \frac{1}{2}\cot^2 x - \log \sin x + c$$

(v) $$\int \sec^7 x\, dx = \frac{\tan x \sec^3 x}{4} + \frac{3}{4}\int \sec^3 x\, dx$$

$$= \frac{1}{4}\tan x \sec + \frac{3}{4}\left[\frac{\tan x \sec x}{2} + \frac{1}{2}\int \sec x dx\right]$$

$$= \frac{1}{4}\tan x \sec^3 x + \frac{3}{8}\tan x \sec x + \frac{3}{8}\log(\sec x + \tan x) + c$$

(vi) Evaluate $\int \sec^6 x\, dx$

$$\int \sec^6 x\, dx = \frac{\tan x \sec^4 x}{5} + \frac{4}{5}\int \sec^4 x\, dx$$

$$= \frac{1}{5}\tan x \sec^4 x + \frac{4}{5}\left[\frac{\tan x \sec^2 x}{3} + \frac{2}{3}\int \sec^2 x dx\right]$$

$$= \frac{1}{5}\tan x \sec^4 x + \frac{4}{15}\tan x \sec^2 x + \frac{8}{15}\tan x + c$$

(vii) Evaluate $\int \operatorname{cosec}^7 x dx$

Now

$$\int \operatorname{cosec}^7 x dx = -\frac{\cot x \operatorname{cosec}^5 x}{6} + \frac{5}{6}\int \operatorname{cosec}^5 x\, dx$$

$$= \quad -\frac{\cot x \operatorname{cosec}^5 x}{6} + \frac{5}{6}\left[\frac{-\cot x \operatorname{cosec}^3 x}{4} + \frac{3}{4}\int \operatorname{cosec}^3 x dx\right]$$

$$= \quad -\frac{1}{6}\cot x \operatorname{cosec}^5 x - \frac{5}{24}\cot x \operatorname{cosec}^3 x$$

$$+\frac{15}{24}\left[\frac{-\cot x \operatorname{cosec} x}{2} + \frac{1}{2}\int \operatorname{cosec} x\, dx\right]$$

$$= \quad -\frac{1}{6}\cot x \operatorname{cosec}^5 x - \frac{5}{24}\cot x \operatorname{cosec}^3 x - \frac{15}{48}\cot x \operatorname{cosec} x$$

$$+\frac{15}{48}\log(\operatorname{cosec} x - \cot x) + c$$

(viii) Evaluate $\int \operatorname{cosec}^6 x dx$

Now

$$\int \operatorname{cosec}^6 x dx = -\frac{\cot x \operatorname{cosec}^4 x}{5} + \frac{4}{5}\int \operatorname{cosec}^4 x\, dx$$

$$= -\frac{1}{5}\cot x \operatorname{cosec}^4 x + \frac{4}{5}\left[\frac{-\cot x \operatorname{cosec}^2 x}{3} + \frac{2}{3}\int \operatorname{cosec}^2 x\, dx\right]$$

$$= \frac{1}{5}\cot x \operatorname{cosec}^4 x - \frac{4}{15}\cot x \operatorname{cosec}^2 x + \frac{8}{15}(-\cot x) + c$$

$\in$

$$= \frac{1}{5}\cot x \operatorname{cosec}^4 x - \frac{4}{15}\cot x \operatorname{cosec}^2 x - \frac{8}{15}\cot x + c.$$

EXERCISE

Using reduction formulae to prove the following :

(i) $\int \tan^3 x dx = \frac{1}{2} \tan^2 x - \log \sec x + c$

(ii) $\int \cot^5 x\, dx = -\frac{1}{4} \cot^4 x + \frac{1}{2} \cot^2 x + \log \sin x + c$

(iii) $\int \tan^4 x\, dx = \frac{1}{3} \tan x + x + c$

(iv) $\int \sec^4 x dx = \frac{1}{3} \tan x (\sec^2 x + 2) + c$

(v) $\int \operatorname{cosec}^4 x dx = -\frac{1}{2} \operatorname{cosec} x \cot x + \frac{1}{2} \log (\operatorname{cosec} x - \cot x) + c$

(vi) $\int_0^{\pi/4} \tan^5 x dx = \frac{1}{2} \log 2 - \frac{1}{4}$

(vii) $\int_0^{\pi/4} \sec^3 x dx = \frac{1}{2}\left[\sqrt{2} + \log\left(\sqrt{2}+1\right)\right]$

(viii) If $I_m = \int_0^{\pi/4} \tan^n x dx, (n \in N)$ (*Imp.*)

prove that (i) $I_n + I_{n-2} = \frac{1}{n-1}$

(ii) $n [I_{n+1} + I_{n-1}] = 1$

(iii) Deduce the value of I_5

Reduction Formula for $\int \sin^m x \cos^n x dx$

where m, n $\in$ N

The integral $\int \sin^m x \cos^n x\, dx$ can be connected with any of the following integrals.

1. $\int \sin^{m-2} x \cos^n x dx$
2. $\int \sin^{m+2} x \cos^n x dx$
3. $\int \sin^m x \cos^{n-2} x dx$
4. $\int \sin^m x \cos^{n+2} x dx$
5. $\int \sin^{m-2} x \cos^{n+2} x dx$
6. $\int \sin^{m+2} x \cos^{n-2} x dx$ (In general)

Procedure of Connections

(1) First write down the given integral and the connector. Suppose λ and μ be the smaller of the two indices of sin x and cos x respectively in the two expressions of integral & connector.

(2) Put P = $\sin^{\lambda+1} x \cos^{\mu+1} x$

(3) Find $\frac{dP}{dx}$ and arrange it into two expressions of integral and connector.

(4) Integrate back to get required result.

Example : Let us connect

$\int \sin^m x \cos^n x dx$ with $\int \sin^{m-2} x \cos^n x dx$

Sol. : Here $\lambda = m - 2$, $\mu = n$

So $P = \sin^{\lambda+1} x \cdot \cos^{\mu+1}$

$= \sin^{m-2+1}x \cdot \cos^{n+1}x$

$= \sin^{m-1}x \cdot \cos^{n+1}x$

$\frac{dP}{dx} = (m\text{-}l) \sin^{m-2} \cdot \cos x \cdot \cos^{n+1}x + \sin^{m-1} x \cdot (n+l) \cos^{n} x \cdot (-\sin x)$

$= (m-l) \sin^{m-2} \cdot \cos^{n+2} x - (n+l) \sin^{m}x \cdot \cos^{n} x$

$= (m-l) \sin^{m-2}x \cdot \cos^{n} x - (n+l) \sin^{m}x \cdot \cos^{n} x$

$= (m-l) \sin^{m-2}x \cdot \cos^{n} x - \cos^{2}x(l-\sin^{2}x) - (n+1) \sin^{m}x \cdot \cos^{n} x$

$= (m-l) \sin^{m-2}x \cdot \cos^{n} x - (m-l) \sin^{m}x \cos^{n}x - (n+1) \sin^{m}x \cos^{n} x$

$= (m-l) \sin^{m-2}x \cos^{n} x - (m-l+n+l) \sin^{m}x \cos^{n}x$

$= (m-l) \sin^{m-2}x \cos^{n} x - (m+n) \sin^{m}x \cos^{n}x$

Integrating on both sides, we get

$$P = (m-l) \int \sin^{m\text{-}2}x \cdot \cos^{n}xdx - (m+n) \int \sin^{m}x \cos^{n}xdx$$

$$\sin^{m-1}x \cos^{n+1} x = (m-l) \int \sin^{m\text{-}2}x \cdot \cos^{n}xdx - (m+n) \int \sin^{m}x \cos^{n}xdx$$

$$\Rightarrow (m+n) \int \sin^{m}x \cos^{n}xdx = -\sin^{m\text{-}1} x \cos^{n+1}x + (m-l) \int \sin^{m\text{-}2}x \cos^{n}xdx$$

$$\Rightarrow \boxed{\int \sin^{m}x \cos^{n}xdx = \frac{-\sin^{m-1}x \cdot \cos^{n+1}x}{m+n} + \frac{m-l}{m+l} \int \sin^{m-2}x \cdot \cos^{n}xdx}$$

which is required reduction formula.

Example : Let us connect

$\int \sin^{m}x \cos^{n}xdx$ with $\int \sin^{m}x \cos^{n-2}xdx$

Sol. : Here $\lambda = m$ and $\mu = n-2$

So $\quad P = \sin^{\lambda+1} x \cos^{\mu+1} x$

$= \sin^{m+1}x \,.\, \cos^{n-2+1}$

$= \sin^{m+1}x \,.\, \cos^{n-1}$

$$\frac{dP}{dx} = \sin^{m+1} x.\ (n-1) \cos^{n-2}x\ (-\sin x) + (m+1) \sin^{m}x \,.\, \cos x \,.\, \cos^{n-1}x$$

$= -(n-1) \sin^{m+2}x \,.\, \cos^{n-2} x + (m+1) \sin^{m} x \,.\, \cos^{n} x$

$= (m+1) \sin^{m} x \cos^{n} x - (n-1) \sin^{m}x \,.\, \cos^{n-2} x \,.\, \sin^{2}x$

$= (m+1) \sin^{m} x \cos^{n} x - (n-1) \sin^{m}x \cos^{n-2} x \,.\, (1 - \cos^{2}x)$

$= (m+1) \sin^{m} x \cos^{n} x - (n-1) \sin^{m}x \cos^{n-2}x + (n-1) \sin^{m}x \cos^{n} x$

$= (m+1+n-1) \sin^{m} x \cos^{n} x - (n-1) \sin^{m}x \cos^{n-2}x$

$$\frac{dP}{dx} = (m+n) \sin^{m} x \cos^{n} x - (n-1) \sin^{m} x \cos^{n-2} x$$

By integrating both sides, we get

$$P = (m+n) \int \sin^{m}x \cos^{n}x dx - (n-1) \int \sin^{m}x \cos^{n-2}x dx$$

$$\Rightarrow \sin^{m+1}x \,.\, \cos^{n-1} x = (m+n) \int \sin^{m}x \cos^{n}x dx - (n-1) \int \sin^{m}x \cos^{n-2}x dx$$

$$\Rightarrow (m+n) \int \sin^{m}x \cos^{n}x dx = \sin^{m+1} x \cos^{n-1}x + (n-1) \int \sin^{m}x \cos^{n-2}x dx$$

$$\Rightarrow \boxed{\int \sin^{m}x \cos^{n}x dx = \frac{\sin^{m+1}x \,.\, \cos^{n-1}x}{m+n} + \frac{n-1}{m+n} \int \sin^{m}x \,.\, \cos^{n-2}x dx}$$

This is required reduction formula.

Example : (1) Evaluate $\int \sin^{2}x \cos^{4}x dx$

Now

$$\int \sin^2 x \cos^4 x dx = \frac{\sin^3 x \cos^3 x}{6} + \frac{3}{6} \int \sin^2 x \cos^2 x dx$$

$$= \frac{\sin^3 x \cos^3 x}{6} + \frac{1}{2}\left[\frac{\sin^3 x \cos x}{4} + \frac{1}{4}\int \sin^2 x \cos^0 x dx\right]$$

$$= \frac{\sin^3 x \cos^3 x}{6} + \frac{1}{8} \sin^3 x \cos x + \frac{1}{8}\int \sin^2 x\, dx$$

$$= \frac{\sin^3 x \cos^3 x}{6} + \frac{1}{8} \sin^3 x \cos x + \frac{1}{8}\left[\frac{\cos x \sin x}{2} + \frac{1}{2}\int \sin^0 x\, dx\right]$$

$$= \frac{\sin^3 x \cos^3 x}{6} + \frac{1}{8} \sin^3 x \cos x - \frac{1}{16} \sin x \cos x + \frac{1}{16}\int 1.\, dx$$

$$= \sin^3 x \cos^3 x + \frac{1}{8} \sin^3 x \cos x - \frac{1}{16} \sin x \cos x + \frac{1}{16} x + c$$

(2) Evaluate $\int \sin^4 x \cos^3 x dx$

$$\int \sin^4 x \cos^3 x dx = \frac{\sin^5 x \cos^2 x}{7} + \frac{2}{7} \int \sin^4 x \cos x dx$$

$$= \frac{\sin^5 x \cos^2 x}{7} + \frac{2}{7} \frac{\sin^5 x}{5} + c$$

$$\left(\because \int f^n(x) f'(x)\, dx = \frac{f^{n+1}(x)}{n+1}\right)$$

$$= \sin^5 x \left[\frac{1}{7} \cos^2 x + \frac{2}{35}\right] + c$$

(3) Evaluate $\int \sin^2 x \cos^2 x dx$

$$\int \sin^2 x \cos^2 x dx = \frac{-\sin x \cos^3 x}{4} + \frac{1}{4} \int \sin^0 x \cos^2 x dx$$

$$= \frac{-\sin x \cos^3 x}{4} + \frac{1}{4} \int \cos^2 x dx$$

$$= \frac{-\sin x \cos^3 x}{4} + \frac{1}{4}\left[\frac{\sin x \cos x}{2} + \frac{1}{2} \int \cos^0 x \, dx\right]$$

$$= \frac{-\sin x \cos^3 x}{4} + \frac{1}{8} \sin x \cos x + \frac{1}{2} x + c$$

(4) Evaluate $\int \sin^4 x \cos^2 x dx$

$$\int \sin^4 x \cos^2 x dx = \frac{-\sin^3 x \cos^3 x}{6} + \frac{3}{6} \int \sin^2 x \cos^2 x dx$$

$$= \frac{-\sin^3 x \cos^3 x}{6} + \frac{1}{2}\left[\frac{-\sin x \cos^3 x}{4} + \frac{1}{4} \int \sin^0 x \cos^2 dx\right]$$

$$= \frac{-\sin^3 x \, . \cos^3 x}{6} - \frac{1}{8} \sin x \cos^3 x + \frac{1}{8} \int \sin^0 x \cos^2 x dx$$

$$= \frac{-\sin^3 x \cos^3 x}{6} - \frac{1}{8} \int \sin x \cos^3 x$$

$$+ \frac{1}{8}\left[\frac{\sin x \cos x}{2} + \frac{1}{2} \int \cos^0 x \, dx\right]$$

$$= \frac{-\sin^3 x \cos^3 x}{6} - \frac{1}{8} \sin x \cos^3 x + \frac{1}{16} \int (\sin x \cos x + x) + c$$

Note : In $\int \sin^m x \cos^n x dx$ it is very easy (in practice) to solve various Questions as under :

(i) If m and n are even m < n, then use the formula

$$\int \sin^m x \cos^n x dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x dx$$

(ii) If m and n are even and m > n, then use the formula

$$\int \sin^m x \cos^n x dx = \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x dx.$$

(iii) If m is even and n is odd, then use the formula

$$\int \sin^m x \,.\, \cos^n x dx = \frac{\sin^{m+1} x \,.\, \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \,.\, \cos^{n-2} x dx.$$

(iv) If m is odd and n is even then use the formula

$$\int \sin^m x \,.\, \cos^n x dx = \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-n}{m+n} \int \sin^{m-2} x \,.\, \cos^n x dx$$

Reduction Formula for $\int_0^{\pi/2} \sin^m x \,.\, \cos^n x dx$

First connect $\int \sin^m x \,.\, \cos^n x dx$ with $\int \sin^{m-2} x \cos^n x dx$

.... as done above .... we get

$$\int \sin^m x \cos^n x dx = \frac{-\sin^{m-1}x \cos^{n+1}x}{m+n} + \frac{m+1}{m+n}\int \sin^{m-2}x \cos^n x dx$$

Now let $l_{m,n} = \int_0^{\pi/2} \sin^m x \,.\cos^n x dx$

Then

$$l_{m,n} = \left|\frac{-\sin^{m-1}x\,.\cos^{n+1}x}{m+n}\right|_0^{\pi/2} + \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2}x\,.\cos^n xdx$$

$$= (0-0) + \frac{m-1}{m+n}\left[\left|\frac{-\sin^{m-3}x\,.\cos^{n+1}x}{m+n-2}\right|_0^{\pi/2} + \frac{m-3}{m+n-2}\int_0^{\pi/2} \sin^{m-4}x\,.\cos^n xdx\right]$$

$$= \frac{m-1}{m+n}\left[(0-0) + \frac{m-3}{m+n-2}\ l_{m-4,n}\right]$$

$$= \frac{(m-1)(m-3)}{(m+n)(m+n-2)}\ l_{m-4,n} \qquad ......(1)$$

Again connecting $\int \sin^m x \cos^n x dx$ with $\int \sin^m x \cos^{n-2} x dx$, we get

$$\int \sin^m x\ .\ \cos^n x dx = \frac{\sin^{m+1}x\ .\ \cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\ .\ \cos^{n-2}xdx$$

$$\text{So } l_{m,n} = \left|\frac{\sin^{m+1}x \cos^{n-1}}{m+n}\right|_0^{\pi/2} + \frac{n-1}{m+n}\int \sin^{m-2}x\,.\cos^{n-2} xdx$$

$$1_{m,\,n} = (0–0)+\frac{n-1}{m+n}\left[\left|\frac{\sin^{m+1}x\cos^{n-3}x}{m+n-2}\right|_0^{\pi/2}+\frac{n-3}{m+n-2}\int_0^{\pi/2}\sin^m x\cos^{n-4}x\,dx\right]$$

$$= \frac{n-1}{m+n}\left[(0-0)+\frac{n-3}{m+n-2}1_{m,n-4}\right]$$

$$= \frac{(n-1)(n-3)}{(m+n)(m+m-2)}\,1_{m,\,n-4} \qquad \text{....... (2)}$$

Generalising (1) & (2), we get

$$1_{m,n} = \begin{cases} \dfrac{(m-1)(m-3)\text{.........}4.2}{(m+n)(m+n-2)\text{..........}}1_{1,n} \text{ if m is odd} \\ \dfrac{(m-1)(m-3)\text{.............}3.1}{(m+n)(m+n-2)\text{...........}}1_{0,n} \text{ if m is even} \\ \dfrac{(n-1)(n-3)\text{...............}4.2}{(m+n)(m+n-2)\text{...........}}1_{m,1} \text{ if n is odd} \\ \dfrac{(n-1)(n-3)\text{..............}3.1}{(m+n)(m+n-2)\text{.............}}1_{m,0} \text{ if n is even} \end{cases}$$

where $1_{1,n} = \int_0^{\pi/2}\sin x\cos^n x\,dx$

$$= \left|\frac{-\cos^{n+1}x}{n+1}\right|_0^{\pi/2}$$

$$= -\left(0 - \frac{1}{n+1}\right)$$

$$= \frac{1}{n+1}$$

$$1_{0,\ n} = \int_0^{\pi/2} \sin^0 x \cdot \cos^n x dx$$

$$= \int_0^{\pi/2} \cos^n x dx$$

$$= \begin{cases} \dfrac{(n-1)(n-3)...........4.2}{n(n-2)................3.1} \text{ if n is odd} \\ \dfrac{(n-1)(n-3).............3.1}{n(n-2)..................4.2}\dfrac{\pi}{2} \text{ if n is even} \end{cases}$$

$$1_{m,1} = \int_0^{\pi/2} \sin^m x \cos x dx$$

$$= \left|\frac{\sin^{m+1} x}{m+1}\right|_0^{\pi/2}$$

$$= \frac{1}{m+1} - 0$$

$$= \frac{1}{m+1}$$

$$1_{m,0} = \int_0^{\pi/2} \sin^m x \cdot \cos^0 x dx$$

$$= \int_0^{\pi/2} \sin^m x dx$$

$$=$$

Thus combining all the above cases we have the following result :

$$1m,n = \begin{cases} \dfrac{(m-1)(m-3)..........4.2}{(m+n)(m+n-2)........(n+1)} \text{ if m is odd \& n is even} \\ \dfrac{(n-1)(n-3)................4.2}{(m+n)(m+n-2)................(m+1)} \text{ if m is even \& n is odd} \\ \dfrac{(m-1)(m-3)..........4.2(n-1)(n-3)........4.2}{(m+n)(m+n-2).............4.2} \text{ if both m \& n are odd} \\ \dfrac{(m-1)(m-3)..........3.1(n-1)(n-3).......3.1}{(m+n)(m+n-2)..............4.2}\dfrac{\pi}{2} \text{ if both m \& n are even} \end{cases}$$

Example : (i) Evaluate $\int_0^{\pi/2} \sin^6 x \cos^9 x dx$

$$\int_0^{\pi/2} \sin^6 x \cdot \cos^9 x dx = \frac{8 \cdot \cancel{6}^{2} \cdot 4 \cdot 2}{15 \cdot 13 \cdot 11 \cdot \cancel{9}_{3} \cdot 7}$$

$$= \frac{128}{45045}$$

(ii) Evaluate $\int_0^{\pi/2} \sin^{11} x \cos^4 x dx$

$$\int_0^{\pi/2} \sin^{11} x \cos^4 x dx = \frac{1\cancel{0}^2 \cdot 8 \cdot \cancel{6}^2 \cdot 4 \cdot 2}{15 \cdot 13 \cdot 11 \cdot \cancel{9}_3 \cdot 7 \cdot \cancel{5}}$$

$$= \frac{256}{45045}$$

(ii) Evaluate $\int_0^{\pi/2} \sin^4 x \cos^6 x dx$

$$\int_0^{\pi/2} \sin^4 x \cos^6 x dx = \frac{3 \cdot 1 \cdot \cancel{5} \cdot 3 \cdot 1}{1\cancel{0}_2 \cdot 8 \cdot \cancel{6}_2 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{3\pi}{512}$$

(iv) Evaluate $\int_0^{\pi/2} \sin^3 x \cos^5 x dx$

$$\int_0^{\pi/2} \sin^3 x \cos^5 x dx = \frac{2 \cdot 4 \cdot 2}{8 \cdot 6 \cdot 4 \cdot 2} = \frac{1}{24}$$

(v) Evaluate $\int_0^1 x^5 \left(1 - x^2\right)^{5/2} dx$

Sol. : Let $I = \int_0^1 x^5 \left(1-x^2\right)^{5/2} dx$

Put $x = \sin\theta$

diff $dx = \cos\theta \, d\theta$

when $x \to 0$, $\theta \to 0$

$x \to 1$, $\theta \to \frac{\pi}{2}$

$$\therefore I = \int_0^{\pi/2} \sin^5\theta \cdot \left(1-\sin^2\theta\right)^{5/2} \cdot \cos\theta d\theta$$

$$= \int_0^{\pi/2} \sin^5\theta \left(\cos^2\theta\right)^{5/2} \cdot \cos\theta d\theta$$

$$= \int_0^{\pi/2} \sin^5\theta \cdot \cos\theta^5 \cdot \cos\theta d\theta$$

$$= \int_0^{\pi/2} \sin^5\theta \cdot \cos^6 d\theta$$

$$= \frac{4 \cdot 2}{11 \cdot 9 \cdot 7}$$

$$= \frac{8}{693}$$

(vi) Evaluate $\int_0^{\infty} \frac{x^5}{\left(1+x^2\right)^6} dx$

Let $I = \int_0^{\infty} \frac{x^5}{\left(1+x^2\right)^6} dx$

Put $x = \tan\theta$

Diff

$dx = \sec^2\theta \, d\theta$

when $x \to 0, \theta \to 0$

$x \to \infty, \theta \to \frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\tan^5\theta}{\left(1+\tan^2\theta\right)^6} \cdot \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{\tan^5\theta}{\left(\sec^2\theta\right)^6} \cdot \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{\tan^5\theta}{\sec^{12}\theta} \cdot \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{\sin^5\theta}{\cos^5\theta} \cdot \cos^{12}\theta \frac{1}{\cos^2\theta} d\theta$$

$$= \int_0^{\pi/2} \sin^5\theta \cdot \cos^5\theta d\theta$$

$$= \frac{4\cdot2\cdot4\cdot2}{10\cdot8\cdot6\cdot4\cdot2}$$

$$= \frac{1}{60}$$

(vii) Evaluate $\int_0^{2a} x^2 \left(2ax - x^2\right)^{1/2} \cdot dx$

Let $I = \int_0^{2a} x^2 \left(2ax - x^2\right)^{1/2} \cdot dx$

Put $x = 2a \sin^2\theta$

Diff.

$dx = 4a \sin\theta \cdot \cos\theta$

when $x \to 0, \theta \to 0$

$x \to 2a, \theta \to \frac{\pi}{2}$

$$\therefore\ I = \int_0^{\pi/2} 4a^2 \sin^4\theta \, (2a\cdot 2a \sin^2\theta - 4a^2\sin^2\theta)^{1/2} \cdot 4a \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} 4a^2 \sin^4\theta \cdot 2a \sin^2\theta(1-\sin^2\theta)^{1/2} \cdot 4a \sin\theta \cos\theta \, d\theta$$

$$= 4a^2 \cdot 2a\cdot4a \int_0^{\pi/2} \sin^4\theta \cdot \sin\theta \cdot (\cos^2\theta)^{1/2} \sin\theta \cos\theta \, d\theta$$

$$= 32a^4 \int_0^{\pi/2} \sin^6\theta \cdot \cos\theta \cdot \cos\theta \, d\theta$$

$$= 32a^4 \int_0^{\pi/2} \sin^4\theta \cdot \cos^2\theta \, d\theta$$

$$= 32a^4 \cdot \frac{5 \cdot 3 \cdot 1 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \frac{\pi}{2}$$

$$= \frac{5\pi a^4}{8} 3$$

EXERCISE

Q 1. Obtain Reduction formula for

$\int \sin^m x \cos^n x \, dx$ and hence evaluate $\int \sin^4 x \cos^5 x \, dx$.

Q 2. Obtain Reduction formula for $\int \sin^m x \cos^n x \, dx$ in terms of

$\int \sin^{m-2} x \cdot \cos^n x \, dx$.

Q 3. Obtain reduction formula for

$\int \sin^m x \cdot \cos^n x \, dx$ in terms of $\int \sin^m x \cdot \cos^{n-2} x \, dx$

Q 4. Obtain reduction formula for $\int \tan^n x dx$ and hence evaluate $\int \tan^5 x dx$.

Q 5. Obtain reduction formula for $\int_0^{\pi/2} \sin^m x \cdot \cos^4 x dx$ and hence evaluate :

(i) $\int_0^{\pi/2} \sin^{10} x \,.\, \cos^4 x dx$ (ii) $\int_0^{\pi/2} \sin^7 x \,.\, \cos^5 x dx$

(iii) $\int_0^{\pi/2} \sin^6 x.\cos^7 x dx$ (iv) $\int_0^{\pi/2} \sin^5 x.\cos^4 x dx$

Q6. If m and n are +ve integers, show that

$$\int_0^1 x^m (1-x)^n dx = \frac{m!n!}{(m+n+1)}$$

Q7. Show that $\int_0^\infty \frac{x^m dx}{(1+x)^{m+n+2}} = \frac{m!n!}{(m+n+1)!}$

Q8. Show that $\int_0^1 x^4 (1-x^2)^{3/2} dx = \frac{3\pi}{256}$

Q9. Show that $\int_0^1 x^{3/2} (1-x)^{3/2} dx = \frac{3\pi}{128}$

Q10. Show that $\int_0^{2a} x\sqrt{2ax - x^2} dx = \frac{\pi a^3}{2}$

Q11. Show that $\int_0^\infty \frac{dx}{(a^2 + x^2)^4} = \frac{5\pi}{32a^7}$

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Rajouri

RECTIFICATION

Definition

Formulae for finding Rectification

Solved Examples

Exercise

Def. : The process of finding the length of an are of a curve between two given points is called rectification.

Any formula expressing the differential coefficient of are length S established in different differential calculas at once give rise by integration to a formula for find the are length depending upon the nature of the equation representing the curve.

For Cartesian equations of the form y = f(x)

If the length of an are of a curve measured from a fixed point A for which x = a to another point P(x, y) on the curve be S, then

$$\frac{dS}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\therefore \quad S = \int_a^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Thus the length of any are AB of the curve between two points A and B for which abscissae are a and b respectively is

$$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

For Cartesian equation of the type x = f (y)

As explained above in this case

$$\frac{dS}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

Thus if the ordinates of any two points C and D on the curve be c and d respectively then the length of are CD is given by d

$$S = \int_c^d \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dy$$

Illustrative Examples

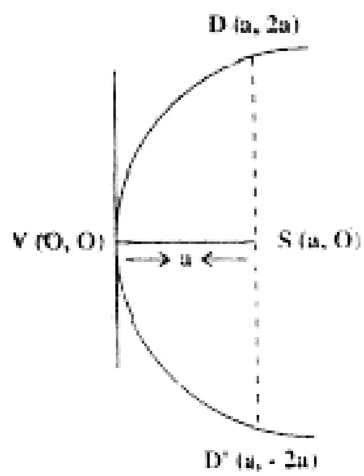
(1) Find the length of the parabola $y^2 = 4ax$ from the vertex to one of the extremity of the latus rectum.

Sol. : Given parabola is $y^2 = 4ax$ .... (1)

Let S(a, 0) be the focus, D and D' be the ends of latus rectum.

Let V (0, 0) be the vertex.

If S denotes the are length from vertex to one of the extremity of the latus rectum. Then the required length is VD.



From $y^2 = 4ax$, we have

$$x = \frac{y^2}{4a}$$

Diff. w. r to y, we get

$$\frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$$

$$\therefore \quad S = \int_{V(0,0)}^{D(a,2a)} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^{2a} \sqrt{1+\frac{y^2}{4a^2}}\, dy$$

$$= \frac{1}{2a}\int_0^{2a} \sqrt{4a^2+y^2}\, dy$$

$$= \frac{1}{2a}\left[\left|\frac{y}{2}\sqrt{4a^2+y^2}+\frac{4a^2}{2}\log\left(\frac{y+\sqrt{4a^2+y^2}}{2a}\right)\right|_0^{2a}\right]$$

$$\left(\because \int\sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left(\frac{x+\sqrt{a^2+x^2}}{a}\right)\right)$$

$$= \frac{1}{2a}\left\{\frac{2a}{2}\sqrt{4a^2+4a^2}+\frac{4a^2}{2}\log\left(\frac{2a+\sqrt{4a^2+4a^2}}{2}\right)\right\}$$

$$-\left\{0+\frac{4a^2}{2}\log\left(0+\frac{\sqrt{4a^2}}{2a}\right)\right\}$$

$$= \frac{1}{2a}\left[a.2a\sqrt{2}+2a^2\log\left(1+\sqrt{2}\right)-0\right] \qquad (\because \log 1 = 0)$$

$$= \frac{2a^2}{2a}\left[\sqrt{2}+\log\left(\sqrt{2+1}\right)\right]$$

$$= a\left[\sqrt{2}+\log\left(\sqrt{2}+1\right)\right]$$

Example 2 : Find the length of the curve y = log sec x from x = 0 to x $= \frac{\pi}{3}$

Sol. : Given equation of the curve y = log sec x

Diff. w.r. to x, we get

$$\frac{dy}{dx} = \frac{1}{\sec x}\frac{d}{dx}(\sec x)$$

$$= \frac{1}{\sec x}\cdot\sec x \tan x$$

$$= \tan x$$

If S denotes the are length of the given curve from x = 0 to x = $\frac{\pi}{3}$.

Then

$$S = \int_0^{\pi/3} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\pi/3} \sqrt{1+\tan^2}\, dx$$

$$= \int_0^{\pi/3} \sec x\, dx$$

$$= \left|\log(\sec x + \tan x)\right|_0^{\pi/3}$$

$$= \log\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right) - \log\ (\sec\ 0 + \tan\ 0)$$

$$= \log\ \left(2+\sqrt{3}\right) - \log\ (1+0)$$

$$= \log\ \left(2+\sqrt{3}\right).$$

Example 3 : Show that the length of the curve y = log $\left(\frac{e^x - 1}{e^x + 1}\right)$ from x = 1 to x = 2 is log $\left(e + \frac{1}{e}\right)$

Sol. : The equation of the curve is

$$y = \log\left(\frac{e^x - 1}{e^x + 1}\right)$$

Diff. w.r. to x, we get

$$\frac{dx}{dy} = \frac{1}{\frac{e^x - 1}{e^x + 1}} \frac{d}{dx}\left(\frac{e^x - 1}{e^x + 1}\right)$$

$$= \frac{e^x + 1}{e^x - 1} \cdot \frac{\left(e^x + 1\right).e^x - \left(e^x - 1\right).e^x}{\left(e^x + 1\right)^2}$$

$$= \frac{e^x \cdot e^x + e^x - e^x.e^x + e^x}{\left(e^x - 1\right)\left(e^x + 1\right)} = \frac{2e^x}{e^{2x} - 1}$$

If S denotes the length of the curve then

$$S = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \left(\frac{2e^x}{e^{2x} - 1}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{\frac{\left(e^{2x} - 1\right)^2 + 4e^{2x}}{e^{2x} - 1}}\, dx$$

$$= \int_1^2 \frac{e^{2x}+1}{e^{2x}-1}\,dx$$

$$= \int_1^2 \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}\,dx$$

$$= \left|\log\left(e^{x}-e^{-x}\right)\right|_1^2 \qquad \left(\because \int \frac{f`(x)}{f(x)}dx = \log f(x)\right)$$

$$= \log\ (e^2 – e^{-2}) – \log(e – e^{-1})$$

$$= \log\ \left(e^2 - \frac{1}{e^2}\right) – \log\ \left(e - \frac{1}{e}\right)$$

$$= \log\ \left(\frac{e^2 - \frac{1}{e^2}}{e - \frac{1}{e}}\right)$$

$$= \log\ \left(\frac{\left(e + \frac{1}{e}\right)\left(e - \frac{1}{e}\right)}{e - \frac{1}{e}}\right)$$

$$= \log\ \left(e + \frac{1}{e}\right)$$

Example 4 : Find the length of the are of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$.

Sol. : Given parabola $y^2 = 4ax$ .... (1)

Given line is $3y = 8x$ .... (2)

$$\Rightarrow \quad y = \frac{8}{3}x \qquad \text{....... (3)}$$

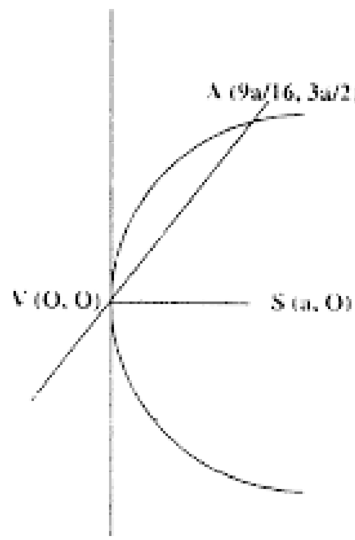
Making use of (3) in (1), we get

$$\frac{64}{9}x^2 = 4ax$$

$$\frac{16}{9}x^2 - 9x = 0$$

$$x\left(\frac{16}{9}x - a\right) = 0$$

$$\Rightarrow \quad x = 0, \frac{9a}{16}$$



when $x = 0, y = 0$

$$x = \frac{9a}{16} \text{ then}$$

$$y = \frac{8}{3} \cdot \frac{9a}{16} = \frac{\cdot 3a}{2}$$

From (1) $x = \dfrac{y^2}{4a}$

Diff. w.r. to y, we get

$$\frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$$

If S denotes the are length of parabola cut off by the line. Then

$$S = \int_0^{3a/2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy$$

$$= \int_0^{3a/2} \sqrt{1+\frac{y^2}{(2a)^2}}\cdot dy$$

$$= \frac{1}{2a}\int_0^{3a/2} \sqrt{(2a)^2+y^2}$$

$$= \frac{1}{2a}\left[\left|\frac{y}{2}\sqrt{4a^2+y^2}+\frac{4a^2}{2}\log\left(\frac{y+\sqrt{4a^2}+\sqrt{y^2}}{2a}\right)\right|_0^{3a/2}\right]$$

$$= \frac{1}{2a}\left\{\frac{3a}{4}\sqrt{4a^2+\frac{9a^2}{4}}+2a^2\log\frac{\left(\frac{3}{2}a+\sqrt{4a^2+\frac{9a^2}{4}}\right)}{2a}\right\}$$

$$-\left\{0+2a^2\log\left[\frac{0+\sqrt{4a^2+0}}{2a}\right]\right\}$$

$$S = \frac{1}{2a}\left[\frac{3a}{4}\cdot\frac{5a}{2} + 2a^2 \log\left(\frac{\frac{3}{2}a + \frac{5a}{2}}{2a}\right) - 2a^2 \log 1\right]$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + 2a^2 \log 2 - 0\right]$$

$$= a\left[\frac{15}{16} + \log 2\right]$$

Example 5 : Find the length of the are of $y^2 = x^3$ from $x = 0$ to $x = 5$.

Sol. : Given equation of the curve

$$y^2 = x^3$$

$$\Rightarrow \quad y = x^{3/2}$$

Diff. w.r. to x, we get

$$\frac{dx}{dy} = \frac{3}{2}x^{1/2}$$

If S denotes the are length from $x = 0$ to $x = 5$, then

$$S = \int_0^5 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dx$$

$$= \int_0^5 \sqrt{1 + \frac{9x}{4}}\, dx$$

$$= \frac{1}{2}\int_0^5 (4+9x)^{1/2}\, dx$$

$$= \frac{1}{2}\left|\frac{(4+9x)^{3/2}}{\frac{3}{2}}\cdot\frac{1}{9}\right|_0^5$$

$$= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{9}\,[(49)^{3/2}-(4)^{3/2}]$$

$$= \frac{1}{27}\left[\left(7^2\right)^{3/2}-\left((2)^2\right)^{3/2}\right]$$

$$= \frac{1}{27}\,[7^3-2^3]$$

$$= \frac{1}{27}\,[343-8]$$

$$= \frac{335}{27}$$

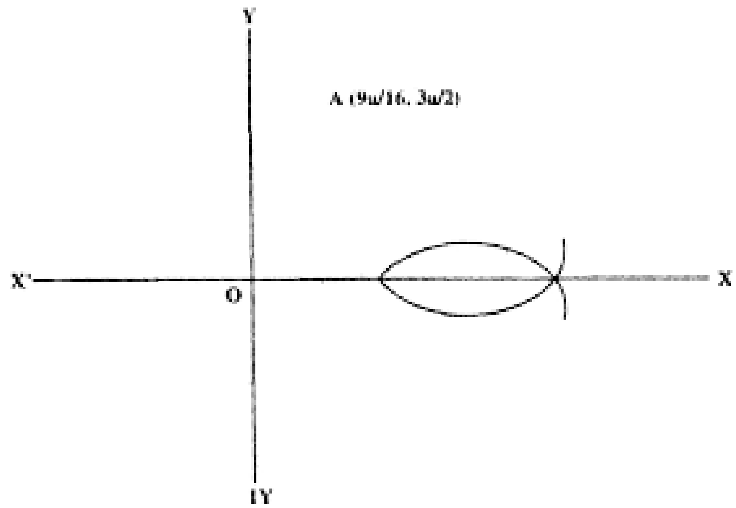
Example 6 : Find the perimeter of the loop fo the curve

$9ay^2 = (x-2a)\,(x-5a)^2$

Sol. : The equation of the curve is

$9\,ay^2 = (x-2a)\,(x-5a)^2$ .... (1)

The curve is symmetrical about x-axis and meet x-axis at x = 2a and x = 5a and hence the rough sketch of the curve is shown in the figure.



Also from (1)

$$y = \frac{(x-2a)^{-1/2}(x-5a)}{3\sqrt{a}}$$

Diff. w. r. to x, we get

$$\frac{dy}{dx} = \frac{1}{3\sqrt{a}}\left[(x-2a)^{1/2} + (x-5a)\cdot\frac{1}{2}(x-2a)^{-1/2}\right]$$

$$= \frac{1}{3\sqrt{a}}\left|\frac{2x-4a+x-5a}{2\sqrt{x-2a}}\right|$$

$$= \frac{1}{3\sqrt{a}}\left[\frac{3x-9a}{2\sqrt{x-2a}}\right]$$

$$= \frac{x-3a}{2\sqrt{a}\cdot\sqrt{x-2a}}$$

If S denotes the perimeter of loop of the curve then, $$S = 2\int_{2a}^{5a}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dx$$

$$= 2\int_{2a}^{5a}\sqrt{1+\frac{(x-3a)^{\ 2}}{4a(x-2a)}}\ dx$$

$$S = \frac{2}{2\sqrt{a}}\int_{2a}^{5a}\frac{\sqrt{4ax-8a^2+x^2+9a^2-6ax\cdot dx}}{\sqrt{x-2a}}$$

$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a}\frac{x-a}{\sqrt{x-2a}}dx$$

$$= \frac{1}{\sqrt{a}}\int_{2a}^{5n}\frac{x-2a+a}{\sqrt{x-2a}}dx$$

$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a}\left[(x-2a)^{1/2}+a(x-2a)^{-1/2}\right]dx$$

$$= \frac{1}{\sqrt{a}}\left|\frac{(x-2a)^{3/2}}{\frac{3}{2}}+\frac{a(x-2a)^{1/2}}{\frac{1}{2}}\right|_{2a}^{5a}$$

$$= \frac{1}{\sqrt{a}}\left[\left\{\frac{2}{3}(3a)^{3/2}+2a\cdot(3a)^{1/2}\right\}-\left\{\frac{2}{3}(0)+2a(0)\right\}\right]$$

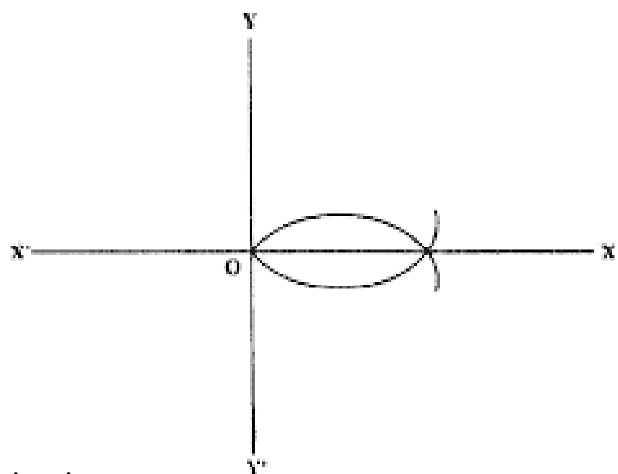
$$= \frac{1}{\sqrt{a}}\left[\frac{2}{3}\cdot 3a\sqrt{3a} + 2a\sqrt{3a}\right]$$

$$= \frac{1}{\sqrt{a}}\cdot 4a\sqrt{3a}$$

$$= \frac{4a\cdot\sqrt{3}\cdot\sqrt{a}}{\sqrt{a}} = 4\sqrt{3a}$$

Example 7 : Find the perimeter of the loop of the curve $3ay^2 = x\,(x-a)^2$ $(a<0)$

Sol. : The curve is symmetrical about x-axis and it meet x-axis at x = 0 and x = a. Hence a rough sketch of the curve is shown in the figure.



The given equation is

$$3ay^2 = x(x-a)^2$$

$$y^2 = \frac{x(x-a)^2}{3a}$$

$$y = \frac{\sqrt{x}\cdot(x-a)}{\sqrt{3a}}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{\sqrt{3a}}\left[\sqrt{x} + (x-a)\cdot\frac{1}{2}x^{1/2}\right]$$

$$= \frac{1}{\sqrt{3a}}\left[\sqrt{x} + \frac{x-a}{2\sqrt{x}}\right]$$

$$= \frac{3x-a}{2\sqrt{x}\cdot\sqrt{3a}}$$

If S denotes the length of the loop of curve then

$$S = 2\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$$

$$= 2\int_0^a \sqrt{1+\frac{(3x-a)^2}{3a.4x}}\cdot dx$$

$$= 2\int_0^a \frac{\sqrt{12ax+9x^2+a^2-6ax}}{2\sqrt{a}\cdot\sqrt{x}\cdot\sqrt{3}}\cdot dx$$

$$= \frac{1}{\sqrt{3}\cdot\sqrt{a}}\int_0^a \left(\frac{3x+a}{\sqrt{x}}\right)\cdot dx$$

$$= \frac{1}{\sqrt{a}}\frac{1}{\sqrt{3}}\int_0^a \left(\frac{3x}{\sqrt{x}} + \frac{a}{\sqrt{x}}\right)\cdot dx$$

$$= \frac{1}{\sqrt{3}}\frac{1}{\sqrt{a}}\int_0^a 3x^{1/2} + ax^{1/2}\, dx$$

$$= \frac{1}{\sqrt{3}}\frac{1}{\sqrt{a}}\left|\frac{3x^{3/2}}{\frac{3}{2}} + a\cdot\frac{x^{1/2}}{\frac{1}{2}}\right|_0^a$$

$$= \frac{1}{\sqrt{3}}\frac{1}{\sqrt{a}}\left[\frac{2}{3}\cdot 3a^{3/2} + 2a\cdot a^{1/2} - 0\right]$$

$$= \frac{1}{\sqrt{3}}\frac{1}{\sqrt{a}}\left[4a\sqrt{a}\right]$$

$$= \frac{4a}{\sqrt{3}}$$ ***Ans.***

Example 8 : If S be the length of are of the curve $y = c \cosh\left(\frac{x}{c}\right)$ from the vertex (0, 0) to the point (x, y), then show that $y^2 = c^2+s^2$

Sol. : The equation of the curve is $y = c \cosh\left(\frac{x}{c}\right)$ .... (1)

Diff. w.r. to x, we get

$$\frac{dy}{dx} = c \sinh\left(\frac{x}{c}\right)\frac{dy}{dx}\left(\frac{x}{c}\right)$$

$$= c\ \sinh\left(\frac{x}{c}\right)\cdot\frac{1}{c}$$

$$= \sinh\ \left(\frac{x}{c}\right)$$

Since S denotes the length of are measured from x = 0 to x = x

$$\therefore \quad S = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \int_0^x \sqrt{1+\sin^2 h\left(\frac{x}{c}\right)}\, dx$$

$$= \int_0^x \sqrt{\cosh^2\left(\frac{x}{c}\right)}\, x$$

$$= \int_0^x \sqrt{\cosh\left(\frac{x}{c}\right)}\, dx$$

$$= \left|\frac{\sinh\left(\frac{x}{c}\right)}{\frac{1}{c}}\right|_0^x$$

$$= c\left[\sinh\left(\frac{x}{c}\right) - \sinh(0)\right]$$

$$S = c \sinh\left(\frac{x}{c}\right)$$

Sq. on both sides, we get

$$S^2 = c^2 \sin^2 h\left(\frac{x}{c}\right)$$

$$= c^2\left[\cos^2 h\left(\frac{x}{c}\right) - 1\right]$$

$$= c^2\left[\frac{y^2}{c^2} - 1\right] \qquad \text{from (1)}$$

$$S^2 = y^2 - c^2$$

$$\Rightarrow \boxed{y^2 = c^2 + S^2} \qquad \text{Hence proved.}$$

EXERCISE

Q 1. Find the length of the arc of the curve $y^2 = x^3$ from (0, 0) to (4, 8).

Q 2. Find the length of the arc of the circle $x^2 + y^2 - 2ax = 0$ in the first quadrant.

Q 3. Prove that whole length of the curve $8a^2y^2 = x^2 (a^2 - x^2)$ is $\pi a\sqrt{2}$

Q 4. Find the length of the loop of the curve

$$9y^2 = (x - 2)(x - 5)^2$$

Q 5. Find the length of the parabola $y = x^2$ from $x = 0$ and $x = 1$.

Q 6. Show that the perimeter of the curve :

$x^{2/3} + y^{2/3} = a^{2/3}$ is 6a.

Q 7. Find the length between $x = a$ and $x = b$ of the curve $e^y = \frac{e^x - 1}{e^x + 1}$

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Rajouri

VOLUMES AND SURFACES OF SOLIDS OF REVOLUTIONS

Volume of a Solid of Revolution

Introduction & Definition

Formulae for Volumes

Solved Examples

Exercise

Surface of a Solid of revolution

Introduction

Formulae for finding surface

Solved Examples

Exercise

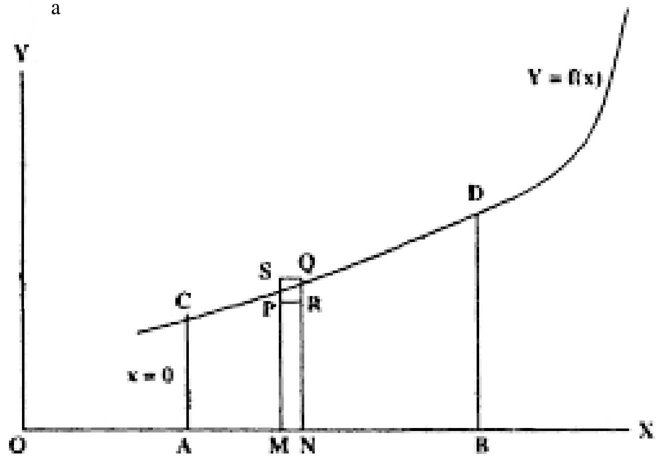
Volumes of Solids of Revolution

INTRODUCTION

If a plane area is revolved about a fixed straight line in its own plane, then the body so generated by the area is called the solid of revolution. The fixed line about which the plane area rotates is called the axis of revolution.

Art :- Prove that the volume of the solid generated by the revolution of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ about x-axis is

$$\int_a^b \pi y^2 \, dx$$



Proof :- Let the equation of the curve CD be y = f(x) and let the abscissae of C and D be *a* and *b* respectively. We have to find the volume generated by the revolution about the x-axis, of the area bounded by the given curve, the ordinates AC and BD and the x-axis. Take any point P(x, y) on the curve and let PM be its ordinates. Take another point $Q(x + \delta x, y + \delta y)$ on the curve close to P and let QN be its ordinate.

Let v and $v + \delta v$ respectively be the volumes of the solids generated by the revolution of the areas AMPC and ANQC about the x-axis.

Then clearly δv is the volume of the solid generated by the revolution of the elementary area MNQP about the x-axis and it lies between the volumes generated by the revolution of the rectangles MNRP and MNQS respectively about x-axis.

$\therefore \quad \pi y^2 \, \delta x < \delta v < \pi \, (y + \delta y)^2 \, \delta x$

or $\quad \pi y^2 < \dfrac{\delta v}{\delta x} < \pi \, (y + \delta y)^2 \qquad \text{......... (1)}$

Taking limit when $\delta x \to 0$, $\delta y \to 0$, then we have

$$\frac{dv}{dx} = \pi y^2$$

By integrating w.r. to x, we get

$$v = \int_a^b xy^2 dx$$

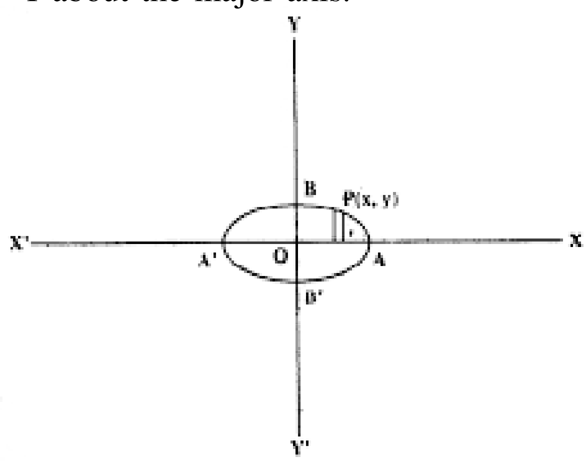
Hence proved.

Remark :- When the axis of revolution is the y-axis then the foumla becomes

$$V = \int_a^b \pi y^2 dx$$

Solved Examples

Example No. 1 : Find the volume generated by the revolution of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the major axis.



Sol. : The solid generated by the revolution of an ellipse about the major axis is called a prolate period.

$\therefore$ It is clear that the major axis is obtained by the area APA' about the axis. Also since the curve is symmetrical about both axis.

$$\therefore \quad \text{Required volume} = 2\int_0^a \pi\, y^2 dx$$

$$= 2\pi\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx$$

$$= 2b^2\pi\left|x-\frac{1}{a^2}\cdot\frac{x^3}{3}\right|_0^a$$

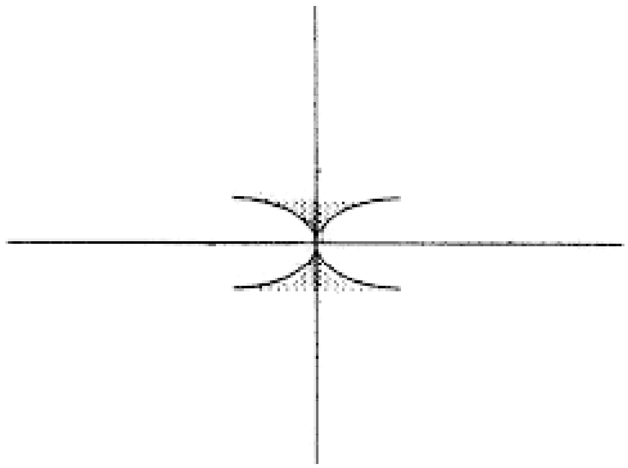
$$= 2b^2\pi\left[a-\frac{1}{a^2}\cdot\frac{a^3}{3}-0\right]$$

$$= 2b^2\pi\left(a-\frac{a}{3}\right)$$

$$= \frac{4ab^2\pi}{3}$$

Example 2 : The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the volume of the reel then generated.

Sol. : The tangent at the vertex is $x = 0$ *i.e.* y-axis. Since curve is symmetrical about x-axis.



$\therefore$ Required volume $= 2\int_0^{2a} \pi x^2 dy$

Required volume $= 2\pi\int_0^{2a} \cdot\left(\frac{y^2}{4a}\right)\cdot dy$

$$= \frac{2\pi}{16a^2}\int_0^{2a} y^4 dy$$

$$= \frac{\pi}{8a^2}\left|\frac{y^5}{5}\right|_0^{2a}$$

$$= \frac{\pi}{8a^2}\left[\frac{32a^5}{5} - 0\right]$$

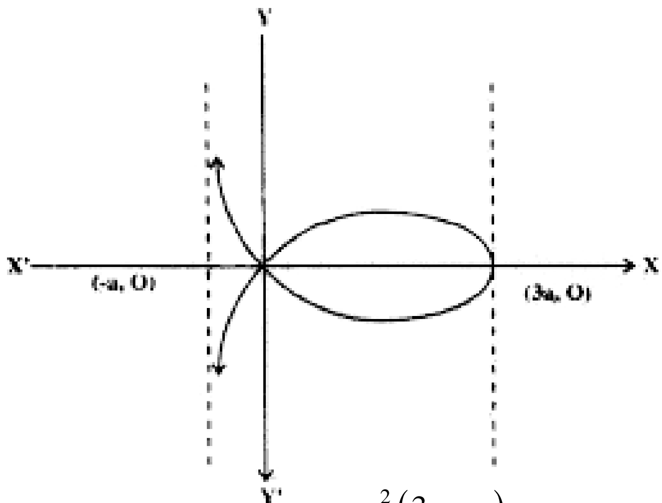
$$= \frac{4a^3\pi}{5}$$

Example 3 : Find the volume of the solid formed by the revolution about the x-axis of the loop of the curve $y^2 (a + x) = x^2 (3a - x)$.

Sol. : The given equation of the curve is

$$y^2 (a + x) = x^2 (3a - x) \qquad \text{........ (1)}$$

Since the curve is symmetrical about x-axis and meet x-axis at $x = 0$ & $x = 3a$.



Since from (1), $y^2 = \dfrac{x^2(3a - x)}{a + x}$

$\therefore$ $x \not< -a$. Hence a rough shape of the curve is as shown.

$\therefore$ Required Volume $= \displaystyle\int_0^{3a} \pi\, y^2 dx$

$$= \pi \int_0^{3a} \frac{x^2(3a - x)}{a + x} dx$$

$$= \pi \int_0^{3a} \frac{3ax^2 - x^3}{x + a}$$

$$= \pi \int_0^{3a} \left(-x^2 + 4ax - 4a^2 + \frac{4a^3}{x+a} \right) dx$$

$$= \pi \left| -\frac{x^3}{3} + \frac{4ax^2}{2} - 4a^2 \cdot x + 4a^3 \log(x+a) \right|_0^{3a}$$

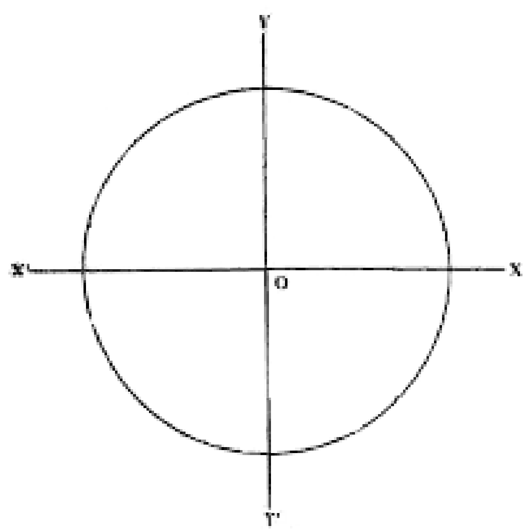
$$= \pi \left[-9a^3 + 18a^3 - 12a^3 + 4a^3 \log 4a - 4a^3 \log a \right]$$

$$= \pi \left[4a^3 \cdot \log \frac{4a}{a} - 3a^3 \right]$$

$$= a^3\pi\ [4\ \log\ 2^2 - 3]$$

$$= \pi a^3\ [8\ \log\ 2 - 3]$$

Example 4. Show that the volume of a shpere of a radius a is $\frac{4}{3}\pi a^3$.



Sol. : Since the equation of a sphere whose radius is 'a' is $x^2 + y^2 = a^2$. Since this curve is symmetrical about both axis. It meets x-axis at (a, 0) and (–9, 0)

$$\therefore \quad \text{Required volume} = \pi \int_{-a}^{a} y^2 \, dx$$

$$= \pi \int_{-a}^{a} \left(a^2 - x^2\right) dx$$

$$= \pi \left| a^2 x - \frac{x^3}{3} \right|_{-a}^{a}$$

$$= \pi \left[\left(a^3 - \frac{a^3}{3} \right) - \left(-a^3 + \frac{a^3}{3} \right) \right]$$

$$= \pi \left[\frac{2a^3}{3} - \left(-\frac{2a^3}{3} \right) \right]$$

$$= \frac{4a^3}{3} \pi$$

Example 5 : Find the volume of the solid generated by revolving about x-axis the loop of

(i) $y^2 (a - x) = x^2 (a + x)$

(ii) $y^2 (a + x) = x^2 (a - x)$.

Sol. : (1) Exercise for Students.

(2) It is symmetrical about x-axis and it meet x-axis at $x = 0$ & $x = a$.

$$\therefore \quad \text{Required volume} = \pi \int_{0}^{a} y^2 \cdot dx$$

$$= \pi\int_0^a \frac{ax^2 - x^3}{a+x}\,dx$$

$$= \pi\int_0^a \left(-x^2 + 2ax - 2a^2 + \frac{2a^3}{a+x}\right)dx$$

$$= \pi\left[-\frac{x^3}{3} + \frac{2ax^2}{2} - 2a^2x + 2a^3\log(a+x)\right]_0^a$$

$$= \pi\left[-\frac{a^3}{3} + a^3 - 2a^3 + \log 2a - 2a^3 \log a\right]_0^a$$

$$= a^3\pi\left[-\frac{4}{3} + 2\log 2\right]$$

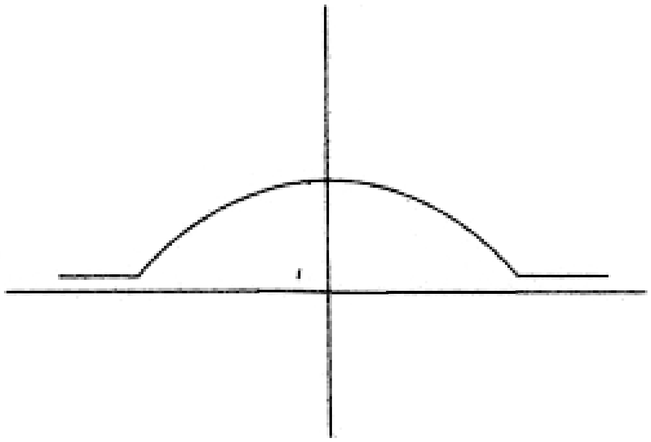
$$= 2\alpha^3\pi\left[\log 2 - \frac{2}{3}\right]$$

Example 6 : Find the volume of the solid formed by the revolution of the curve $y\ (a^2 + x^2) = a^3$ about the asymptote.

Sol. : The given equation of the curve

$$y = \frac{a^3}{a^2 + x^2} \qquad \text{.......... (1)}$$

Since the equation of the curve is symmetrical about y-axis and it meet y-axis at $y = a$. The asymptote parallel to x-axis is $y = 0$ while the curve has no other asymptote.



$$\therefore \quad \text{Required volume} = 2\int_0^\infty \pi\, y^2 dx$$

$$= 2\pi\int_0^\infty \left(\frac{a^3}{a^2+x^2}\right)^2 dx$$

$$= 2\pi\int_0^\infty \frac{a^6}{\left(a^2+x^2\right)} dx$$

Put $\quad x = a\tan\theta$

Diff. $\quad dx = a\sec^2\theta d\theta$

When $\quad x \to 0, \quad \theta \to 0$

$\quad x \to \infty, \quad \theta \to \frac{\pi}{2}$

$$\therefore \quad \text{Required Volume} = 2\pi a^6 \int_0^{\pi/2} \frac{a\sec^2\theta d\theta}{\left(a^2 + a^2\tan^2\theta\right)^2}$$

$$= \frac{2\pi a^6 \cdot a}{a^4} \int_0^{\pi/2} \frac{\sec^2 \theta d\theta}{\left(\sec^2 \theta\right)^2}$$

$$= 2\pi a^3 \int_0^{\pi/2} \frac{1}{\cos^2 \theta} \cdot \cos^4 \theta d\theta$$

$$= 2\pi a^3 \int_0^{\pi/2} \cos^2 \theta d\theta$$

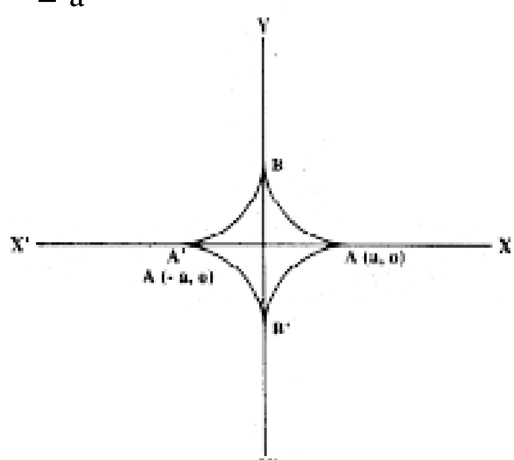
$$= 2\pi a^3 \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2 a^3}{2}$$

Example 7 : Find the volume of the solid generated by revolving the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ ahout x–axis.

Sol. : Given equation of the curve

$$x^{2/3} + y^{2/3} = a^{2/3} \qquad \text{........ (1)}$$



Since the curve is symmetrical about both axis. It meet x-axis at x = a and y-axis at x = a.

$\therefore$ Required volume = $2\int_0^a \pi y^2\, dx$

Now from (1), $y^{2/3} = a^{2/3} - x^{2/3}$

$\therefore$ $y^2 = (a^{2/3} - x^{2/3})^3\, dx$

$\therefore$ Volume = $2\pi\int_0^a \left(a^{2/3} - x^{2/3}\right)^3 dx$

Put $x = a\sin^3\theta$

when $x \to 0$, $\theta \to 0$

$x \to a$, $\theta \to \frac{\pi}{2}$

Diff.

$dx = 3a\sin^2\theta \cdot \cos\theta\, d\theta$

$\therefore$ Volume $= 2\pi\int_0^{\pi/2}\left(a^{2/3} - a^{2/3}\cdot\sin^2\theta\right)\cdot 3a\sin^2\theta\cdot\cos\theta\, d\theta$

$= 2\pi\cdot 3a\cdot a^2\int_0^{\pi/2}\left(1-\sin^2\theta\right)^3\cdot\sin^2\theta\cdot\cos\theta\, d\theta$

$= 6\pi a^3\int_0^{x/2}\cos^6\theta\cdot\sin^2\theta\cdot\cos\theta\, d\theta$

$= 6\pi a^3\int_0^{x/2}\cos^7\theta\cdot\sin^2 d\theta$

$$= 6\pi a^3 \cdot \frac{6.4.2}{9.7.5.3}$$

$$= \frac{32\pi a^3}{105}$$

EXERCISE

Q 1. Find the volume of the solid generated by revolving about x-axis the loop of $a^2y^2 = x^2 (a^2–x^2)$.

Q 2. Find the volume by the revolution of the loop of the curve

$$y^2 (a - x) = x^2 (a + x).$$

Q 3. The loop of the curve w $ay^2 = x (x–a)^2$ revolves about x-axis. Find the volume of the solid generated.

Q 4. Find the volume of the solid formed by the revolution of the curve

$y^2 (2a–x) = x^3$ about its asymptote.

Q 5. The part of the curve $y^2 = x^2 (1–x^2)$ between x = 0 and x = 1 rotates about the x-axis. Obtain the volume of the solid thus generated.

SURFACE OF A SOLID OF REVOLUTION

INTRODUCTION :

The surface generated by the boundary of a plane area is called the surface of revolution.

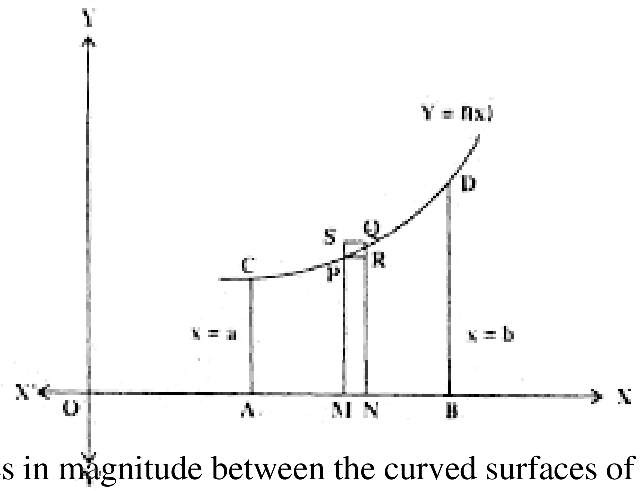
Art : Prove that the curved surface of solid generated by the revolution about the x-axis, of the area bounded by the curve y = f(x), the x-axis and the ordinates x = a & x = b is $\int_a^b 2\pi y dS$; where S is the length of the arc of the curve measured from a fixed point on it to any point P(x, y).

Sol. : Let CD be the arc of the curve y = f(x) lying between the ordinates x = a and x = b. Take two points P (x, y) and Q(x + δx, y + δy) on the curve close to each other and let PM and QN be their ordinates.

let δs = elementary arc length PQ

and δS = Area of the elementary zone traced out by the revolution of the arc PQ about OX.

To know the surface generated by the revolution of the arc CD, we have the curved surface of the solid generated by the revolution of the area MNQP about the



x-axis lies in magnitude between the curved surfaces of the two right circular cylinder, each of thickness δs, one of which has the radius PM (= y) and the other QN (= y + δy)

$\therefore \quad 2\pi y\delta s < \delta S < 2\pi (y + \delta y)\, \delta s$

$\Rightarrow \quad 2\pi y < \dfrac{\delta S}{\delta s} < 2\pi (y + \delta y)$

$\therefore$ When $\delta S \to 0$, $\delta y \to 0$ and we get

$$\frac{dS}{ds} = 2\pi y$$

$\Rightarrow \quad dS = 2\pi y \; ds$

By integrating we get

$$\int_a^b dS = \int_a^b 2\pi y ds$$

$$\Rightarrow \quad S = \int_a^b 2\pi y ds$$

Remarks : Since we proved that the surface area of the solid generated by the revolution about the x-axis of the area bounded by the curve y = f(x), the x-axis, the ordinates x = a and x = b is

$$\int_a^b 2\pi y ds \text{ or } \int_a^b 2\pi y \frac{ds}{dx} \cdot dx \text{ where } \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Hence the surface area of the solid generated by the revolution about the y = axis of the area bounded by the curve y = f(x), the y-axis, the absissae y = c and y = d is

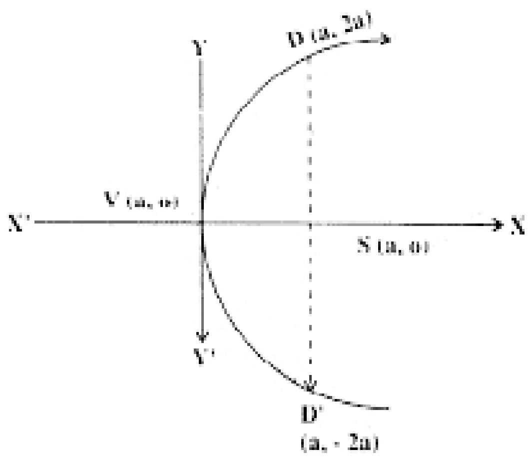
$$\int_c^b 2\pi x ds$$

Solved Examples

Example 1. Find the area of the surface formed by the revolution of $y^2 = 4ax$ about x-axis by the arc from vertex. to the end of the latus rectum.

Sol. : Given =n of the curve is

$$y^2 = 4ax \qquad \text{......... (1)}$$



$$\Rightarrow \quad y = 2\sqrt{a}\cdot\sqrt{x} = 2\sqrt{a}\cdot x^{1/2}$$

$$\frac{dy}{dx} = 2\sqrt{a}\cdot\frac{1}{2}x^{-1/2}$$

$$= \frac{\sqrt{a}}{\sqrt{x}}$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1+\frac{a}{x}}$$

$$= \frac{a+x}{\sqrt{x}}$$

If S denotes the surface, then

$$\text{Required surface} = \int_0^a 2\pi y dx$$

$$= 2\pi\int_0^a 2\sqrt{a}\cdot\sqrt{x}\cdot\frac{\sqrt{a+x}}{\sqrt{x}}\,dx$$

$$= 4\pi\sqrt{a}\int_0^a (a+x)^{1/2}\,dx$$

$$= 4\pi\sqrt{a}\left|\frac{(a+x)^{1/2+1}}{\frac{1}{2}+1}\right|_0^a$$

$$= 4\pi\sqrt{a}\cdot\frac{2}{3}\left[(2a)^{3/2} - a^{3/2}\right]$$

$$= \frac{8\pi\sqrt{a}}{3}\,[a^{3/2}((2)^{3/2}-1)]$$

$$= \frac{8}{3}\pi a^2\left(2\sqrt{2}-1\right)$$

Example 2 : Show that the surface area of a sphere of radius 'a' is $4\pi\ a^2$.

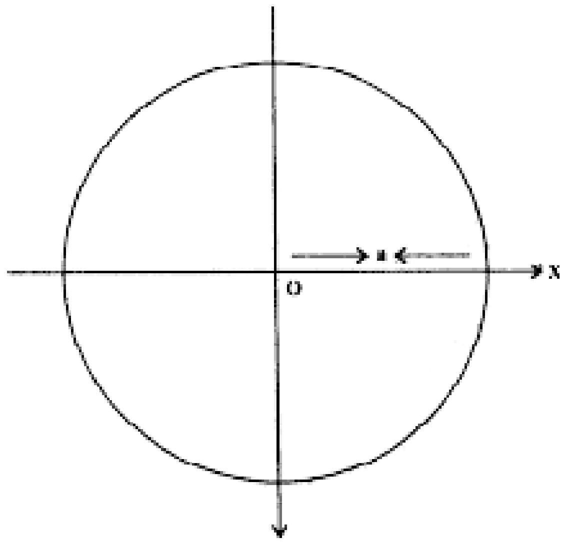
Sol. : Given equation of sphere is

$$x^2 + y^2 = a^2$$

$$y^2 = a^2 - x^2$$

$$y = (a^2 - x^2)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}(a^2 - x^2)^{-1/2}(-2x)$$



If S denotes the surface. Then

Required surface $= 2\int_0^a 2\pi y ds$

$$= 2\cdot 2\pi\int_0^a \sqrt{a^2 - x^2}\cdot\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$$

$$= 4\pi\int_0^a \sqrt{a^2 - x^2}\cdot\sqrt{1+\frac{x^2}{a^2 - x^2}}\, dx$$

$$= 4\pi\int_0^a \sqrt{a^2 - x^2}\cdot\frac{\sqrt{a^2 - x^2 + x^2}}{\sqrt{a^2 - x^2}}\, dx$$

$$= 4\pi\int_0^a a dx$$

$$= 4\pi a\ \left|x\right|_0^a$$

$= 4\pi a\ [a - 0]$

$= 4\pi a^2$

Example 3 : Find the surface of the solid generated by the revolution of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis.

Sol. : Given equation of the curve

$$x^{2/3} + y^{2/3} = a^{2/3}$$

$$y^{2/3} = a^{2/3} - x^{2/3}$$

$$\therefore \quad y = (a^{2/3} - x^{2/3})^{3/2}$$

$$\frac{dy}{dx} = \frac{3}{2}(a^{2/3} - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{1/3}\right)$$

$$= \frac{-\left(a^{2/3} - x^{2/3}\right)^{1/2}}{x^{1/3}}$$

$\therefore$ If S denotes the required surface, then

$$S = 2\int_0^a 2\pi y dx$$

$$= 2\pi\int_0^a \left(a^{2/3} - x^{2/3}\right)^{3/2} \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\pi\int_0^a \left(a^{2/3} - x^{2/3}\right)^{3/2} \cdot \sqrt{1 + \frac{a^{2/3} - x^{2/3}}{x^{2/3}}\, dx}$$

$$= 4\pi \int_0^a \left(a^{2/3} - x^{2/3}\right)^{3/2} \cdot \frac{a^{1/3}}{x^{1/3}} dx$$

$$= -4\pi a^{1/3} \int_0^a \left(a^{2/3} - x^{2/3}\right)^{3/2} \left(-x^{-1/3}\right) dx$$

$$= -4\pi a^{1/3} \left| \frac{\left(a^{2/3} - x^{2/3}\right)}{\frac{3}{2}+1} \cdot \frac{3}{2} \right|_0^a$$

$$= -4\pi a^{1/3} \cdot \frac{2}{5} \cdot \frac{3}{2} [0-(a^{2/3})^{5/2}]$$

$$= -4\pi a^{1/3} \cdot \frac{3}{5} \cdot a^{5/3} = \frac{12\pi a^2}{5}$$

EXERCISE

Q. 1. Find the area of the surface formed by revolving $x^2 + 4y^2 = 16$ about x-axis.

Q. 2. A loop of $8a^2 y^2 = x^2 (a^2 - x^2)$ revolved about x-axis. Find the surface of solid formed.

Q. 3. Find the curved surface of the solid generated by the revolution about the x-axis of the area bounded by the parabola $y^2 = 4ax$, the ordinate $x = 3a$ and y-axis.
